ON THE EXISTENCE OF TRIANGULATED SPHERES IN 3-GRAPHS, AND RELATED PROBLEMS

by

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To the memory of A. RÉNYI

1. An *r-graph*, $H^{(r)}$ consists of a set $V(H^{(r)})$ of *vertices*, and a class $E(H^{(r)})$ of *r*-subsets of $V(H^{(r)})$ (i.e. unordered *r*-tuples of vertices). We shall use various letters (in place of H) to name *r*-graphs, but the superscript (r) will always be included; except possibly when $r = 2$ — a 2-graph is simply a *graph*. If we follow the symbol $H^{(r)}$ by (n) or $(n; k)$ it will mean that the *r*-graph has exactly n vertices, and, in the second case, *at least k r-tuples* (as we shall name the members of $E(H^{(r)})$). The letter G is reserved to denote *all* *r*-graphs with the properties indicated by any appendices to the symbol: thus $G^{(r)}$ represents any *r*-graph; $G^{(r)}(n)$ any $G^{(r)}$ with n vertices; and $G^{(r)}(n; k)$ any $G^{(r)}(n)$ with at least k *r*-tuples.

The extremal theory of graphs was initiated by P. TURÁN [20, 21]: he determined for every positive integer t the smallest integer k for which every $G(n; k)$ contains a complete *t*-gon $\left(\text{i.e. a } G\left(t; \binom{t}{2}\right)\right)$. Extremal problems for ordinary graphs have been studied extensively [2–9, etc.]; many general results have been obtained, but many unsolved problems remain. For *r*-graphs very little is known [5, 11]. Even the original problem of TURÁN [20] (to determine the smallest k for which for fixed t, every $G^{(r)}(n; k)$ contains all $\binom{t}{r}$ *r*-tuples which can be formed from some t vertices) remains unsolved [14, 16].

In this paper we shall confine our interest to 3-graphs. The problem described in the title represents an analogue of the well known property of graphs that any $G(n; n)$ contains a polygon. That result could be restated, in topological terms, as saying that any simplicial 1-complex with at least as many 1-simplexes as 0-simplexes must contain a triangulation of the 1-sphere. (It is *best possible* in the sense that there exist $G(n; n - 1)$'s containing no polygon.) In Theorem 3 we shall determine asymptotically the maximum number of 2-simplexes a simplicial 2-complex may contain without containing

a subcomplex which is a triangulation of the 2-sphere. More precisely, we shall prove that there exist constants c_1 and c_2 such that every $G^{(3)}(n; c_2\, n^{3/2})$ contains a double pyramid (i.e. a polytope whose faces are all triangles having two of their vertices as adjacent vertices of a fixed polygon, and whose third vertex is one of two fixed points); but that some $G^{(3)}(n; c_1\, n^{3\,2})$ contains no triangulation of the sphere. Also, we discuss several related results.

2. Extremal numbers. Let H be any fixed family of r-graphs. Then $\mathrm{ex}(n; H)$ will denote the largest integer k for which there exists a $G^{(r)}(n; k)$ containing none of the members of H as a sub-r-graph. (When H consists of a single r-graph $H^{(r)}$ we may abbreviate $\mathrm{ex}(n; \{H^{(r)}\})$ to $\mathrm{ex}(n; H^{(r)})$). For s less than r the *s-tuples* of an r-graph will be *any* set of s vertices; in particular, the *edges* will be any unordered pair of vertices. The *star* of an s-tuple S in a $G^{(r)}$ ($s < r$) will be the $(r-s)$-graph whose vertices are all vertices of the $G^{(r)}$ not in S, and whose $(r-s)$-tuples are all sets $U - S$ where U ranges over the r-tuples of the $G^{(r)}$ which contain S as a subset; the *valency* of an s-tuple is the number of $(r-s)$-tuples in its star. The *product* of an r-graph $A^{(r)}$ and an s-graph $B^{(s)}$ will be an $(r+s)$-graph whose vertex set is the union of those of $A^{(r)}$ and $B^{(s)}$ and whose $(r+s)$-tuples are all unions of an r-tuple of the first and a *disjoint* s-tuple of the second; in particular, a *cone* over $A^{(r)}$ is a product of $A^{(r)}$ with a disjoint $G^{(1)}(1; 1)$; a *double pyramid* (mentioned above) is a product of a polygon (graph) with a disjoint $G^{(1)}(2;2)$.

