COLLOQUIUM MATHEMATICUM

VOL. VII 1960 FASC. 2

ON A PROBLEM OF S. HARTMAN ABOUT NORMAL FORMS

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With the usual notation let x, y be integers and

$$\min_y |x\alpha-\beta-y| = \|x\alpha-\beta\|.$$

A pair of numbers (α,β) is called *normal*, *positively normal* and *negatively normal* if the inequality

$$\|x\alpha-\beta\| < 1/t \tag{1}$$

is soluble for *any* $t > t_0$ with $|x| < ct$, $0 < x < ct$ and $-ct < x < 0$ respectively, where t_0, c depend only on β and α.

S. Hartman [2] raised the question, whether or not a normal pair is necessarily positively or negatively normal. In this note we shall give a negative answer to this question constructing a normal pair (α,β) which is neither positively nor negatively normal. Before the proof we remark the following:

1. Suppose that $0 < \alpha < 1$, α is irrational,

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_k+\cdots},$$

a_k $(k = 1, 2, \ldots)$ are positive integers,

$$p_0 = -1, \quad q_0 = 0, \quad p_1 = 0, \quad q_1 = 1,$$

$$\frac{p_k}{q_k} = \frac{1}{a_1+}\cdots\frac{1}{a_{k-1}} \quad (k = 2, 3, \ldots)$$

the convergents of α, and $d_k \stackrel{\mathrm{def}}{=} q_k\alpha - p_k$ (1) $(d_0 = -1)$. Then we have

(1) The sequence d_k has alternative signs for $k = 1, 2, \ldots$ and $|d_k|$ is monotonically decreasing.

the well-known recursive formulas

$$q_{k+1} = a_k q_k + q_{k-1} \quad (k = 1, 2, \ldots), \tag{2}$$

$$d_{k+1} = a_k d_k + d_{k-1} \quad (k = 1, 2, \ldots). \tag{3}$$

We note the identity

$$1 = d_1 + \sum_{k=1}^{\infty} a_k d_k \tag{4}$$

as a consequence of (2) and (3), and

$$\frac{1}{a_k+1} < |d_k| q_k < \frac{1}{a_k} \quad (k = 1, 2, \ldots). \tag{5}$$

In [1] and [3] it is proved that for any $0 < \beta < 1$ with the d_k's defined above, it is possible to determine uniquely a sequence b_k of non-negative integers – which we call throughout this note *digits* of β according to α – with the following properties:

(a) $$\beta = \sum_{\nu=1}^{\infty} b_\nu d_\nu;$$

(b) $$1 \leqslant b_1 \leqslant a_1 + 1, \quad 0 \leqslant b_\nu \leqslant a_\nu \quad (\nu = 2, 3, \ldots);$$

(c) if ξ is defined by

$$\min_{0<x<t} \|x\alpha - \beta\| = \|\xi\alpha - \beta\|,$$

then with suitable l, r

$$\xi = b_1 q_1 + \ldots + b_{l-1} q_{l-1} + r q_l \quad (0 \leqslant r < b_l). \tag{6}$$

As in [3], (2.13), it is easy to see that

$$\|\xi\alpha - \beta\| = \left| b_1 d_1 + \ldots + b_{l-1} d_{l-1} + r d_l - \sum_{\nu=1}^{\infty} b_\nu d_\nu \right| \tag{7}$$

$$= \left| (b_l - r) d_l + \sum_{\nu=l+1}^{\infty} b_\nu d_\nu \right|.$$

According to (b) and footnote (1) we have, using (6),

$$\left| \sum_{\nu=l}^{\infty} b_\nu d_\nu \right| \leqslant \sum_{\nu=0}^{\infty} a_{l+2\nu} |d_{l+2\nu}| = |d_{l-1}|.$$

Therefore we get from (7)

$$(b_l - \nu - 1)|d_l| \leqslant \|\xi\alpha - \beta\| \leqslant (b_l - r + 1)|d_l| \tag{8}$$

and in the case $r = b_l - 1$, when $a_{l+1} > b_{l+1}$,

$$\|\xi\alpha-\beta\| = \Big|d_l + \sum_{\nu=l+1}^{\infty} b_\nu d_\nu\Big| = \Big|d_{l+2} - a_{l+1}d_{l+1} + b_{l+1}d_{l+1} + \sum_{\nu=l+2}^{\infty} b_\nu d_\nu\Big|,$$

and consequently, since d_{l+1} and $-d_{l+2}$ have the same sign,

(9) $$(a_{l+1}-b_{l+1})|d_{l+1}| \leqslant \|\xi\alpha-\beta\| \leqslant (a_{l+1}-b_{l+1}+2)|d_{l+1}|.$$

