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On two problems of Erdös, Szüsz and Turán concerning diophantine approximations

by

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1. Introduction. The present paper concerns itself with the following pair of problems posed by Erdös, Szüsz and Turán [2]:

Problem 1. *For* $A > 0$, $c \geqslant 1$, *let*

$$S(N, A, c) = \text{set of } \xi \in [0,1] \text{ which satisfy } |b\xi - a| \leqslant Ab^{-1} \text{ for some integers } a, b \text{ with } N \leqslant b \leqslant cN, (a, b) = 1.$$

Does

$$\lim_{N\to\infty} |S(N, A, c)| \tag{1.1}$$

exist, and if so, what is its value? (If C is a set, $|C|$ denotes its Lebesgue measure.)

If $|b\xi - a| \leqslant (2b)^{-1}$, then a/b must be a continued fraction convergent of ξ. ([5], Chapter 10.) The next problem is therefore closely related to problem 1.

Problem 2. *For* $c \geqslant 1$, *let*

$$T(N, c) = \text{set of } \xi \in [0,1] \text{ which have at least one continued fraction convergent } p_n/q_n \text{ with } N \leqslant q_n \leqslant cN.$$

Does

$$\lim_{N\to\infty} |T(N, c)| \tag{1.2}$$

exist, and if so, what is its value?

Originally, these problems were treated by means of the methods of the article immediately following this one [7]. It was noticed, however, by the second author that a much simpler, almost self contained treatment of these problems is possible and it is our aim to present this treatment here.

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As one could more or less expect, the limits (1.1) and (1.2) indeed exist. We give the explicit value for a more general expression than (1.2) in Theorem 1 and use this to show the existence of (1.1). The explicit value of (1.1), however, is not found. The limit (1.1) has been evaluated though for $A \leqslant e^{-1}$ by another method ([2], [6]). Estimates for (1.1) have also been given in [1] and [4]. We introduce some notation to give a more precise statement of the results.

Denote the regular continued fraction of an irrational(¹)(²) $\xi \in (0,1)$ by

$$[a_1(\xi), a_2(\xi), \ldots] = \cfrac{1}{a_1(\xi) + \cfrac{1}{a_2(\xi) \ldots}}$$

and its nth convergent by $p_n(\xi)/q_n(\xi)$. One has the well-known recursion formulae ([5], chapter 10)

$$q_0 = 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \tag{1.3}$$

$$p_0 = 0, \quad p_1 = 1, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}. \tag{1.4}$$

Introduce also

$$\begin{aligned} a'_{n+1} = a'_{n+1}(\xi) &= a_{n+1} + [a_{n+2}, a_{n+3}, \ldots] \\ &= a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} \ldots}} = a_{n+1} + \frac{1}{a'_{n+2}} \end{aligned} \tag{1.5}$$

and

$$q'_{n+1} = q'_{n+1}(\xi) = a'_{n+1}q_n + q_{n-1} = q_{n+1} + \frac{q_n}{a'_{n+2}} = \frac{q'_{n+2}}{a'_{n+2}}. \tag{1.6}$$

The main tool we use is

LEMMA 1. *Let $k_2 > k_1 \geqslant 1$, $(k_1, k_2) = 1$ and $z \geqslant 1$. Put*

$$A(k_1, k_2, z) = \{\xi : 0 \leqslant \xi \leqslant 1, \text{ there exists an } n \geqslant 1 \text{ for which } q_{n-1} = k_1, q_n = k_2, a'_{n+1} \geqslant z\}.$$

Then

$$|A(k_1, k_2, z)| = \frac{2}{k_2(zk_2 + k_1)}. \tag{1.7}$$

By means of this lemma it is easy to solve problem 2. In fact, we prove a more general result.

(¹) We shall ignore rational ξ's all the time. They form a set of measure zero and therefore do not influence the metric results.

(²) We use the notation of chapter 10 of [5] except that we drop $a_0(\xi) = [\xi]$ from our formulae, since $a_0(\xi) = 0$ in all our considerations.