It will be helpful to use geometrical language from time to time interpreting the triples of a 3-graph as the 2-simplexes of a simplicial 2-complex (which contains all possible 1-simplexes). A *wheel* will be a cone over a polygon. An *octahedron* will be a double pyramid over a 4-gon.

3. An elementary result on wheels. Let W denote the set of all wheels.

THEOREM 1. $\lim\limits_{n \to \infty} n^{-2}\, \mathrm{ex}\,(n; W) = 1/3$.

PROOF.

a. In any $G^{(3)}(n; [n(n-2)+1]/3)$ some vertex must have valency at least $n-1$, hence its star must contain a polygon. Thus for all n, $\mathrm{ex}\,(n; W) < (n-1)^2/3$.

b. We construct for every integer n a 3-graph $A_n^{(3)}\,(n; n(n-4)/3)$ containing no wheels. The vertices will be the residues modulo n; $\{x, y, z\}$ is selected as a triple if and only if $x + y + z \equiv 0$ or $1 \pmod{n}$, *and* x, y, z are distinct. For fixed x, the number of solutions (x, y, z) of these congruences is $2n$; of these, at most four are of the form (x, y, y), and at most two of each of the forms (x, y, x) and (x, x, z). Thus the number of triples in the 3-graph is at least $n(2n-8)/3!$. In the star of any vertex x, every vertex has valency at most 2. Any polygon there is of the form $y_1, y_2, \ldots, y_r, y_1$ where $y_1 + y_2 \equiv -x$, $y_2 + y_3 \equiv 1 - x$, $y_3 + y_4 \equiv -x$ etc. i.e.

$\ldots \equiv y_5 - y_3 \equiv y_3 - y_1 \equiv 1 \equiv y_2 - y_4 \equiv y_4 - y_6 \equiv \ldots$. But this implies that all vertices, including x are in the polygon, which is impossible. We conclude that the star is a union of open arcs.

As a related result we have the following. Let $X^{(3)}$ denote the $G^{(3)}(4; 2)$ with two triples having an edge in common, i.e. the product of a $G^{(2)}(2; 1)$ and a disjoint $G^{(1)}(2; 2)$.

THEOREM 2. $\lim\limits_{n \to \infty} n^{-2} \operatorname{ex}(n; X^{(3)}) = 1/6$.

PROOF.

a. In any $G^{(3)}\left(n, \binom{n}{2}\Big/ 3 + 1\right)$ some edge lies in more than one triple; this yields an upper bound of

$$n^{-2} \operatorname{ex}(n; X^{(3)}) \leqq (1 - n^{-1})/6 .$$

b. The Steiner triple systems [12, 17] which exist for $n \equiv 1$ or 3 (mod 6) ($n \geq 7$), are example of a $G^{(3)}(n; n(n-1)/6)$ where every edge has unit valency. Lower bounds for the other congruence classes of n are easily obtained from these 3-graphs by erasing up to three vertices (and all triples containing them) from these 3-graphs.

4. Triangulations of the 2-sphere. Let T denote the set of triangulations of the 2-sphere. (We rely on geometric intuition in part of the following discussion. A rigorous definition of a triangulation of the 2-sphere might be: a $G^{(3)}(n; t)$ such that

(i) the star of every vertex is a polygon;

(ii) every edge is in zero or two triples;

(iii) (orientability) the triples can be each cyclically ordered so that the two orientations induced on any edge of valency two are in opposite directions;

(iv) (genus 0) the number of edges of valency 2 is exactly $n + t - 2$.)

THEOREM 3. *There exist positive constants c_1 and c_2 such that for all n,*

$$c_1 < n^{-5 \, 2} \operatorname{ex}(n; T) < c_2 .$$

(In fact we prove that for $\varepsilon > 0$ and n sufficiently large, $c_1 > 3^{-5 \, 2} - \varepsilon$. $c_2 < 18^{-1 \, 2} + \varepsilon$).