As to the uniqueness of the representation (a) we remark that if

(10) $$0 < b_k < a_k, \quad 0 \leqslant b_k^* \leqslant a_k,$$

and

$$\gamma = \sum_{\nu=1}^{\infty} b_\nu d_\nu = \sum_{\nu=1}^{\infty} b_\nu^* d_\nu,$$

then

(11) $$b_\nu = b_\nu^* \quad (\nu = 1, 2, \ldots).$$

Namely, if there exists an index l — and we take without loss of generality the first one for which $b_l \neq b_l^*$ — then we consider

$$\gamma' = \sum_{\nu=l}^{\infty} b_\nu d_\nu = \sum_{\nu=l}^{\infty} b_\nu^* d_\nu.$$

According to footnote (1)

$$(-1)^{l+1}\sum_{\nu=l}^{\infty} b_\nu^* d_\nu = b_l^*|d_l| - \sum_{\nu=0}^{\infty} b_{l+2\nu+1}^*|d_{l+2\nu+1}| + \sum_{\nu=1}^{\infty} b_{l+2\nu}^*|d_{l+2\nu}|$$

$$\overset{\text{def}}{=} b_l^*|d_l| - \sum\nolimits_1^* + \sum\nolimits_2^*,$$

$$(-1)^{l+1}\sum_{\nu=l}^{\infty} b_\nu d_\nu = b_l|d_l| - \sum_{\nu=0}^{\infty} b_{l+2\nu+1}|d_{l+2\nu+1}| + \sum_{\nu=1}^{\infty} b_{l+2\nu}|d_{l+2\nu}|$$

$$\overset{\text{def}}{=} b_l|d_l| - \sum\nolimits_1 + \sum\nolimits_2.$$

From (3) and (10)

$$0 \leqslant \sum\nolimits_1^* \leqslant |d_l|, \quad 0 \leqslant \sum\nolimits_2^* \leqslant |d_{l+1}|,$$

and consequently

(12) $$(b_l^*-1)|d_l| \leqslant (-1)^{l+2}\gamma' \leqslant b_l^*|d_l| + |d_{l+1}|.$$

Similarly, taking into account that $0 < b_\nu < a_\nu$ and using (3), we get

$$|d_{l+1}| < \sum\nolimits_1 < |d_l| - |d_{l+1}|, \quad 0 < \sum\nolimits_2 < |d_{l+1}|,$$

and consequently

(13) $$(b_l-1)|d_l| + |d_{l+1}| < (-1)^{l+1}\gamma' < b_l|d_l|.$$

Since b_l and b_l^* are integers, (12) and (13) cannot be satisfied simultaneously, which means that (11) holds.

2. According to our remark in 1, we show that if α is the number defined by

$$(14) \qquad a_k = k^4$$

and β is the number for which

$$b_1 = 1,$$
$$(15) \qquad b_k = \begin{cases} k & \text{if} \quad k = 2\nu, \\ k^4 - k & \text{if} \quad k = 2\nu+1, \end{cases} \qquad (\nu = 1, 2, \ldots),$$

then the pair α, β is normal, but neither positively nor negatively normal.

From (4) and (a)

$$(16) \qquad \beta' = 1-\beta = (a_1+1-b_1)d_1 + \sum_{k=2}^{\infty}(a_k - b_k)d_k.$$

As we proved in 1, from (15) it follows that the digits of β' according to α are uniquely determined by (16) and

$$b_1' = (a_1+1-b_1),$$
$$b_k' = a_k - b_k = k^4 - k \quad \text{if} \quad k = 2\nu,$$
$$b_k' = a_k - b_k = k \quad \text{if} \quad k = 2\nu+1. \qquad (\nu = 1, 2, \ldots),$$

Let

$$c_k \overset{\text{def}}{=} b_1 q_1 + \ldots + b_k q_k, \qquad c_k' \overset{\text{def}}{=} b_1' q_1 + \ldots + b_k' q_k.$$

For any $c_1 < t$ we determine the index k by

$$c_{k-1} < t < c_k.$$

At first we prove that the pair (α, β) defined by (14) and (15) is normal. For the proof we distinguish three cases.

