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Problem. Let $\mathcal{G}$ be a regular $n \times n$ bipartite graph of degree $t$ and $t \leq n$. Is it true that the number of all different one factors of $\mathcal{G}$ (which might contain multiple edges) is not less than $t!$ ?

REMARKS.

1. We allow that $\mathcal{G}$ contains multiple edges. For graphs $\mathcal{G}$ not containing multiple edges the answer is trivially affirmative.

2. Van der Warden's famous conjecture concerning the minimum of permanents of double stochastic matrices is equivalent to the following problem in graph theory:

(1) Let $\mathcal{G}$ be a regular $n \times n$ bypartite graph of degree $k \cdot n$ $(k, n \geq 1)$. Then the number of one factors is not less than $k^n \cdot n!$

The Problem and the above statement seem to be incomparable but the special case $t = n$ of the Problem is equivalent to the special case $k = 1$ of the graph formulation of Van der Waerden's conjecture.

3. The answer to the Problem is affirmative in case $t = 3$ (The cases $t = 1, 2$ are trivial.)

4. P. Erdős conjectured and proved [1], [2] that if $A$ is an $n \times n$ double stochastic matrix then the permanent has a member of value $\geq \frac{1}{n^n}$.

It is easily seen that this statement is equivalent to the following weakening of (1).

(2) Under the conditions of (1) $\mathcal{G}$ has a one-factor which occurs with multiplicity $\geq k^n$.

To indicate that the graph theoretic approach might be useful we outline a short proof of (2). By the assumption and by König's theorem $\mathcal{G}$ is the sum of $kn$ disjoint first grade factors $F_1, \dots, F_{kn}$ .

Let $g$ denote the set of edges of $\mathcal{G}$, and let $m(x)$ denote the multiplicity of the edge $x \in g$ in $\mathcal{G}$. We prove that there is a $1 \le i \le kn$ with

$$\prod_{x \in F_i} m(x) \ge k^n$$

Put $A = \prod_{i=1}^{kn} \prod_{x \in F_i} m(x)$ . Considering that each $x \in g$ occurs in exactly $m(x)$ of the $F_i$ , we have

$$A = \prod_{x \in g} m(x)^{m(x)}$$

We also have

$$\sum_{x \in g} m(x) = kn^2; \qquad m(x) \ge 1 \qquad \text{for } x \in g$$

and $|g| \le n^2$. Thus, by a well-known inequality we get

$$A \ge \left( \frac{kn^2}{|g|} \right)^{\frac{kn^2}{|g|} \cdot |g|} \ge k^{kn^2}$$

It follows that there is $1 \le i \le kn$ with

$$\prod_{x \in F_i} m(x) \ge A^{\frac{1}{kn}} \ge k^n$$

REFERENCE

[1] M. MARCUS-H. MINE: Some results on doubly stochastic matrices. Proc. Amer. Math. Soc. (13) 1962.