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INTERSECTION THEOREMS FOR GRAPHS II.

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ABSTRACT

If G and H are graphs on the same vertex set, let $G \cap H$ be the graph with $E(G \cap H) = E(G) \cap E(H)$ and $V(G \cap H) = \{$the end-points of the edges of $G \cap H\}$. For a given family $\mathscr{L}$ of graphs $f(n, \mathscr{L})$ is the maximum number of graphs $G_1, \dots, G_N$ defined on the same n-element set for which $G_i \cap G_j \in \mathscr{L}$, $(1 \leqslant i < j \leqslant N)$. In a previous paper of ours [9] we have given bounds on $f(n, \mathscr{L})$ when $\mathscr{L}$ is the family of paths (including the empty graph ϕ in one case, excluding it in the other case), and when $\mathscr{L}$ is the family of cycles (and ϕ is included). In the present paper we determine the exact value of $f(n, \mathscr{L})$ for the cases when $\mathscr{L}$ is the family of cycles,

(a) ϕ included,

(b) ϕ excluded.

We also prove that if $\mathscr{L}$ is the class of graphs topologically equivalent with one of the graphs $L_1, \dots, L_k$ where L_i is connected and has no

vertices of valency 1 $(1 \leq i \leq k)$, then there exists an integer r such that $f(n, \mathscr{L}) = O(n^r)$.

INTRODUCTION

Intersection properties of finite set systems have an extensive literature. Some of the first results of this area are due to Fisher [4], de Bruijn and Erdős [2], Bruck and Ryser [1], some of the last ones to Ray-Chaudhuri and Wilson [6], Deza, Erdős and Singhi [3], Deza, Erdős and Frankl [5].

In a lecture on the Rome conference, 1973, one of us [8] formulated a general class of problems related to the above ones. In a special form it is the following:

Let M be a finite set of integers, $|X|$ denote the cardinality of X and let $A_1, \ldots, A_N$ be a family of subsets of $S = \{1, \ldots, n\}$ satisfying the condition

$$|A_i \cap A_j| \in M \quad \text{if} \quad 1 \leq i < j \leq N.$$

How large can N be?

Instead of considering intersection properties of set systems one can consider intersection properties of given structures, e.g. of graphs, hypergraphs, partially ordered sets, groups, subsets of the integers, ... and instead of putting conditions on the size of the intersection one may put conditions on its structure.

In the present paper we shall prove some intersection theorems on finite graphs without loops and multiple edges.

Definition 1. If G is a graph, $E(G)$ will denote its edge-set, $V(G)$ its vertex-set. For given graphs G and H with $V(G) = V(H)$ $G \cap H$ is the graph with $E(G \cap H) =: E(G) \cap E(H)$ and $V(G \cap H) =$ the set of end-vertices of the edges in $E(G) \cap E(H)$.

For a given family $\mathscr{L}$ of graphs a set $G_1, \ldots, G_N$ of graphs with the same vertex set V will be called an $\mathscr{L}$-intersection family if

(1) $G_i \cap G_j \in \mathcal{L}$ if $1 \leqslant i < j \leqslant N$.

Problem 1. How large can N be for a given family $\mathcal{L}$ and V, $|V| = n$? The maximum of N will be denoted by $f(n, \mathcal{L})$.

In [9] we have proved that

(a) If $\mathcal{A}_1$ is the family of cycles and ϕ, then

$$f(n, \mathcal{A}_1) = O(n^4).$$

(Here $G \cap H = \phi$ means that $E(G) \cap E(H) = \phi$.)

(b) If $\mathcal{A}_3$ is the family of paths, ϕ included, then

$$f(n, \mathcal{A}_3) = O(n^5).$$

(c) If $\mathcal{A}_4$ is the family of nonempty paths, then

$$f(n, \mathcal{A}_4) = O(n^4).$$

The exponents in (a), (b) and (c) are sharp. In the first part of this paper we determine the exact value of $f(n, \mathcal{A}_1)$ and $f(n, \mathcal{A}_2)$, where $\mathcal{A}_2$ is the family of nonempty cycles.