THEOREM 1. *Put*

$$m = m(N, \xi) = \text{largest } n \text{ with } q_n(\xi) \leqslant N$$

and let

(1.8) $$U(N, x, y, z) = \{\xi : 0 \leqslant \xi \leqslant 1,\ q_{m(N,\xi)} \leqslant xN,\ q_{m(N,\xi)+1} > yN,\ a'_{m(N,\xi)+2} \geqslant z\}.$$

Then

$$\lim_{N\to\infty} |U(N, x, y, z)| = G(\bar{x}, \bar{y}, \bar{z}) = \frac{12}{\pi^2}\int_{\bar{y}}^{\infty} \frac{1}{t}\log\frac{\bar{z}t+\bar{x}}{\bar{z}t}\,dt$$

where

(1.9) $$\bar{x} = \min(1, x), \quad \bar{y} = \max(1, y) \quad \text{and} \quad \bar{z} = \max(1, z).$$

Since $T(N, c)$ is the complement with respect to $[0, 1]$ of the set of ξ for which the last $q_n(\xi) \leqslant cN$ is actually $\leqslant N$, one has $T(N, c) = [0, 1] - U(cN, c^{-1}, 1, 1)$ (recall that $q_{m+1} \geqslant N$ and $a'_{m+2} \geqslant 1$ for all ξ). Thus the answer to problem 2 is given by

$$\lim_{N\to\infty} |T(N, c)| = 1 - G(c^{-1}, 1, 1) = 1 - \frac{12}{\pi^2}\int_{1}^{\infty} \frac{1}{t}\log\left(1+\frac{1}{ct}\right)dt$$

$$= \frac{12}{\pi^2}\int_{c^{-1}}^{1} \frac{1}{v}\log(1+v)\,dv.$$

In section 3 we shall indicate how theorem 1 can be used to prove the existence of the limit in (1.1). In principle this existence proof even points the way how to compute the value of this limit for specific values of A and c but the necessary computations are too complicated to be carried out.

2. Solution of problem 2. We begin with the

Proof of lemma 1. We use the well-known formulae (see chapter 10 of [5] and formula II.11.3 in [9])

(2.1) $$\xi = \frac{p_n(\xi)}{q_n(\xi)} + \frac{(-1)^n}{q_n(\xi)q'_{n+1}(\xi)} = \frac{p_n}{q_n} + \frac{(-1)^n}{q_n(a'_{n+1}q_n + q_{n-1})}$$

and

(2.2) $$\frac{q_n}{q_{n-1}} = \frac{a_n q_{n-1} + q_{n-2}}{q_{n-1}} = a_n + \frac{1}{q_{n-1}/q_{n-2}} = a_n + \cfrac{1}{a_{n-1} + \ddots + \cfrac{1}{a_1}}$$

$$= a_n + [a_{n-1}, \ldots, a_1].$$

Now k_2/k_1 has exactly two expansions as a simple continued fraction, one with an even number of convergents and one with an odd number at convergents (theorem 158 of [5]). Let the two possible expansions for k_2/k_1 be

$$\frac{k_2}{k_1} = a^1_{2m} + [a^1_{2m-1}, \ldots, a^1_1] \tag{2.3}$$

and

$$\frac{k_2}{k_1} = a^2_{2m+1} + [a^2_{2m}, \ldots, a^2_1]$$

$$(a^1_{2m} \geqslant 1 \text{ and } a^2_{2m+1} \geqslant 1 \text{ since } \frac{k_2}{k_1} > 1). \tag{2.4}$$

