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ON THE DISCREPANCY OF THE SEQUENCE $\{n\alpha\}$

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Let $\{x\}$ denote the fractional part of x, I a subinterval of $[0, 1)$, $|I|$ the length of the interval I and $S_N(I; \alpha)$ the number of $\{n\alpha\}$'s, for which

$$\{n\alpha\} \in I \quad \text{and} \quad 0 < n \leqslant N, \quad (n \text{ integer}).$$

The object of the present paper is to sketch a new proof of an "explicit" formula for

$$\Delta_N(I; \alpha) =: S_N(I; \alpha) - N|I|,$$

when $I = [0, \beta)$. The formula was found independently by S. Monteferrante and by the author and stated at first in a lecture in Oberwolfach, at the number theory meeting during 25-31 March, 1968.

The expression "explicit" is of course a relative one, it refers for certain expansion on N resp. β. One must say in advance that the formula seems to be more useful than aesthetic. Its usefulness is shown by the fact that one can deduce from it quite a lot of known and some new results.

NOTATIONS

Let $\alpha = [0, a_1 \dots]$ be the continued fraction expansion of an irrational $\alpha \in (0, 1)$

(1) $$\frac{P_k}{Q_k} = [0, a_1 \dots a_k] \quad \text{and} \quad D_k = Q_k \alpha - P_k .$$

Then we have

(2) $$P_k = a_k P_{k-1} + P_{k-2}, \quad Q_k = a_k Q_{k-1} + Q_{k-2} \qquad (k \geqslant 0)$$

with $P_{-2} = 0$, $P_{-1} = 1$, $Q_{-2} = 1$, $Q_{-1} = 0$, $a_0 = 0$. Further we have

(3) $$D_k = a_k D_{k-1} + D_{k-2} .$$

It is known, ([1], [2]) that any real number $\beta \in (-\alpha, 1 - \alpha)$ can be represented in the form

(4) $$\beta = \sum_{i=0}^{\infty} c_i D_i$$

where the c_i's are integers,

(5) $$0 \leqslant c_0 < a_1, \quad 0 \leqslant c_i \leqslant a_{i+1} \quad \text{for} \quad i > 0$$

and

(6) $$c_i = a_{i+1} \quad \text{only if} \quad c_{i-1} = 0.$$

This representation of β is unique if we do not allow the case

(7) $$c_{\nu_0 + 2i} = a_{\nu_0 + 2i} \quad \text{for some} \quad \nu_0 \quad \text{and} \quad i = 1, 2, \dots .$$

(Conversely, to any sequence (c_n) under the above conditions we get a number $\beta \in (-\alpha, 1 - \alpha)$.)

It is known also and often used ([1], [2], [5]) that there is a unique expansion of any natural number X in the form

$$X = \sum_{i=0}^{n} x_i Q_i$$

where the x_i's are integers $0 \leqslant x_0 < a_1$,

(8) $$0 \leqslant x_i \leqslant a_{i+1} \quad \text{for} \quad i > 0$$

and

(9) $$x_i = a_{i+1} \quad \text{only if} \quad x_{i-1} = 0.$$

Let $N = \sum_{i=0}^{m} b_i Q_i$ $(b_m > 0)$ be the expansion of N in the above form,

(10) $$N_k =: \sum_{i=0}^{k} b_i Q_i \quad \text{and} \quad Q_k^* =: \sum_{i=0}^{k} c_i Q_i.$$

For the sake of simplicity let

(11) $$\beta = \sum_{i=i_0}^{\infty} c_i D_i, \quad c_{i_0} > 0, \quad i_0 \text{ even}$$

(i.e. $c_i = 0$ for $i < i_0$ and $0 < \beta < 1 - \alpha$.)

Our formula in question qualitatively says, that $\Delta_N((0, \beta); \alpha)$ is "almost additive" in the terms $b_i Q_i$ resp. $c_i D_i$. More exactly the following theorem holds:

Theorem.