Theorem 1. *If* $n \geqslant 4$ *then* $f(n, \mathcal{A}_2) = \binom{n}{2} - 2$ *and the only extremal system, that is the only* $\mathcal{A}_2$*-intersection system* $G_1, \dots, G_N$ *for* $N = f(n, \mathcal{A}_2)$ *is the following one:* $E(G_1)$ *forms a triangle, and* $E(G_2), \dots, E(G_n)$ *contain the edges of* $E(G_1)$ *and exactly one more edge in all the possible* $\binom{n}{2} - 3$ *ways.*

Remark 1. A system $G_1, \dots, G_N$ is called a strong Δ-system if the intersection of any two of these graphs is the same. The construction in Theorem 1 is a strong Δ-system. It would be interesting to know, how large can N be if $G_1, \dots, G_N$ form an $\mathcal{A}_2$-intersection system but not a strong Δ-system. The following constructions are conjectured to yield the maximum.

Construction 1. Let $T_1, \dots, T_s$ be a maximal system of edge-disjoint cycles in a complete graph K_n, where maximality means that s is the maximum possible. If the graphs G_i, $1 \leqslant i \leqslant s$ are defined by

$$E(G_i) = E(T_1) \cup E(T_i) \quad \text{for} \quad i = 2, 3, \dots, s$$

and

$$E(G_1) = \bigcup_{2}^{n} E(T_i),$$

$V(G_i) = V(K_n)$, $(i = 1, \dots, s)$, then $G_i \cap G_j$ is always a cycle $(T_1, T_i$ or $T_j)$. If $n = 6k + 1$ or $6k + 3$, $T_1, \dots, T_s$ must be a Steiner triple system by the maximality and in this case the number of graphs is $s = \frac{1}{3}\binom{n}{2}$. If $n = 6k$ or $n = 6k + 2$, one can obtain the maximal systems of cycles from the above ones by omitting the triplets incident to the $(6k + 1)$-th or the $(6k + 3)$-th vertex. The cases $n = 6k + 4$, $6k + 5$ shall not be discussed here.

Construction 2. Again, we start with a maximal system of independent cycles $T_1, \dots, T_s$ and assume that p is the largest integer for which $p^2 + p + 1 \leqslant s$ and a finite geometry $PG(p, 2)$ exists on the $p^2 + p + 1$ elements. One may assume that the points of this $PG(p, 2)$ are $1, 2, \dots, p^2 + p + 1$. We consider the graphs $G_1, \dots, G_{p^2+p+1}$ defined as follows: let $L_1, \dots, L_{p^2+p+1}$ be the lines of this finite projective geometry, let $E(G_j) = \bigcup_{i \in L_j} E(T_i)$. Again, the family G_j is an $\mathscr{A}_2$-intersection system with nearly as many graphs as in Construction 1 and with exactly the same number of graphs if $s = p^2 + p + 1$ for a prime power p.

Conjecture 1. Let $G_1, \dots, G_N$ be graphs forming an $\mathscr{A}_2$-intersection system. If $G_1, \dots, G_N$ is not a strong Δ-system, then $N \leqslant \frac{n^2}{6} + o(n^2)$.

Conjecture 2. If $G_1, \dots, G_N$ form an $\mathscr{A}_2$-intersection system on n vertices and for each edge e we know that the graphs $G_{i}, \dots, G_{i_m}$ containing this edge e do not form a strong Δ-system, then $N = o(n^2)$.

We shall prove the weaker

Theorem 2. *If* $G_1, \ldots, G_N$ *form an* $\mathscr{A}_2$*-intersection system on* n *vertices but not a strong* Δ*-system, then* $N \leqslant \binom{n^2/\sqrt{6}}{} + n$.

Theorem 3. *Let* $n \geqslant 10$, $\binom{x}{2} =: \frac{x(x-1)}{2}$ *for any real number* x, *and* $s =: \frac{1}{3}\binom{n}{2}$. *Then*

$$\binom{s-n}{2} < f(n, \mathscr{A}_1) \leqslant \binom{s}{2} + s + 1. \tag{2}$$

The upper bound is the best possible if $n = 6k + 1$ *or* $n = 6k + 3$.

For the structure of extremal systems the following holds: There exists an n_0 such that if $n > n_0$, $G_1, \ldots, G_N$ form an $\mathscr{A}_1$-intersection system and $N = f(n, \mathscr{A}_1)$, then upto a permutation of $V = V(G_i)$ the graphs G_i are defined as follows:

Let $T_1, \ldots, T_s$ be a maximal system of edge disjoint cycles in K_n. Let $P_1, \ldots, P_{\binom{s}{2}+s}$ be the family of singletons and pairs on $\{1, \ldots, s\}$ and G_i be defined by $V(G_i) = V$, $E(G_i) = E(T_j)$ if $P_i = \{j\}$ and $E(G_i) = E(T_j) \cup E(T_{j'})$ if $P_i = \{j, j'\}$. Let $E(G_{\binom{s}{2}+s+1}) = \phi$. If the cycles $T_1, \ldots, T_s$ do not cover the whole K_n, then let each edge of $E(K_n) - \bigcup E(T_i)$ form a further graph G_i. These graphs G_i form an extremal system and all the extremal systems can be obtained in this way.

