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THE MEAN-SQUARE DISCREPANCIES OF SOME TWO-DIMENSIONAL LATTICES

by

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§ 1. Introduction

Let Q^2 denote the square defined by

$$0 \leqq x < 1; \quad 0 \leqq y < 1,$$

and let $\mathbf{Z}$ be any finite set of points $\mathbf{z}_0, \dots, \mathbf{z}_{m-1}$ contained in Q^2, $\mathbf{z}_i = (x_i, y_i)$ $(i = 0, \dots, m-1)$. The degree of equidistribution of $\mathbf{Z}$ can be described by the function

$$g(\mathbf{z}) = m^{-1}\nu(\mathbf{z}) - xy,$$

where $\mathbf{z} = (x, y)$ is in the closure $\bar{Q}^2$ of Q^2, and $\nu(\mathbf{z})$ is the number of points of $\mathbf{Z}$ for which $x_i < x$ and $y_i < y$.

Clearly, if the equidistribution of $\mathbf{Z}$ is good, $|g(\mathbf{z})|$ should be small throughout $\bar{Q}^2$. If we want a single number to measure the equidistribution in question, the obvious choice is a norm of $g(\mathbf{z})$. The two most natural norms are

(i) $$D^*(\mathbf{Z}) = \sup_{\mathbf{z} \in \bar{Q}^2} |g(\mathbf{z})|$$

and

(ii) $$D^{(2)}(\mathbf{Z}) = \Big(\int_{Q^2} g(\mathbf{z})^2 \, d\mathbf{z}\Big)^{1/2}.$$

The first norm is known as the *extreme discrepancy* of X, or, more simply, its discrepancy. For the second, the name of L^2 discrepancy, or *mean-square discrepancy* was introduced in 1968 ZAREMBA [10], although its concept appeared as early as 1954 ROTH [7]. The definitions of $D^*(\mathbf{Z})$ and $D^2(\mathbf{Z})$ can be extended in an obvious manner to any number of dimensions; however, the present paper deals only with the case of two dimensions.

The two concepts of discrepancy, apart from their intrinsic number-theoretical interest, play an important part in numerical analysis: If we regard the expressions

$$m^{-1}\big(f(\mathbf{z}_0) + \dots + f(\mathbf{z}_{m-1})\big)$$

as approximate values of the integral

$$\int_{Q^2} f(\mathbf{z}) \, d\mathbf{z},$$

the absolute values of the errors have, under suitable conditions of smoothness imposed on f, upper bounds of the form

$$C^* D^*(\mathbf{Z}) \quad \text{or} \quad C^{(2)} D^{(2)}(\mathbf{Z}),$$

where the coefficients C^* and $C^{(2)}$ depend only on f (see, for instance HLAWKA [3], ZAREMBA [10] or [11] or KOROBOV [5]). If $\mathbf{Z}$ is a suitable lattice, then, depending on the smoothness of f, the error of integration can be of a much smaller order of magnitude than the bounds indicated above (see, for instance HLAWKA [4], KOROBOV [5], ZAREMBA [11] or VILENKIN [9]).

K. F. ROTH [7] proved that

$$D^{(2)}(\mathbf{Z}) \geqq c_2 m^{-1} (\log m)^{1/2}$$

for every finite set $\mathbf{Z}=\{\mathbf{z}_0, \ldots, \mathbf{z}_{m-1}\}\subset Q^2$, where c_2 is an absolute constant. W. SCHMIDT [8] proved that for any such set $\mathbf{Z}$

$$D^*(\mathbf{Z}) \geqq cm^{-1} \log m,$$

where c is again an absolute constant.

Sequences of sets $\mathbf{Z}\subset Q^2$ for which

$$D^*(\mathbf{Z}) = O(m^{-1} \log m), \tag{1.1}$$

in particular sequences of such *lattices* $\mathbf{Z}$ are well-known (see, for instance HLAWKA [4], KOROBOV [5], ZAREMBA [11] or VILENKIN [9]). Sequences of sets $\mathbf{Z}\subset Q^2$ for which

$$D^2(\mathbf{Z}) = O\big(m^{-1} (\log m)^{1/2}\big) \tag{1.2}$$

have also been known (DAVENPORT [1], HALTON—ZAREMBA [2], VILENKIN [9]). But none of these sets formed a lattice, although the one considered by DAVENPORT [1] was a symmetric union of two lattices. In view of the theoretical and practical importance of lattices, it was felt that it was worth investigating which lattices $\mathbf{Z}$, if any, had an L^2 discrepancy of the order of $m^{-1}(\log m)^{1/2}$.

At this stage it should be recalled that if A is an upper bound of the partial quotients of the finite or infinite continued-fraction expansion of a number α, if m does not exceed the denominator of α in the case when α is rational, and if $\mathbf{Z}$ consists of the points

$$\langle 0, 0\rangle,\ \langle m^{-1}, \{\alpha\}\rangle,\ \langle 2m^{-1}, \{2\alpha\}\rangle, \ldots, \langle (m-1)m^{-1}, \{(m-1)\alpha\}\rangle,$$

$\{x\}$ denoting the fractional part of x, then

$$D^*(\mathbf{Z}) \leqq K^* m^{-1} \log m, \tag{1.3}$$

where K^* is a constant depending only on A; this is an immediate consequence of Proposition 4.3 in Zaremba [12].

