ON A PROBLEM IN THE THEORY OF SIMULTANEOUS APPROXIMATION

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To the memory of Professor L. FEJÉR

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ irrational numbers, p, q positive integers, and

$$\min_p |q\alpha - p| \stackrel{\text{def}}{\equiv} \|q\alpha\|.$$

As a theorem of Dirichlet it is well known, that there exist infinitely many q 's, so, that

(1) $$\|q\alpha_\nu\| < \frac{1}{q^{\frac{1}{k}}} \qquad \nu = 1, 2, \ldots, k$$

which means, that the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ are simultaneously well approxi mable with rational numbers $\frac{p_\nu}{q}$ (with common denominators). In the case $k = 1$ the continued fraction algorithm gives a satisfactory process for the construction and characterization of the q 's, for which

(2) $$\|q\alpha\| < \frac{1}{q}.$$

In case $k > 1$ until now no algorithm corresponding to the continued fraction algorithm is known, and this is why „exact theorem" of simultaneous approximation in case $k > 1$ are not known.

P. TURÁN raised the question what sort of a localisation can be given for the integers q satisfying (1). After Turán's first results P. Szüsz solved in [1] that for $k = 1$, $0 < \vartheta < 1$

$$\|q\alpha\| < \frac{1}{q^\vartheta}$$

has a solution in the interval $N \leq q \leq N^{1+\vartheta}$ for any N. He proved moreover, that this result is best-possible in the sense, that the exponent $1 + \vartheta$ generally cannot be replaced by any universal $\vartheta' < 1 + \vartheta$ (and even a slightly stronger result.)

We prove a theorem corresponding the general case $k \geq 1$. This theorem goes for $k = 1$ into Szüsz's result, apart from the constant factor. In contrary to the case $k = 1$ (i. e Szüsz's result) we cannot prove that our localisation is best possible.

Theorem. *Let* $\alpha_1, \ldots, \alpha_k$ *be irrational* $0 < \vartheta \leq \frac{1}{k}$. *The inequalities*

$$\|q\,\alpha_\nu\| < \frac{2^{k+1}}{q^\vartheta} \qquad \nu = 1, 2, \ldots, k$$

have for all N always a solution with q integers satisfying the condition

$$N \leq q \leq N^{\frac{1+\vartheta}{1-\vartheta(k-1)}}.$$

This means for example, that the „generally best possible" approximation

$$\|q\,\alpha_\nu\| < \frac{2^{k+1}}{q^{\frac{1}{k}}} \qquad \nu = 1, 2, \ldots, k$$

is attainable under the condition

$$N \leq q \leq N^{k+1}.$$

For the proof we need the following

Lemma. Let α irrational, $0 < \varrho < 1$, $q_1 < q_2 < \ldots < q_M$ positive integers. There are at least $[M\,\varrho]$ positive integers

$$q_1' < q_2' < \ldots < q'_{[M\varrho]}$$

of the form $q_\nu' = q_{e_\nu} - q_i\ (\leq q_M)$ (i fixed, $\nu = 1, 2, \ldots, [M\varrho]$) for which

$$\|q_\nu'\,\alpha\| < \delta.$$

(For example in case the $q_\nu = \nu$, ($\nu = 1, \ldots, M$) there are at least $[M\varrho]$ integers under the condition $0 \leq q' \leq M$ for which

$$\|q'\,\alpha\| < \varrho.)$$

For the proof of this Lemma we consider the circle with unitperiphery. From a starting point 0 we put on the arcs with length α in positive direction once, twice, ..., n-times. We shall call the end-points of these arcs as α-, 2α-, ..., $n\,\alpha$-points. If we consider the $q_1\alpha$-, $q_2\alpha$-, ... $q_M\,\alpha$-points on the circle, there is somewhere on the circle an arc with length ϱ, which contains at least $M\varrho$ points among these. Let these points be the $q_{\nu_1}\,\alpha$-, $q_{\nu_2}\,\alpha$-, ..., $q_{\nu_{[M\varrho]}}\,\alpha$-points, ($q_{\nu_1} < q_{\nu_2} < \ldots < q_{\nu_{[M\varrho]}}$). Since the $m\,\alpha$-, and $n\,\alpha$-points ($m < n$) on the circle have the same relative position and distance, as the 0 and $(n-m)$ α-points, the

$$0, (q_{\nu_2} - q_{\nu_1})\,\alpha\text{-}, \ldots, (q_{\nu_{[M\varrho]}} - q_{\nu_1})\,\alpha\text{-}$$

points all are in an arc with length ϱ and containing the point 0. This means, that

$$\|(q_{\nu_i} - q_{\nu_1})\,\alpha\| < \varrho \qquad i = 1, 2, \ldots, [M\,\varrho]$$

and this proves the Lemma.