In the examples above for each considered case we have a polynomial upper bound on $f(n, \mathscr{L})$. Perhaps the most interesting problem in connection with graph intersection theorems is the

Problem of polynomial upper bounds. Which properties of the family $\mathscr{L}$ does primarily influence the order of magnitude of $f(n, \mathscr{L})$? For instance, which conditions on $\mathscr{L}$ ensure that $f(n, \mathscr{L}) = O(n^r)$ for some $r > 0$? Can one find good necessary and sufficient conditions for this?

Some introductory examples.

(a) Let $\mathscr{L}$ be a *finite* family of graphs. Then, as it is easy to see, $f(n, \mathscr{L}) = O(n^r)$ for $r = \max\{|V(L_i)| + 2: L_i \in \mathscr{L}\}$.

(b) Let $\mathscr{L}$ be the family of complete graphs, then $f(n, \mathscr{L}) = 2^n$ and the extremal system consists of the spanned subgraphs of a K_n.

(c) Let $\mathscr{L}$ be the family of stars, that is of the trees, where each edge is incident to the same vertex. Now $f(n, \mathscr{L}) = 2^{n-1}$.

In this paper we shall not go into deeper details but prove just one theorem in connection with the existence of polynomial upper bounds.

Theorem 4. *Let $\mathscr{L}$ be a family of graphs with minimum valency $\geqslant 2$ and maximum $\leqslant K$, for which the number of components and the number of vertices of valency $\neq 2$ are also $\leqslant K$. There exist an $r = r_K$ such that $f(n, \mathscr{L}) = O(n^r)$.*

Remark 2. Another way of formulating the condition of Theorem 4 is to assume that there exists a finite family $\{L_1, \ldots, L_k\}$ of graphs with minimum valence $\geqslant 2$ such that the family of graphs, topologically equivalent with some of $L_1, \ldots, L_k$, contains $\mathscr{L}$.

Remark 3.

(a) Let $\mathscr{L}$ be the family of graphs consisting of vertex-disjoint K_3's. Fixing an $L_n \in \mathscr{L}$ on n vertices we define $G_1, \ldots, G_N$, $N = 2^{[\frac{n}{3}]}$ as the system of graphs consisting of some triangles of L_n and isolated vertices. $G_1, \ldots, G_N$ is an $\mathscr{L}$-intersection family showing that the condition on the boundedness of the number of components is necessary.

(b) If $K_2(p, q)$ denotes the complete bipartite graph with p and q vertices in its classes, let $\mathscr{L}$ be the family of $K_2(2, q)$'s. Since $K_2(2, n-2)$ contains $2^{n-2} - 1$ such subgraphs $(n - 2 \geqslant 1)$, and the intersection of any two of them is again such a subgraph, $f(n, \mathscr{L}) \geqslant$ $\geqslant 2^{n-2} - 1$. This example shows that the condition on the maximum valency is also necessary.

(c) Let $\mathscr{L}$ be the family of graphs having only vertices of valency 2 and 3. Let G be a cubic graph on n vertices and assume that P is a one-factor of G. Let us form, for all the subsets P' of P, the graph $G_{P'} =: G - P'$. Thus we get an $\mathscr{L}$-intersection family of $2^{[\frac{n}{3}]}$ graphs. Hence the condition on the number of vertices of valency $\neq 2$ is necessary.

PROOFS OF THEOREMS 1, 2, 3

Definition 2. Let $G_1, \ldots, G_M$ be an $\mathscr{A}_1$ or an $\mathscr{A}_2$-intersection system on the vertex-set V. A sequence $(y_1, \ldots, y_k)$ will be called a *walk* if the edges (y_i, y_{i+1}) are all different $(i = 1, \ldots, k-1)$ and there exists a G_{j_i} containing the path (y_{i-1}, y_i, y_{i+1}) [for $i = 2, \ldots, k-1$] in which y_i has valency 2. Moreover, assume that $(y_1, \ldots, y_k)$ is maximal under the above conditions, which means that either $y_1 = y_k$, when the walk is called *closed*, or each G_j containing (y_1, y_2) (or (y_{k-1}, y_k)) has valency $\neq 2$ at y_1 (or y_k), when the path is called *open*. (It will turn out that in the most essential cases a closed walk is a K_3, an open walk is an edge.)