Then we conclude from the above that if k_1, k_2 are denominators of two consecutive convergents of ξ, say p_{n-1}/q_{n-1} and p_n/q_n, then one must have either $n = 2m$ and $a_i(\xi) = a^1_i, 1 \leqslant i \leqslant 2m$ or $n = 2m+1$ and $a_i(\xi) = a^2_i, 1 \leqslant i \leqslant 2m+1$. In either case p_n is determined by (1.4) with a_i replaced by a^1_i resp. a^2_i. Denote the two possible values of p_n by p^1 and p^2 and put

$$I_1 = \left[\frac{p^1}{k_2}, \frac{p^1}{k_2} + \frac{1}{k_2(zk_2+k_1)}\right]$$

and

$$I_2 = \left[\frac{p^2}{k_2} - \frac{1}{k_2(zk_2+k_1)}, \frac{p^2}{k_2}\right].$$

We conclude from (2.1) that

$$A(k_1, k_2, z) \subset I_1 \cup I_2. \tag{2.5}$$

Observe that $I_1 \cap I_2$ consists of at most one point. This is obvious if $p^1 = p^2$, and in case $p^1 \neq p^2$ it follows from

$$\left|\frac{p^1}{k_2} - \frac{p^2}{k_2}\right| \geqslant \frac{1}{k_2} \geqslant \frac{2}{k_2(k_2+k_1)} \geqslant \frac{2}{k_2(zk_2+k_1)}$$

which is valid because $k_2 > k_1 \geqslant 1$ and $z \geqslant 1$. Thus

$$|I_1 \cup I_2| = \frac{2}{k_2(zk_2+k_1)}$$

and the lemma will follow once we show that

$$|I_1 \cup I_2 - A(k_1, k_2, z)| = 0.$$

For this purpose, take

$$\eta = \cfrac{1}{a_1^1+\cfrac{1}{a_2^1+\ddots\cfrac{1}{a_{2m}^1+\cfrac{1}{y}}}} = \left[a_1^1, \ldots, a_{2m-1}^1, a_{2m}^1+\frac{1}{y}\right]$$

for some $y \geqslant z$. Then $a_1(\eta) = [1/\eta] = a_1^1$ and in general, by the continued fraction algorithm (see [5], chapter 10.6),

$$a_i(\eta) = a_i^1, \quad 1 \leqslant i \leqslant 2m \quad \text{and} \quad a'_{2m+1}(\eta) = y.$$

Moreover,

$$[a_1^1, \ldots, a_{2m-1}^1] = \frac{r_1}{s_1} \quad \text{and} \quad [a_1^1, \ldots, a_{2m}^1] = \frac{r_2}{s_2}$$

are the $(2m-1)$st and $2m$th convergent to η. Hence s_1, s_2 are the values obtained in (1.3) for q_{2m-1} resp. q_{2m} when a_i is replaced by a_i^1. As a result,

(2.6) $\quad r_2 s_1 - r_1 s_2 = -1 \quad$ and $\quad (s_1, s_2) = 1 \quad$ (theorem 150 of [5])

and

(2.7) $\quad \dfrac{s_2}{s_1} = a_{2m}^1 + [a_{2m-1}^1, \ldots, a_1^1] = \dfrac{k_2}{k_1} \quad$ (see (2.2) and (2.3)).

Since also $(k_1, k_2) = 1$, (2.6) and (2.7) imply $k_1 = s_1$, $k_2 = s_2$, and together with $a'_{2m+1}(\eta) = y \geqslant z$ this implies $\eta \in A(k_1, k_2, z)$. On the other hand (see p. 140 of [5]),

$$\eta = \left[a_1^1, \ldots, a_{2m}^1 + \frac{1}{y}\right] = \frac{yr_2+r_1}{ys_2+s_1} = \frac{r_2}{s_2} + \frac{1}{s_2(ys_2+s_1)}$$

and r_2 is the value of p_{2m} obtained in (1.4) when a_i is replaced by a_i^1. But this is precisely the number we denoted by p^1 so that

$$\eta = \frac{p^1}{k_2} + \frac{1}{k_2(yk_2+k_1)}.$$

This is a generic element of I_1, and as y varies from z to ∞, η runs through all of I_1, except for an endpoint, i.e. $I_1 - A(k_1, k_2, z)$ consists of one point only. A similar argument for I_2 completes the proof of lemma 1.