PROOF.

a. *The upper bound:* Let c be a constant (to be further specified below). The sum of the valencies of the edges of a $G^{(3)}(n, cn^{5/2})$ is at least $3cn^{5/2}$; hence the number of unordered pairs of triples having an edge in common must be at

least $\dfrac{1}{2}\left[(3cn^{5/2})^2 \Big/ \dbinom{n}{2} - 3cn^{5/2}\right]$ which is greater than $(n-3)\dbinom{n}{2}$ when $c >$
$> (n^{\frac{1}{3}} + (8n^2 - 31n + 24)^{\frac{1}{3}})/12n$, in particular when $c > 18^{-\frac{1}{2}} + (144n)^{-\frac{1}{3}}$,
For such c there must exist an edge xy whose star contains at least $n - 2$ edges.
hence contains a polygon. The product of this polygon with the $G^{(1)}$ $(2; 2)$
formed by x and y is a double pyramid.

b. *The lower bound*: Let q be a prime power and $m = q^2 + q + 1$. We
construct a 3-graph $S_q^{(3)}(3m, m(q+1))$ as follows. Let A denote the set of
residues modulo m. By a well known theorem of SINGER [19] there exists a
perfect difference set D for A, i.e. a subset D of A such that every non-zero
element of A is uniquely representable modulo m as a difference $d_1 - d_2$ where
$d_1, d_2 \in D$. The vertex set of $S_q^{(3)}$ will be the union of three disjoint copies of A,
labelled A_1, A_2, A_3. The triples will be precisely those (a_1, a_2, a_3) with $a_i \in A_i$
$(i = 1, 2, 3)$ and $a_1 + a_2 + a_3 \in D$. (As always, addition is modulo m.) Suppose
now that $M^{(3)}$ is a sub-3-graph of $S_q^{(3)}$ whose every vertex has a polygon for its
star. Without limiting generality, consider a vertex a_1 in A_1. Its star must be
an *even* polygon of the form $a_2^1, a_3^1, a_2^2, a_3^2, \ldots, a_2^r, a_3^r, a_2^1$ where $a_2^i \in A_2$, $a_3^i \in A_3$.
Suppose $r = 2$, i.e. that a_1 has valency 4. Then the following triples must have
their sums in D: $a_1\, a_2^1\, a_3^1,\ a_1\, a_3^1\, a_2^2,\ a_1\, a_2^2\, a_3^2,\ a_1\, a_3^2\, a_2^1$. Distinctness of the vertices
of the polygon forces the four sums here to be distinct. But then $a_3^2 - a_3^1 =$
$= (a_1 + a_2^i + a_3^2) - (a_1 + a_2^i + a_3^1)$ $(i = 1, 2)$ a contradiction, in that a non-
zero element cannot be expressible in two ways as a difference of elements of
the difference set. Thus the only triangulations $M^{(3)}$ of surfaces contained in
$S_q^{(3)}$ must have all vertices of valency at least 6. But it is well known ([13], p.
104) that any triangulation of the sphere must include vertices of valency less
than 6. Thus ex $(3m, T) > m^{5/2}$.

Let now ε be given between 0 and 1. For n sufficiently large there exists
a prime p between $((1 - \varepsilon)^{2/5} n/3 - 3/4)^{1/3} - 1/2$ and $(n/3 - 3/4)^{1/2} - 1/2$, i.e.
$(1 - \varepsilon)^{2/5} n < 3(p^2 + p + 1) < n$. We may construct a 3-graph with n
vertices by adjoining isolated vertices to $S^{(3)}$. The resulting $G^{(3)}(n)$ has more
than $(1 - \varepsilon)3^{-5/2} n^{5/2}$ triples and contains no triangulation of the sphere.

5. Further extremal problems for 3-graphs. We now commence a study
similar to that begun for graphs in [4]. Namely, if $\mathcal{G}_f^{(r)}(l; k)$ denote the set of
all r-graphs $G^{(r)}(l; k)$, we inquire as to value of ex $(n, \mathcal{G}_f^{(3)}(l; k))$, which value
we denote by $f^{(3)}(n; l, k) - 1$ to be consistent with [4]. Below we determine
the asymptotic behavior of some of these numbers for $l \leq 6$.