Lemma 1.

(a) *If* G_j *contains an edge of a walk* P, *then* $P \subseteq G_j$.

(b) *A walk is either a path or a cycle.*

(c) *If* $P = (y_1, \ldots, y_k)$ *is an open walk contained in at least two* G_i*'s, then* y_1 *and* y_k *have valence* $\geqslant 3$ *in any* G_i *containing* P.

(d) *Each edge of* $\bigcup_{i=1}^{N} E(G_i)$ *is contained in exactly one walk.*

Proof. Let $P = (y_1, \ldots, y_k)$ be a walk and $(y_s, y_{s+1}) \in E(G_j)$ for some s, $1 \leqslant s \leqslant k-2$. To prove (a) it is enough to show that $(y_{s+1}, y_{s+2}) \in E(G_j)$, too. There exists a G_m containing the path (y_s, y_{s+1}, y_{s+2}) in which y_{s+1} has valency 2. Since y_{s+1} has valency 2 in $G_j \cap G_m$ as well and (y_s, y_{s+1}) is one of the edges of $G_j \cap G_m$, (y_{s+1}, y_{s+2}) must be the other one incident to y_{s+1}: $(y_{s+1}, y_{s+2}) \in E(G_j)$.

To prove (b) we have to show that no 3 vertices y_{i-1}, y_{i+1} and y_j can be joined to the same y_i. Indirectly, if there are 3 vertices y_{i-1}, y_{i+1}, y_j joined to y_i, we consider the G_m containing (y_{i-1}, y_i, y_{i+1}) in which y_i has valency 2 and observe, that by (a) G_m must also contain (y_j, y_i), which is a contradiction.

(c) and (d) are obvious consequences of the above ones.

Lemma 2. *An arbitrary open walk is contained in at most* $n-2$ G_j*'s if* $|V| = n \geqslant 4$.

Proof. Let $G_{i_1}, \ldots, G_{i_s}$ be the graphs containing the open walk $P = (y_1, \ldots, y_k)$. Let E_{i_h} be the set of edges of G_{i_h} incident to y_1 and different from (y_1, y_2). Here $n \geqslant 4$ is assumed and thus we may assume that $s \geqslant 2$. By (d) of Lemma 1, $|E_{i_h}| \geqslant 2$, and $|E_{i_h} \cap E_{i_{h'}}| = 1$ if $h \neq h'$. Hence by the de Bruijn – Erdős theorem [2], $s \leqslant n-2$.

Lemma 3.

(a) *If* P *is a closed walk and the* $\mathscr{A}_2$*-intersection system* $G_1, \ldots, G_M$ *is not a strong* Δ*-system, then* P *is contained in at most* $\frac{1}{3}\binom{n}{2} - 1$ G_j*'s.*

(b) *If* P_i *are closed walks* $(i = 1, 2;\ P_1 \neq P_2)$ *contained by* s_i *graphs from the considered* $\mathscr{A}_2$*-intersection system, then*

$$(s_1 - 1)(s_2 - 1) \leqslant \frac{1}{3}\binom{n}{2} - 3.$$

Proof.

(a) Since the system is not a strong Δ-system, there exists a G_h not containing P. For any G_i containing P the cycles $G_i \cap G_h$ and P are edge independent by (a) of Lemma 1: otherwise G_h would also contain P. If G_i and G_j contain P, then $E(G_i) - E(P)$ and $E(G_j) - E(P)$ are disjoint, hence at most $\frac{1}{3}\left(\binom{n}{2} - 3\right)$ G_i's contain P.

(b) Let $A_1, \ldots, A_{s_1}$ and $B_1, \ldots, B_{s_2}$ be the G_j's containing P_1

and P_2 respectively. At most one G_{j_0} can contain both P_1 and P_2 and for the other graphs the cycles $C_{i,j} =: A_i \cap B_j$ are edge-disjoint and disjoint also from P_1 and P_2. This proves (b).

Proof of Theorem 2. Let $G_1, \ldots, G_N$ be an extremal $\mathscr{A}_2$-intersecting system. Let $e(G)$ denote the number of edges of G. We shall distinguish the following cases:

(i) Each closed walk is contained by at most $n-2$ G_j's and there exists a G_h with $e(G_h) < \sqrt{\frac{3}{2}}\, n$.

(ii) Each closed walk is contained in at most $n-2$ G_j's and each G_j has at least $\sqrt{\frac{3}{2}}\, n$ edges.