We now turn to the

Proof of theorem 1. Let U be as in (1.8) and $\bar{x}, \bar{y}, \bar{z}$ as in (1.9). Since, by definition of $m = m(N, \xi)$,

$$q_m \leqslant N < q_{m+1}$$

and, by (1.5), $a'_{m+2} \geqslant 1$, one has

$$U(N, x, y, z) = U(N, \bar{x}, \bar{y}, \bar{z}).$$

Therefore, we may and will assume $x \leqslant 1 \leqslant y, 1 \leqslant z$. Notice now that if

$$q_n(\xi) \leqslant xN \leqslant N \leqslant yN < q_{n+1}(\xi), \tag{2.8}$$

then automatically

$$m(N, \xi) = n. \tag{2.9}$$

If we also take into account that

$$A(k_1, k_2, z) = \emptyset \quad \text{if} \quad (k_1, k_2) \neq 1$$

(by theorem 150 of [5]), then we find

$$U(N, x, y, z) = \bigcup_{\substack{(k_1,k_2)=1 \\ k_1 \leqslant xN \leqslant yN < k_2}} A(k_1, k_2, z) \tag{2.10}$$

At the same time we see from (2.8), (2.9) that the summands in (2.10) are disjoint since $q_m(\xi)$ and $q_{m+1}(\xi)$ are uniquely determined by ξ and N Thus

$$|U(N, x, y, z)| = \sum_{\substack{(k_1,k_2)=1 \\ k_1 \leqslant xN \leqslant yN < k_2}} |A(k_1, k_2, z)| \tag{2.11}$$

$$= \sum_{k_2 > yN} \frac{2}{k_2} \sum_{k_1 \leqslant xN}' \frac{1}{zk_2 + k_1}$$

where $\sum'$ involves only k_1 for which $(k_1, k_2) = 1$. We evaluate the asymptotic behavior of the primed sum for fixed k_2 in a slightly more general setting.

Lemma 2. *There exists a constant K, independent of k_2, such that* (3)

$$\left| \sum_{B < k_1 \leqslant C}' \frac{1}{D + Ek_1} - \frac{\Phi(k_2)}{Ek_2} \log \frac{D + EC}{D + EB} \right| \leqslant K \frac{d(k_2)}{D + EB} \tag{2.12}$$

whenever $B \leqslant C$, $E > 0$ and $D + EB \geqslant 0$.

Proof. Recall that the sum $\sum'$ contains only those k_1 for which $(k_1, k_2) = 1$. In view of (3) ([5], theorem 263)

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1 \end{cases}$$

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(3) Φ denotes Euler's function and μ the Möbius function; $d(k_2)$ = the number of divisors of k_2.

one has therefore

$$\sum_{B<k_1\leqslant C}{}' \frac{1}{D+Ek_1} = \sum_{B<k_1\leqslant C} \frac{1}{D+Ek_1} \sum_{d|(k_1,k_2)} \mu(d)$$
$$= \sum_{d|k_2} \mu(d) \sum_{\substack{B<k_1\leqslant C\\ d|k_1}} \frac{1}{D+Ek_1}$$
$$= \frac{1}{E} \sum_{d|k_2} \frac{\mu(d)}{d} \sum_{Bd-1<n\leqslant Cd-1} \frac{1}{D(Ed)^{-1}+n}.$$