For the proof of the theorem in what follows let $\gamma > 1$, $\delta \leq \frac{\gamma}{k}$. γ and δ will to be determined later. According the Lemma — applied with $M_1 \overset{\text{def}}{\equiv} [N^\gamma]$ and $\delta = \frac{1}{N^\vartheta}$ — we can assert that

$$\|q\,\alpha_1\| < \frac{1}{N^\delta} \tag{3}$$

has at least $M_2 \overset{\text{def}}{\equiv} [N^{\gamma-\delta}]$ solution in q's with $1 \leq q \leq N^\gamma$. Let these q' s be

$$q_1^{(1)} < q_2^{(1)} < \ldots < q_{M_2}^{(1)}.$$

If we consider the $q_\nu^{(1)}\alpha_2$-points for $\nu = 1, 2, \ldots, M_2$, we can assert —according to Lemma — that there exist at least $M_3 \overset{\text{def}}{\equiv} [N^{\gamma-2\delta}]$ multipla, for which the inequality

$$\|q\,\alpha_2\| < \frac{1}{N^\delta} \tag{4}$$

holds. We denote these by

$$q_1^{(2)} < q_2^{(2)} < \ldots < q_{M_3}^{(2)}.$$

Owing to the Lemma, we have with a suitable i

$$q_\nu^{(2)} = q_{e_\nu}^{(1)} - q_j^{(1)}, \qquad \nu = 1, 2, \ldots, M_3.$$

Since the $q_\nu^{(2)}$'s satisfy (4) and the $q_\nu^{(1)}$'s satisfy (3), we have

$$\|q_\nu^{(2)}\,\alpha_i\| < \frac{2}{N^\delta}, \qquad \begin{array}{l} \nu = 1, 2, \ldots, M_3 \\ i = 1, 2. \end{array}$$

Continuing this construction k-times, we get the

$$q_\nu^{(k)} \overset{\text{def}}{=} q_\nu \qquad \nu = 1, 2, \ldots, \quad M_{k+1} = [N^{\gamma-k\delta}]$$

multipla, with $1 \leq q_\nu \leq N^\gamma$ having the property

$$\|q_\nu\,\alpha_i\| < \frac{2^{k-1}}{N^\delta} \qquad \begin{array}{l} \nu = 1, 2, \ldots, M_{k+1} \\ i = 1, 2, \ldots, k. \end{array} \tag{5}$$

If there among these is a q_0 with $N \leq q_0 \leq N^\gamma$, then for this we get

$$\|q_0\,\alpha_i\| < \frac{2^k}{N^\delta} \leq \frac{2^k}{q_0^{\frac{\delta}{\gamma}}} \qquad i = 1, 2, \ldots, k. \tag{6}$$

If we have

$$1 \leq q_\nu \leq N,$$

for any $\nu = 1, \ldots, M_{k+1} = [N^{\gamma - k\delta}]$
then we have among these at least one q_0 with

$$N^{\gamma - k\delta} \leq q_0 \,(< N). \tag{7}$$

We define integers q_0' and p with

$$N < p\, q_0 \overset{\text{def}}{\equiv} q_0' < 2N.$$

From this and (7) we have

$$1 \leq p < 2 N^{1-(\gamma - k\delta)} \tag{8}$$

and therefore, according to (5) and (8)

$$\|q_0' \alpha_i\| < 2 N^{1-(\gamma - k\delta)} \|q_0 \alpha_i\| < \frac{2^k}{N^{\gamma-(k-1)\delta-1}} < \frac{2^{k-1+\gamma+(k-1)\delta}}{q_0'^{\gamma-(k-1)\delta-1}}, \qquad i = 1, 2, \ldots, k.$$

So we obtained in any case that there is a q with $N \leq q \leq N^\gamma$ for which either (6) or (9) holds. Restricting our γ and δ by

$$\frac{\delta}{\gamma} = \gamma - (k-1)\delta - 1; \qquad \delta = \frac{\gamma - 1}{k - 1 + \dfrac{1}{\gamma}}$$

we obtain that there is q in any case with $N \leq q \leq N^\gamma$ for which

$$\|q\alpha_i\| < \frac{2^{k+1}}{q^{\frac{\gamma-1}{\gamma(k-1)+1}}} \qquad i = 1, 2, \ldots k.$$

Finally, choosing γ so, that

$$\vartheta = \frac{\gamma - 1}{\gamma(k-1)+1}; \qquad \gamma = \frac{1+\vartheta}{1-(k-1)\vartheta}$$

there is

$$N \leq q \leq N^{\frac{1+\vartheta}{1-(k-1)\delta}}$$

for which

$$\|q\alpha_i\| < \frac{2^{k+1}}{q^\vartheta} \qquad i = 1, 2, \ldots, k,$$

which proves our Theorem.

Reference

[1] P. Szüsz, Bemerkungen zur Approximation einer reellen Zahl durch Brüche, *Acta. Math. Acad. Sci. Hung.*, **6** (1955), 203—212.