(12) $$\Delta_N((0,\beta);\alpha) = \sum_{k=0}^{\infty} ((-1)^k \min(b_k, c_k) - c_k b_k Q_k D_k) - \\ - \sum_{k=0}^{\infty} \sum_{l<k} (c_k b_l + c_l b_k) Q_l D_k + \sum_{l=0}^{\infty} \delta_l$$

where

$$\delta_l = \begin{cases} 1, & \textit{if } l \textit{ even and } Q_{l-1}^* < N_{l-1} \leqslant N_l < Q_l^* \\ -1, & \textit{if } l \textit{ odd and } N_{l-1} \leqslant Q_{l-1}^* \leqslant Q_l^* < N_l \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Using the above notations, we state without proof the following simple lemmas:

1. $X = \sum_{i=0}^{m} x_i Q_i < N = \sum_{i=0}^{m} b_i Q_i$ iff for some $0 \leqslant k \leqslant m$

(13) $\qquad x_i = b_i \quad \text{if} \quad i > k \quad \text{and} \quad x_k < b_k.$

2. Let $X = \sum_{i=j_0}^{m} x_i Q_i$, $x_{j_0} > 0$ (i.e. $x_i = 0$ for $i < j_0$). Then

$$\{X\alpha\} = \left\{\sum_{i=j_0}^{m} x_i Q_i \alpha\right\} = \sum_{i=j_0}^{m} x_i D_i,$$

if j_0 is even; $0 < \sum_{i=j_0}^{m} x_i D_i < 1 - \alpha$ and

$$\{X\alpha\} = \left\{\sum_{i=j_0}^{m} x_i Q_i \alpha\right\} = 1 + \sum_{i=j_0}^{m} x_i D_i$$

if j_0 is odd; $-\alpha < \sum_{i=j_0}^{m} x_i D_i < 0.$

3. For a β satisfying (11) and for a positive integer X

$$\{X\alpha\} \in [0, \beta)$$

iff for some $l \geqslant 0$

(14) $\qquad x_i = c_i \quad \text{if} \quad i < l \quad \text{and} \quad \operatorname{sign}(c_l - x_l) = (-1)^l,$

in addition

(15) $\qquad \text{if} \quad x_0 = \ldots = x_{\nu-1} = 0, \ x_\nu \neq 0 \quad \text{then } \nu \text{ is even.}$

According to these lemmas

(16) $\qquad S_N((0, \beta); \alpha) = \sum_{l,k} S_{l,k}$

where $S_{l,k}$ denotes the number of sequences (x_i) satisfying (8), (9), (13), (14) and (15) with the fixed l and k.

Consequently for the proof of the theorem we have to determine the vaiues of the $S_{l,k}$'s.

(a) *In case* $k > l > i_0$, l *even,* $c_l = 0$ *resp.* $c_l \neq 0$ *we have*

$$S_{l,k} = 0$$

resp.

$$S_{l,k} = c_l b_k (Q_k D_l - D_k Q_l) - b_k (Q_k D_{l+1} - D_k Q_{l+1}).$$

The case $c_l = 0$ is evident.

If $c_l \neq 0$, let A_ν be the number of sequences $(x_l, \ldots, x_{l+\nu})$ which satisfy

$$0 \leqslant x_l < c_l$$

$$0 \leqslant x_i \leqslant a_{i+1}, \quad x_i = a_{i+1} \quad \text{only if} \quad x_{i-1} = 0$$

for $l < i \leqslant l + \nu$.

Obviously, with $A_{-1} =: 1$ we have

$$A_0 = c_l$$

$$A_1 = a_{l+1} c_l + 1 = a_{l+2} A_0 + A_{-1}$$

and

$$A_{j+1} = a_{j+1+l+1} A_j + A_{j-1}.$$

From this, and from the recursive formulas (2), (3) we get

$$A_\nu = c_l (Q_{l+1+\nu} D_l - D_{l+1+\nu} Q_l) - $$
$$- (Q_{l+1+\nu} D_{l+1} - D_{l+1+\nu} Q_{l+1}).$$

Using this with $\nu = k - l - 1$

$$S_{l,k} = b_k A_{k-l-1} = b_k (c_l (Q_k D_l - Q_l D_k) -$$
$$- (Q_k D_{l+1} - D_k Q_{l+1})).$$

(b) *In the case* $k > l = i_0$ *(*i_0 *even!) we have*

$$S_{i_0,k} = b_k (c_{i_0} - 1)(Q_k D_{i_0} - D_k Q_{i_0}) -$$
$$- b_k (Q_k D_{i_0+1} - D_k Q_{i_0+1}) + \delta_k^+ + \delta_k^-$$

where

$$\delta_k^+ = \begin{cases} 1, & \text{if } k \text{ odd}, \ b_k > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_k^- = \begin{cases} -1, & \text{if } b_k > 0, \ b_\nu = 0 \ \text{ for } k < \nu < j, \\ & \qquad\qquad\qquad\quad b_j > 0, \ j \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

For the proof we need a similar reasoning, as in case (a) and beside this we have to take into consideration that for the sequences (x_i) in question (15) must hold.