Inequality (2.12) now follows from the well-known formulae

$$\Bigg|\sum_{Bd^{-1}<n\leqslant Cd^{-1}} \frac{1}{D(Ed)^{-1}+n} - \log\frac{D+EC}{D+EB}\Bigg|$$
$$= \Bigg|\sum_{Bd^{-1}<n\leqslant Cd^{-1}} \frac{1}{D(Ed)^{-1}+n} - \int\limits_{D(Ed)^{-1}+Bd^{-1}}^{D(Ed)^{-1}+Cd^{-1}} \frac{dt}{t}\Bigg|$$
$$\leqslant \frac{K}{D(Ed)^{-1}+Bd^{-1}} \qquad \text{for suitable } K,$$

and ([5], formula (16.3.1))

$$\sum_{d|k_2} \frac{\mu(d)}{d} = \frac{\Phi(k_2)}{k_2}.$$

An application of (2.12), with the proper choices of $B-E$, to the right-hand side of (2.11) leads to the following estimate:

$$(2.13) \qquad |U(N, x, y, z)| = \sum_{k_2>yN} \frac{2}{k_2}\cdot\frac{\Phi(k_2)}{k_2}\log\frac{zk_2+xN}{zk_2+1} + O\Bigg(\sum_{k_2>yN}\frac{d(k_2)}{k_2^2}\Bigg).$$

The error term tends to zero as $N\to\infty$ since $d(k) = O(k^\delta)$ for any $\delta>0$ ([5], theorem 315; a better estimate could be derived from theorem 318). Since, [3],

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\Phi(k)}{k} = \frac{6}{\pi^2},$$

it follows from a simple summation by parts that $\Phi(k_2)k_2^{-1}$ in the right-hand side of (2.13) may be replaced by its "average value" $6\pi^{-2}$. Consequently

$$\lim_{N\to\infty}|U(N,x,y,z)| = \lim_{N\to\infty}\frac{12}{\pi^2}\sum_{k_2>yN}\frac{N}{k_2}\log\frac{zk_2/N+x}{zk_2/N+1/N}\cdot\frac{1}{N} \tag{2.14}$$

$$=\frac{12}{\pi^2}\int_y^\infty\frac{1}{t}\log\left(\frac{zt+x}{zt}\right)dt.$$

This is just the statement of theorem 1 for $x\leqslant 1\leqslant y$, $1\leqslant z$.

3. The existence of $\lim\limits_{N\to\infty}|S(N,A,c)|$ in problem 1. Since no explicit values for the limit in (1.1) can be found by the present method, we restrict ourselves to an indication of the proof of its existence. As a first step we give a lemma which is almost a direct corollary of theorem 1.

LEMMA 3. *For each $k\geqslant 1$, the joint distribution of*

$$\frac{q_{m-1}(\xi)}{N},\ \frac{q_m(\xi)}{N},\ \frac{q_{m+1}(\xi)}{N},\ \frac{q'_{m+1}(\xi)}{N},\ldots,\frac{q_{m+k}(\xi)}{N},\ \frac{q'_{m+k}(\xi)}{N} \tag{3.1}$$

has a limit as $N\to\infty$. I.e. the measure of the set

$$\{\xi: 0\leqslant\xi\leqslant 1, q_{m-1}\xi\leqslant wN, q_m(\xi)\leqslant xN, q_{m+j}(\xi)>y_jN,$$
$$q'_{m+j}(\xi)\geqslant z_jN \text{ for } 1\leqslant j\leqslant k\}$$

has a limit as $N\to\infty$.

Proof. From (1.3), (1.5), and (1.6) one has the following relations

$$\frac{q_{m-1}}{N}=\frac{q_{m+1}}{N}-\left[\frac{q_{m+1}}{N}\cdot\frac{N}{q_m}\right]\frac{q_m}{N}, \tag{3.2}$$

$$\frac{q'_{m+j+1}}{N}=a'_{m+j+1}\frac{q'_{m+j}}{N},\qquad \frac{q'_{m+1}}{N}=\frac{q_{m+1}}{N}+\frac{1}{a'_{m+2}}\cdot\frac{q_m}{N}, \tag{3.3}$$

$$\frac{q_{m+j+1}}{N}=[a'_{m+j+1}]\frac{q_{m+j}}{N}+\frac{q_{m+j-1}}{N}, \tag{3.4}$$