(iii) There exists a closed walk P contained in at least $n-1$ G_j's.

(i) Clearly, each G_i (including G_h) intersects G_h is at least 3 edges and each edge is contained in at most $n-2$ G_j's, hence

$$N \leqslant \sqrt{\frac{3}{2}}\, n \frac{n-2}{3}.$$

This was to be proved.

(ii) Counting, how many times the edges occur in our graphs we get at most $\binom{n}{2}(n-2)$ on the one hand but at least $N \cdot \sqrt{\frac{3}{2}}\, n$ on the other hand. Thus

$$N \leqslant \sqrt{\frac{2}{3}} \frac{(n-1)(n-2)}{2}.$$

This proves the assertion.

(iii) We choose a closed walk P for which the number s of the graphs G_j containing P is the maximum and then a G_h from the graphs containing P for which $e(G_h)$ is the minimum (among the graphs containing P). If $G_1, \ldots, G_s$ are the graphs containing P, then the sets $E(G_j) - E(P)$ are disjoint, therefore

(3) $$M =: e(G_h) - e(P) = \min e(G_i) - e(P) \leqslant \frac{\binom{n}{2}}{s}.$$

For s we have by the assumption and Lemma 3 that

(4) $$n - 1 \leqslant s \leqslant \frac{1}{3}\binom{n}{2} - 1.$$

Let E_1 denote the number of edges of $G_h - E(P)$ belonging to closed walks, E_2 be the number of edges in the open walks. Using Lemmata 2 and 3 N can be estimated as follows:

(5) $$N \leqslant s + \frac{\frac{1}{3}E_1\binom{n}{2}}{3(s-1)} + \frac{E_2 n}{3} \leqslant s + \frac{E_1}{9s-9}\binom{n}{2} + \frac{E_2 n}{3},$$

where s estimates the number of graphs on P, including G_h too and the next term estimates the number of graphs on the other closed walks of G_h, but here G_h is not counted again. The third term stands for the graphs on the open walks of G_h. Observe that $E_1 + E_2 = M$ and

$$\frac{1}{9s-9}\binom{n}{2} < \frac{n}{3}.$$

Hence, and by (5), (3) and (4)

(6) $$N < s + \frac{Mn}{3} \leqslant s + \frac{\binom{n}{2}n}{3s} \leqslant \frac{n^2}{6} + n.$$

This completes the proof.

Proof of Theorem 1. If $n \geqslant 18$ then the upper bound of Theorem 1 is weaker than that of Theorem 2, hence we may assume that the extremal system is a strong Δ-system. This means that each G_i contains a cycle C of k vertices and the edge sets $E(G_h) - E(C)$ are disjoint:

$$N \leqslant \binom{n}{2} - e(C) + 1,$$

which completes the proof. If $n = 3, 4, \ldots, 17$, the theorem remains still valid and can be proved either by carrying out the above proof of Theorem 2 in a more careful way or by changing (simplifying) it at certain points. The details are omitted.

Remark 2. Obviously, Theorem 2 is not only sharper than Theorem 1, but it also expresses the "stability" of the extremal configurations of Theorem 1. Roughly saying, Theorem 2 asserts that if $G_1, \dots, G_N$ is an $\mathcal{A}_2$ intersection system, where N is closer to the maximum than $\left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right) n^2$ then this system is very similar to the optimal one: it is also a strong Δ-system.

Proof of Theorem 3. First we prove the upper bound. Let $G_1, \dots$ $\dots, G_N$ be an extremal $\mathcal{A}_1$-intersecting system.

(a) Let $P_1, \dots, P_q$ be the closed walks of $\bigcup_{i=1}^{n} E_i$ and $G_1, \dots, G_M$ the graphs containing only closed walks. If $N_i =: \{j\colon P_j \subseteq G_i\}$, then by $G_i \cap G_k \in \mathcal{A}_1$ we have $N_i \neq N_k$ and $|N_i \cap N_k| \leq 1$. By [6] we know that the number of these sets is at most $\binom{q}{2} + q$ and the equality holds iff N_i's are the singletons and pairs.

(b) The number of open walks is $\leq \binom{n}{2} - 3q$, hence, by Lemma 2 the number of graphs G_j containing open walks too is at most $\left(\binom{n}{2} - 3q\right)(n-2)$. Hence (counting the empty graph as well), we obtain

$$N \leq \binom{q}{2} + q + 1 + \left(\binom{n}{2} - 3q\right)(n-2) \leq \binom{s}{2} + s + 1, \tag{7}$$

where $s =: \frac{1}{3}\binom{n}{2}$; $q \leq s$ and $n \geq 10$ is used, and the fact that the expression in the middle of (7) achieves its maximum for the maximum value of q. This yields the upper bound.

