Problems

PROBLEM 1. Let H be a given graph, n be a positive integer. Let $g(n,H) = \max$ (k: the edges of K^n can be colored by k colors so that each $H \subseteq K^n$ contains at least two edges of the same color); (K^n is the complete graph on n vertices).

Determine $g(n,C^l)$, where C^l is a cycle of length l.

Is it true that

$$g(n,C^l) = n\left(\frac{l-2}{2} + \frac{1}{l-1}\right) + O(1) \ ?$$

Determine $g(n, K_2(l,l))$, where $K_2(l,l)$ is the complete $l \times l$ bipartite graph.

(It is trivial that there exist positive constants c_1, c_2 such that $c_1 n^{\frac{3}{2}} \leq g(n, K_2(3,3)) \leq c_2 n^{\frac{5}{3}}$ but nothing more is known on $g(n, K_2(3,3))$.)

P.Erdös, M.Simonovits, V.T.Sós

PROBLEM 2. Let G be a digraph with indegrees and outdegrees ≥ 2. Does G have an even cycle (the condition ≥ 2 may be replaced by $\geq 10^{10}$ or by the assumption that the digraph is 10^{10}-connected) ?

(A digraph G is said to be k-connected if the removal of less than k vertices from G always results in a strongly connected digraph with at least two vertices.)

L.Lovász

PROBLEM 3. Does there exist a natural number k such that in each k-connected digraph G any two points are on a cycle ? (False for k = 2.)

J.C.Bermond - L.Lovász

PROBLEM 4. Find a function $f(k)$ (the best function, if possible) with the following property: If G is an $f(k)$-connected graph then G contains a circuit C such that $G - V(C)$ is still k-connected ($f(1) = 3$ by Tutte).

L.Lovász

PROBLEM 5. Let m,n and l be positive integers. Let $A_1, \dots, A_m$ be different subsets of a set S ($|S| = n$), satisfying the following conditions:

$|A_i \cap A_j| \geq l$, $A_i \cup A_j \neq S$ $(i,j = 1,2,\dots,m)$. What is the maximum of m if n and l are fixed? The conjectured optimal system, for $n + l$ odd, is the following. We pick out one element $s \in S$ and take all the subsets of S not containing s and having at least $\frac{n-1+l}{2}$ elements. For the case $n + l$ is even a similar construction can be made. The conjecture is solved for $l = 1$ by Daykin and Lovász, and without the condition $A_i \cup A_j \neq S$ by Katona (Acta Math. Acad. Sci. Hungar. 15(1964),329-337).

G.O.H.Katona

PROBLEM 6. Let $G = (A \cup B, E)$ be an $n \times n$ bipartite graph, that is, $|A| = |B| = n$ and there are edges between A and B only. Let $G_i = (A_i \cup B_i, E_i)$, $(i \in \{1,2,\dots,m\})$ be complete bipartite subgraphs of G such that $A_i \subset A$, $B_i \subset B$, $E_i \subset \mathbf{E}$. Denote by $f(G)$ the minimum of $\sum_{i=1}^{m} |A_i \cup B_i|$, assuming that $\bigcup_{i=1}^{m} E_i = E$ and m runs over all positive integers.

What is the maximum of $f(G)$ if G runs over all the $n \times n$ bipartite graphs? It is not hard to find for every given n a graph G with $f(G) \geq n \log n$ and T.Tarján has found one with $f(G) \geq cn\sqrt{n}$, where c is a positive constant.

A problem of T.Tarján
communicated by G.O.H.Katona

PROBLEM 7. Consider bipartite graphs $\langle X, \Delta, Y \rangle$ where $|X| = |Y| = n$ and the edges from Δ join nodes in X to nodes in Y. Let k_{ij} be the number of 1-factors of the graph that results when the ith node of X and jth node of Y is removed $(1 \leq i,j \leq n)$. The problem is to characterize those graphs for which there is non-zero constant c with $k_{ij} = c$ for all i, j. With T.Foregger we have conjectured that the only such graphs are $K_{n,n}$ and C_{2n} and have proved this for $n \leq 7$.

R.A.Brualdi

PROBLEM 8. Is there a cyclically 7-connected cubic graph with chromatic index 4 ? (A connected loopless graph is called cyclically k-connected, if the removal of less than k edges cannot result in a disconnected graph with two components, each containing a circuit.)

L.S.Mel'nikov

PROBLEM 9. Let G be a map on a closed surface. Denote by $\sigma(G)$ the maximum degree of a vertex of G. Then the chromatic number of G,

$$\chi(G) \leq \sigma(G)$$

provided that no component of G is a complete graph on $\sigma(G) + 1$ vertices; in the case $\sigma(G) = 2$ it is required that no component of G is an odd circuit (a theorem by R.L.Brooks). The chromatic index of G,

$$\chi'(G) \leq \sigma(G) + 1$$

(a theorem by V.G.Vizing). M.Behzad and V.G.Vizing conjecture that the total chromatic number of G,

$$\chi''(G) \leq \sigma(G) + 2.$$

I conjecture that the minimal number of colors for which G has a regular simultaneous coloring of edges and faces of G,

$$\chi^{(3)}(G) \leq \sigma(G) + 3$$

and that the minimal number of colors for which G has a regular simultaneous coloring of vertices, edges and faces of G,

$$\chi^{(4)}(G) \leq \sigma(G) + 4.$$

L.S.Mel'nikov

PROBLEM 10. A well-known result of G.A.Dirac states that any 4-chromatic graph (i.e., graph with chromatic number 4) contains a subdivision of the K_4 as a subgraph (a subdivision of a graph Γ is a graph obtained from Γ by replacing each edge (v_i, v_j) of Γ by a path P_{ij}). The original proof of this result is based on the fact that a critical 4-chromatic graph is 2-connected and has minimal valency ≥ 3, hence the chromatic properties are not used very strongly. It might be of interest to strengthen the result in a way which would bring the chromatic properties more into play, for example by specifying the lengths of the P_{ij}-s. More concretely:

Is it true that any 4-chromatic graph contains a subdivision of the K_4 in which

(i) each of the paths P_{ij} has odd length ?

or

(ii) each of P_{12}, P_{13} and P_{14} has length 1 (i.e., the corresponding three edges of the K_4 have not been subdivided) ?

or

(iii) each of P_{12}, P_{23} and P_{34} has length 1 ?

In connection with (ii) and (iii) it should be noted that the existence of 4-chromatic graphs without short circuits imply that we can hope to leave at most a spanning tree of K_4 not subdivided. The two trees on four vertices give rise to the questions (ii) and (iii).

Other questions could be asked, for example does there exist a subdivision satisfying both (i) and (ii) ?

It is not difficult to prove as a partial result, that any 4-chromatic graph contains a subdivision of the K_4 in which each of P_{12} and P_{23} has length 1 .

B.Toft

PROBLEM 11. For $n \geq 2$ let $f(n)$ denote the least integer for which the following statement is true: If a graph Γ is $f(n)$-connected and contains at least one odd circuit, then for any n vertices of Γ there is an odd circuit of Γ containing these n vertices. What is the value of $f(n)$?

It is not difficult to prove that $n+1 \leq f(n) \leq 2n-1$, hence $f(2) = 3$. I do not know whether $f(3)$ is 4 or 5 .

B.Toft