THEOREM 4. *(The symbols* $c_3, c_4, \ldots$, *will denote positive constants whose
value we may estimate in the course of our proofs.)*

    a. $\lim\limits_{n \to \infty} n^{-2} f^{(3)}(n; 4, 2) = 1/6$

        $c_3\, n^3 < f^{(3)}(n; 4, 3) < f^{(3)}(n; 4, 4)$

b. $$f^{(3)}(n; 5, 2) = [n/3] + 1$$

$$c_4 n^2 \;<\; f^{(3)}(n; 5, 3) < c_5 n^2$$

$$c_6 n^{5/2} < f^{(3)}(n; 5, 4) < c_7 n^{5/2}$$

$$c_8 n^3 \;\;<\; f^{(3)}(n; 5, 5) <\ldots< f^{(3)}(n; 5, 10)$$

c. $$f^{(3)}(n; 6, 2) = 2$$

$$c_9 n^{3/2} < f^{(3)}(n; 6, 3)$$

$$c_{10} n^2 < f^{(3)}(n; 6, 4) < (1/4)n^2$$

$$f^{(3)}(n; 6, 6) < c_{11} n^{5/2}$$

$$f^{(3)}(n; 6, 8) < c_{12} n^{11/4}$$

$$c_{13} n^3 \;<\; f^{(3)}(n; 6, 9) <\ldots< f^{(3)}(n; 6, 20) \, .$$

PROOF. a. The first statement is simply Theorem 2. To prove the second we need only consider the 3-graph we call $T^{(3)}(n, [n^3/27])$ defined as follows: the vertices are partitioned into classes A, B, C, containing respectively $[n/3]$, $[(n + 1)/3]$, and $[(n + 2)/3]$ members; the triples are all $\{a, b, c\}$ such that $a \in A$, $b \in B$, $c \in C$. The case $l = k = 4$ is the first open case of Turán's problem [20].

b. $k = 2$. The lower bound follows from the $G^{(3)}(n, [n/3])$ which has no two triples sharing a vertex. And, in any $G^{(3)}(n, [n/3] + 1)$, some vertex must have valency exceeding 1.

$k = 3$. The lower bound follows from Theorem 2; the upper from Theorem 1. However, some improvement in the constant of the upper bound is possible; cf. the techniques used in proving the case $l = 6$ $k = 4$.

$k = 4$. The lower bound follows from the 3-graphs $S_q^{(3)}$ constructed in the proof of Theorem 3. Any $G^{(3)}(5; 4)$ contained in $S_q^{(3)}$ must have the property that its vertices can be partitioned into three classes so that every triple contains one vertex from each class. It is easily seen by inspection that the only $G^{(3)}(5; 4)$ with this property is a cone over a quadrangle. But we saw in the proof of Theorem 3 that $S_q^{(3)}$ cannot contain such a wheel. The remainder of the proof of the lower bound, viz. the passage from special values of n to all n sufficiently large, is analogous to that used in the proof of Theorem 3.

In any $G^{(3)}(n; cn^{5/2})$ some vertex x must have valency more than $3c(n - 1)^{3/2}$. By a result of KŐVÁRI—SÓS—TURÁN [15] a sufficiently large constant c ensures the existence in the star of x of a 4-gon; hence the 3-graph contains a cone over a 4-gon. (The other examples of a $G^{(3)}(5; 4)$ can all be shown to require (constant) n^3 triples — by the examples $T^{(3)}$ constructed in the proof of a. above.)

$k = 5$. The lower bound follows from the examples $T^{(3)}$ of a. above.

c. The case $k = 2$ is trivial.