(c) Let $T_1, \dots, T_{\hat{s}}$ be (as in Construction 1) a maximal system of edge-disjoint cycles in the complete graph K_n. If $n = 6k+1$ or $6k+3$, the system will be a Steiner triple system with $\hat{s} = \frac{1}{3}\binom{n}{2}$ triangles and the construction gives $\binom{\hat{s}}{2} + \hat{s} + 1$ graphs forming an $\mathcal{A}_1$-intersection system. Hence the upper bound in (7) is sharp. If $n \neq (6k+1)$ or $(6k+3)$, we obtain the lower bound of Theorem 3 by using a lower bound on $\hat{s}$. The details are omitted.

Proof of Theorem 4. Let G_n be an extremal $\mathcal{L}$-intersection system. We shall define the function $F(G_i)$ mapping G_i into some sets of edges as follows:

(a) If G_i contains a vertex v of valency $\geqslant K+1$, we choose such a v and $K+1$ edges of G_i incident to v. This will be $F(G_i)$.

(b) Assume that (a) does not hold but there exist $K+1$ vertices $a_1, \dots, a_{K+1}$ of valence $\geqslant 3$ in G_i. We choose $K+1$ such vertices and for each of them 3 incident edges. This edge-set will be $F(G_i)$.

(c) Assume that neither (a) nor (b) holds but G_i has a vertex of valency 1. Let v be such a vertex and e be the edge incident to v, then we choose $F(G_i) = \{e\}$.

(d) In the other cases we take a set of edges of G_i containing all the edges at least one endpoint of which has valency $\geqslant 3$ in G_i. This set will be denoted by $F_2(G_i)$ and we also choose a set $F_1(G_i)$ which contains exactly one edge from each component of G_i. Put $F(G_i) = F_1(G_i) \cup F_2(G_i)$.

We show that $F(G_i)$ is a monomorphism: if $i \neq j$ then $F(G_i) \neq F(G_j)$. Indeed, first of all, knowing $F(G_i)$ one knows immediately which of the definitions (a), (b), (c) and (d) was used. If $F(G_i)$ was defined by (a) or (b) or (c), then assuming indirectly $F(G_i) = F(G_j)$, $F(G_j)$ was defined in the same way. In this cases $F(G_i) \subseteq G_i \cap G_j$ contradicts $G_i \cap G_j \in \mathcal{L}$. Hence we may assume that (d) was used for $F(G_i) = F(G_j)$. We show that if (u, v, w) is a path in G_i and $(u, v) \in E(G_j)$, then $(v, w) \in E(G_j)$. Since $F(G_i)$ contains edges from each component of G_i, the above assertion will imply that $G_i \subseteq G_j$. By symmetry $G_j \subseteq G_i$, that is $G_i = G_j$. To prove that $(v, w) \in E(G_j)$ we distinguish two cases:

(i) either v has valency 2 in G_i when $(v, w) \notin E(G_j)$ implies that v is of degree 1 in $G_i \cap G_j$ (which is a contradiction), or

(ii) v has degree $\geqslant 3$ in G_i, hence all the edges of G_i incident to v belong to $F(G_i) = F(G_j) \subseteq E(G_j)$, in particular, $(v, w) \in E(G_j)$, what was to be proved.

Now we give an upper bound on the number of $F(G_i)$'s. Using (a), (b) or (c) we always get $|F(G_i)| \leq 3K + 3$, hence the number of these $F(G_i)$'s is only $O(n^{6k+6})$. In case (d) $|F_1(G_i)| \leq K^2 + K$, thus $F_1(G_i)$ can be chosen in at most $O(n^{2K^2+2K})$ ways. Since $|F_2(G_i) \cap \cap F_2(G_j)| \leq K$ if $i \neq j$, any $K+1$ element subset of the edges is contained in at most one $F_2(G_i)$, hence $F_2(G_i)$ can be chosen in at most $O(n^{2K+2})$ ways. Thus $F(G_i)$ can be chosen in at most $O(n^{2(K^2+2K+1)})$ ways. We have $F(G_i) \neq F(G_j)$ if $i \neq j$, therefore $N = O(n^{2K^2+4K+2})$.

Q.E.D.

Added in proof. Recently we have learnt that V. Rödl also had proved some of the results of [7].

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