Case a. If $k = 2\nu$, then from (8) with $\nu = 0$, $l = k$,

$$\|c_{k-1}\alpha - \beta\| \leqq (k+1)|d_k|.$$

According to (2) and (14)

$$(17) \qquad t \leqq b_1 q_1 + \ldots + b_k q_k < a_1 q_1 + \ldots + a_{k-1} q_{k-1} + k q_k < (k+2) q_k.$$

Hence by (5) we get for $k > 3$

$$\|c_{k-1}\alpha - \beta\| < \frac{2}{k^3 q_k} < \frac{1}{t},$$

i. e. $x = c_{k-1}$ is a solution of inequality (1).

Case b. If $k = 2\nu+1$ and

$$c_{k-1} < t \leqslant (k+2)q_k,$$

then, just as before, we get by (9) with $l = k-1$

$$\|(c_{k-1}-q_{k-1})\alpha-\beta\| \leqslant (a_k-b_k+1)|d_k| = (k+1)|d_k| < \frac{k+1}{k^4}\cdot\frac{1}{q_k} < \frac{1}{t},$$

i. e. $x = c_{k-1}-q_{k-1}$ is a solution of inequality (1).

Case c. If $k = 2\nu+1$ and

$$(k+2)q_k < t \leqslant c_k,$$

then, similarly to (17),

$$c_k' < (k+2)q_k < t \leqslant c_k < (k^4-k+2)q_k.$$

Using (8) with $r = 0$ and with $l = k+1$, we get

$$\|c_k'\alpha-\beta'\| < (b_{k+1}'+1)|d_{k+1}| < a_{k+1}|d_{k+1}| < |d_k| < \frac{1}{k^4 q_k} < \frac{1}{t},$$

i. e. $x = c_k'$ is a solution of

$$\|x\alpha-(1-\beta)\| = \|-x\alpha-\beta\| < \frac{1}{t},$$

i. e. in this case (1) has a solution with $x = -c_k'$.

Now we show that our pair α, β defined in (14) and (15) is neither positively nor negatively normal. In order to show the first part, it is sufficient to give to an arbitrary prescribed c a sequence $t_\nu \to \infty$, so that (1) has no solution with $0 < x < ct_\nu$ for $\nu = 1, 2, \ldots$

In order to prove it, let $k = 2\nu+1$, $t_\nu = c_{k-1}+2k^3 q_k$ and

$$\min_{0<x<ct_\nu} \|x\alpha-\beta\| = \|\xi_\nu\alpha-\beta\|.$$

From (8) and (9) it follows that for all $\xi = b_1q_1+\ldots+b_{l-1}q_{l-1}+rq_l$ with $l < k-1$ or with $l = k-1$ and $r < b_{k-1}-1$ we have

$$\|\xi\alpha-\beta\| > |d_{k-1}|,$$

whence, by (5) and the definition of t, we conclude

$$\|\xi\alpha-\beta\| > 1/t_\nu.$$

On the other hand, we have for k large enough the inequality $\xi_\nu \leqslant ct_\nu < c_k$. Thus (6) shows that it remains to examine as values of ξ_ν only the numbers

$$\xi' = b_1q_1+\ldots+b_{k-1}q_{k-1}+rq_k$$

with $0 \leqslant r < 2k^3c$, and

$$\xi'' = b_1q_1+\ldots+b_{k-2}q_{k-2}+(b_{k-1}-1)q_{k-1}.$$

Using (5) and (8) we get for k large enough

$$\|\xi' a-\beta\| > (b_k-r-1)|d_k| > \frac{k^4-3k^3c-1}{(k^4+1)q_k} > \frac{1}{2k^3 q_k} > \frac{1}{t_\nu}$$

and, similarly, using (8) for k large enough

$$\|\xi'' a-\beta\| > (a_k-b_k)|d_k| > \frac{k}{(k^4+1)q_k} > \frac{1}{2k^3 q_k} > \frac{1}{t_\nu},$$

i. e. for $t_\nu = c_{k-1}+2k^3 q_k$, $k=2\nu+1$, $\nu>\nu_0$, the inequality

$$\|x\alpha-\beta\| < 1/t$$

has no solution with $0 < x < ct$.

In an analogous way it is possible to show that for $t = c'_{k-1}+2k^3 q_k$, $k=2\nu$, $\nu>\nu_0$, inequality (1) has no solution with $-ct < x < 0$, which completes the proof.

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Reçu par la Rédaction le 10. 4. 1959