$k = 3$. To prove the lower bound we construct a family of examples $G^{(3)}(n)$ for $n = q^2 + q + 1$ (q any odd prime power) having for q sufficiently large, more than $(1/6)n^{3/2}$ triples but containing no $G^{(3)}(6; 3)$; the proof then may be completed as in the proof of the lower bound in Theorem 3. Our example is the 3-graph "spanned" by the graph exploited in [1], § 3 and [6], Theorem 1. The vertices will be the points of a finite projective plane $PG(2, q)$. Let λ be an orthogonal polarity. A triple $\{x, y, z\}$ of points is selected if and only if the points are distinct and pairwise conjugate with respect to λ i.e. each is incident with the polars of the other two, (sometimes called a *self-polar triangle*). It is easily seen that no triple selected can contain an absolute point of the polarity. (We could "improve" our example by discarding these points, but this improvement will be of no importance asymptotically.) No two distinct triples can have two vertices in common, for the polar line of any point of a triple is determined by the other two. Hence, if this 3-graph is to contain a $G^{(3)}(6; 3)$, that 3-graph must be of the following form: $V = \{1, 2, 3, 1', 2', 3'\}$ $E = \{1'23, 12'3, 123'\}$. The existence of such a sub-3-graph would imply that 1, 2 and 2, 3 and 3, 1 are each conjugate pairs, i.e. that 123 is a triple. But that would yield several pairs of triples sharing an edge; which, as we have remarked above, cannot exist. Every line l of the geometry which is not incident with its pole contains at least $q - 1$ points not on the conic of the polarity. Any two of these points, together with the pole of l form a selected triple. The number of non-self-conjugate lines is q^2: the number of such triples is $(q^2/3) \binom{q-2}{2} >$

$> (1/6)(1 - \varepsilon)q^4$ for q sufficiently large ($0 < \varepsilon < 1$ arbitrary).

We have not succeeded in establishing an upper bound better than (constant) n^2.

$k = 4$. To establish a lower bound (probably capable of some improvement in the constant) let the vertices of a $G^{(3)}(n)$ be the residues modulo n. A triple xyz is selected if and only if x, y, z are distinct, and $x + y + z \equiv 0$ modulo n. This 3-graph has at least $n - 2$ admissible triples containing x, hence at least $n(n - 2)/6$ triples in all. No two triples share two vertices, since the third is uniquely determined by any two. Thus the only type of $G^{(3)}(6; 4)$ which could be present would be of the following form:

$$V = \{1, 2, 3, 1', 2', 3'\}$$

$$E = \{1'23, 12'3, 123', 1'2'3'\} \ .$$

However, the four congruences implied by the existence of these four triples imply the congruence for 123, which is impossible since no two triples share an edge!

For an upper bound we apply (4) of [4]. In any $G^{(3)}(n; (n[(3/4)(n-1)]+1)/3)$ some vertex has a $G(n-1, [(3/4)(n-1)]+1)$ as a star; hence the star contains a $G(5; 4)$. It seems likely that the lower bound could be improved to allow the establishment of $\lim_{n \to \infty} n^{-2} f^{(3)}(n; 6, 4) = 1/4$.

$k = 6$. Let a $G^{(3)}(n; c_{11} n^{5/2})$ be given, where the constant is to be further specified below. By the same sort of reasoning as used in the proof of the upper bound of Theorem 3, there are at least $9c^2 n^3 (1 - (6c \mid \bar{n})^{-1})$ (pairs of triples sharing an edge, which is more than $9c^2(1 - \varepsilon)n^3$ when $n > (36c^2\varepsilon^2)^{-1}$ (ε being any real number strictly between 0 and 1). Hence for such n there exist two vertices whose stars have more than $18c^2(1 - \varepsilon)n$ edges in common; by ([4], (4)) these stars have a $G(4; 3)$ in common if $18c^2(1 - \varepsilon) \geq 2/3$, with n restricted, for example, to exceed ε^{-2}.

$k = 8$. Let a $G^{(3)}(n, c_{12} n^{11/4})$ be given, where the constant is to be further specified below. As in the case $k = 6$ above, there are at least $9c^2 n^{7/2}(1 - (6c \sqrt{\bar{n}})^{-1})$ pairs of triples sharing an edge. Hence for $0 < \varepsilon < 1$ and $n > (36c^2 \varepsilon^2)^{-1}$ there exist two vertices whose stars have more than $18c^2(1 - \varepsilon) n^{3/2}$ edges in common. The proof concludes as in the earlier case, using the result of KŐVÁRI−SÓS−TURÁN [15] cited earlier.

$k = 9$. Any 3-graph $G^{(3)}(6)$ with vertices in three classes such that any triple has one vertex in each class can have no more than eight triples. Thus the 3-graphs $T^{(3)}$ constructed in the proof of a. yield a lower bound.

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