We note that evidently

$$\sum_k (\delta_k^+ + \delta_k^-) = 1 \quad \text{if}$$

(17) for some ν $b_{2\nu+1} > 0$ and $b_j = 0$ if $i_0 < j < 2\nu + 1$,

$$\sum_k (\delta_k^+ + \delta_k^-) = 0 \quad \text{if (17) does not hold.}$$

(c) *In the case* $k > l > i_0$, l *odd,* $c_{l-1} \neq 0$ *resp.* $c_{l-1} = 0$ *we have*

$$S_{l,k} = -(a_{l+1} - c_l - 1) b_k (Q_k D_l - D_k Q_l)$$

resp.

$$S_{l,k} = -(a_{l+1} - c_l) b_k (Q_k D_l - D_k Q_l).$$

For the proof we need a similar reasoning again, as in case (a), and beside this we have to take into consideration, that we may have $x_l = a_{l+1}$ only in case $x_{l-1} = c_{l-1} = 0$.

(d) *In case* $k = l > i_0$, k *even resp.* k *odd, we have evidently*

$$S_{k,k} = \min (c_k, b_k) \quad \text{resp.} \quad S_{k,k} = \max (0, b_k - c_k - 1),$$

$$S_{i_0,i_0} = \min (c_{i_0}, b_{i_0}) + \delta_{i_0}^*$$

where

$$\delta^*_{i_0} = \begin{cases} -1, & \text{if (17) holds} \\ 0 & \text{otherwise.} \end{cases}$$

(e) *In case* $l > k$ $(l > i_0)$ *we have to consider the special sequence* — if it exists at all —

$$x_i = c_i \quad \text{if} \quad i \leqslant l-1$$

$$x_i = b_i \quad \text{if} \quad i \geqslant k+1.$$

We get easily, that

$$S_{l,k} = \begin{cases} 1, & \text{if } l \text{ even and } Q^*_{l-1} < N_{l-1} \leqslant N_l < Q^*_l \text{ or} \\ & \text{if } l \text{ odd and } Q^*_{l-1} < N_{l-1},\ Q^*_l < N_l \\ 0 & \text{otherwise.} \end{cases}$$

The remaining cases are contained in the last case

(f) *In case* $l < i_0$ *we have obviously*

$$S_{l,k} = 0.$$

Now substituting these values of the $S_{l,k}$'s, using the recursive formulas (2), (3) and treating appropriately the ± 1 terms one can complete the proof of the Theorem.

As corollaries of the above theorem we may get e.g. the following results:

Theorem of Hecke [3]. $\Delta_N((0, \beta); \alpha)$ *is bounded in* N *if* $\beta = \{k\alpha\}$ *for some positive integer* k.

Theorem of Kesten [4]. $\Delta_N((0, \beta); \alpha)$ *is bounded in* N *only if* $\beta = \{k\alpha\}$ *for some positive integer* k.

Theorem. *If* $\alpha = [0, a_1, \ldots]$ *has bounded partial quotients;* $a_i \leqslant K$ *for* $i \geqslant 1$, *then*

$$\Delta_N((0, \beta); \alpha) \leqslant C_K \log N$$

where C_k *depends only on* K.

With an explicit value of C_k see [5], (without proof [6]).

Theorem. *If for* $\alpha = [0, a_1 \dots]$ *we have* $a_i \geqslant K$ *for* $i > i_0$, *then*

$$\inf_{\beta, N} \frac{\Delta_N((0, \beta); \alpha)}{\log N} \geqslant C_K^*$$

where C_K^* *depends only on* K *and* $\lim_{K \to \infty} C_K^* = \infty$.

Theorem. *For suitable* α *there exists* $\beta \neq \{k\alpha\}$ *(k integer) such that*

$$\Delta_N((0, \beta); \alpha)$$

is onesidedly bounded in N.

(It is stated without proof e.g. in [6].)

We prove these consequences in a forthcoming paper.

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