and

$$a'_{m+j+1}=\frac{q_{m+j-1}}{N}\cdot\frac{N}{q'_{m+j}-q_{m+j}}. \tag{3.5}$$

These relations recursively express all variables in (3.1) as functions of

$$\frac{q_m}{N},\ \frac{q_{m+1}}{N},\ a'_{m+2}. \tag{3.6}$$

If these functions were continuous, it would follow immediately from the fact that the variables in (3.6) have the joint limiting distribution G (theorem 1) that also the variables in (3.1) have a joint limiting distribution (see [8], p. 425). Even though the functions in (3.2)-(3.5) are not continuous it is possible to show that the conclusion remains valid because the functions in (3.2)-(3.5) are "sufficiently nice" and G is "sufficiently smooth".

We are now able to give a partial answer to problem 1.

THEOREM 2. *The limit*

$$\lim_{N\to\infty}|S(N, A, c)|$$

exists for all $A \geqslant 0, c \geqslant 1$.

Proof. It is well known (see theorem 2.18 of [9]) that

$$\min_{a,\,1\leqslant b\leqslant N}|b\xi-a| = \min_{\substack{1\leqslant b\leqslant N\\(a,b)=1}}|b\xi-a| = |q_m\xi-p_m| = \frac{1}{q'_{m+1}} \tag{3.7}$$

(here again $m = m(N, \xi)$ and in the last step we used (2.1)). This implies for *fixed* b (i.e. the minima in (3.8) are over a *only*) such that

$$q_n(\xi)\leqslant b < q_{n+1}(\xi),$$

$$\frac{q_n}{q'_{n+1}} = q_n|q_n\xi-p_n| \leqslant q_n\min_{(a,b)=1}|b\xi-a| \leqslant b\min_{(a,b)=1}|b\xi-a|. \tag{3.8}$$

Consequently, if $q_{m+1}\leqslant cN$, then (b is the variable in the first min and a in the second min)

$$\min_{q_{m+1}\leqslant b\leqslant cN}\ \min_{(a,b)=1} b|b\xi-a| = \min_{\substack{n\geqslant m+1\\q_n\leqslant cN}}\frac{q_n}{q'_{n+1}}. \tag{3.9}$$

Let us write M_1 for the right-hand side of (3.9) if $q_{m+1}\leqslant cN$ and take $M_1 = \infty$ otherwise. Note that $N\leqslant q_n\leqslant cN$ is possible for at most $2\log_2 c$ values of n because

$$\frac{q_{n+2}}{q_n} > 2.$$

Thus the condition $M_1\leqslant A$ is a condition on the finitely many variables in (3.1) for $k = 2\log c+1$.

For $q_m\leqslant N\leqslant b\leqslant \min(q_{m+1}, cN)$ we use the following lemma which we give without proof.

LEMMA 4. *If*

$$2(A+1) < q_n\leqslant b\leqslant q_{n+1} \quad \textit{and} \quad |b\xi-a|\leqslant\frac{A}{b},$$

then there exist integers r, s such that

$$b = rq_n + sq_{n-1}, \qquad a = rp_n + sp_{n-1}$$

and $|s| \leqslant A+1$. *In addition* $(a, b) = (r, s)$ *and*

$$|b\xi - a| = \frac{rq_n + sq_{n-1} - sq'_{n+1}}{q_n q'_{n+1}}.$$

If we put

$$M_2 = \min_{b} \min_{(a,b)=1} b\,|b\xi - a|,$$

where the first min is only over b satisfying

$$N \leqslant b \leqslant \min(q_{m+1}, cN),$$

then this lemma allows us to express the condition $M_2 \leqslant A$ again as a condition on the variables in (3.1). In fact, since s in lemma 4 is limited to finitely many values, one can write M_2 as a minimum of finitely many simple expressions in these variables in the region $M_2 \leqslant A$.

Since

$$S(N, A, c) = \{\xi : M_1 \leqslant A \text{ or } M_2 \leqslant A\},$$

one can then conclude that $\lim |S(N, A, c)|$ exist from the existence of the limiting distribution in lemma 3.

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