The main purpose of the present paper is to show that if all the partial quotients of the finite or infinite continued fraction expansion of α are equal, $m \geqq 1$ not exceeding the denominator of α when α is rational, then

$$D^{(2)}(\mathbf{Z}) = O\big(m^{-1} (\log m)^{1/2}\big).$$

We obtain this result by examining the expressions

$$\frac{1}{m} \sum_{q=0}^{m-1} S_q^2, \tag{1.4}$$

where

$$S_q = \sum_{j=1}^{q} \left(\{j\alpha\} - \frac{1}{2} \right). \tag{1.5}$$

Propositions about the behaviour of (1.4) and its connection with $D^2(\mathbf{Z})$ may also be of some intrinsic interest.

In a forthcoming paper we are going to prove that if the partial quotients of the continued fraction are not all equal, even if they are bounded, $D^{(2)}(\mathbf{Z})$ can be of the order of $m^{-1} \log m$.

§ 2. A crucial lemma

LEMMA 2.1. *With the previously introduced notations, assuming that the partial quotients of the continued-fraction expansion of α are bounded, and that m does not exceed the denominator of α in the case when α is rational,*

$$D^2(\mathbf{Z}) = O\big(m^{-1}(\log m)^{1/2}\big)$$

if, and only if

$$\frac{1}{m} \sum_{q=0}^{m-1} S_q^2 = O(\log m),$$

where S_q is given by (1.5).

PROOF. We use a technique due to H. DAVENPORT [1]. To simplify some notations, we put

$$G(x, y) = mg(x, y) = v(x, y) - mxy$$

and

$$\psi(\eta) = \{\eta\} - \frac{1}{2}.$$

It is easily verified that for any β and any η in [0, 1]

$$\eta + \psi(\beta - \eta) - \psi(\beta) = \begin{cases} 0 & \text{if } \{\beta\} \geqq \eta, \\ 1 & \text{if } \{\beta\} < \eta. \end{cases}$$

Hence

$$v(x,y) = \sum_{0 \leqq j < mx} (y + \psi(j\alpha - y) - \psi(j\alpha)).$$

Clearly

$$|G(x, y) - \tilde{G}(x, y)| \leqq 1,$$

where

$$\tilde{G}(x, y) = \sum_{0 \leqq j < mx} (\psi(j\alpha - y) - \psi(j\alpha)).$$

Since it is well-known (ROTH [7]), that

$$\int_0^1 \int_0^1 G(x, y)^2 \, dx \, dy$$

is at least of the order of log m, the order of magnitude of the last integral is the same as that of

$$\int_0^1\int_0^1 \tilde{G}(x, y)^2\,dx\,dy.$$

Now we take advantage of the Fourier expansion

$$\psi(\alpha) = -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin 2\pi n\alpha}{n}$$

valid for $\alpha \neq 0$. With this representation,

$$\tilde{G}(x, y) = \sum_{0\leqq j<mx}\left(-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi n(\alpha j-y)}{n}-\psi(j\alpha)\right) =$$

(2.1)
$$= -\frac{1}{\pi}\sum_{n=1}^{\infty} k^{-1}\cos 2\pi ny \sum_{0\leqq j<mx}\sin 2\pi n\alpha j +$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty} k^{-1}\sin 2\pi ny \sum_{0\leqq j<mx}\cos 2\pi n\alpha j - \sum_{0\leqq j<mx}\psi(j\alpha).$$

Now we want to square this expression and integrate it with respect to y from 0 to 1. The three terms of the integrand being orthogonal to each other, the integral of $\tilde{G}(x, y)^2$ is equal to the sums of the integrals of the three squared terms. We shall denote these integrals by I_1, I_2 and I_3, respectively.

We begin with I_2. By the Parseval formula

(2.2)
$$I_2 = \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big(\sum_{0\leqq j<mx}\cos 2\pi n\alpha j\Big)^2.$$

Now we have to distinguish the cases of α being irrational and of α being rational. We begin with the former case, following Davenport [1].

It is well-known (see, e.g., Lemma 6.5 in Zaremba [11]) that if $n\alpha$ is not an integer, then for any m

(2.3)
$$\Big|\sum_{0\leqq j<mx}\cos(2\pi n\alpha j)\Big| \leqq \frac{1}{2\|n\alpha\|}$$

where $\|\xi\|$ denotes the distance of ξ from the nearest integer.

But also

$$\Big|\sum_{0\leqq j<mx}\cos(2\pi i n\alpha j)\Big| \leqq [mx].$$

Thus

(2.4)
$$I_2 \leqq \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\min([mx]^2, 2^{-2}\|n\alpha\|^{-2}).$$

Let p_k/q_k be the successive convergents of the continued-fraction expansion of α, defined by $q_1=1$, $q_2=a_1$, ..., $q_{k+1}=a_kq_k+q_{k-1}$ ($k=2, 3, \ldots$), $p_1=0$, $p_2=1$, ..., $p_{k+1}=$

$=a_k p_k+p_{k-1}$, where $a_1, a_2, \ldots$ are the partial quotients in this expansion. If $q_{k-1}\leqq n\leqq q_k$, by Lagrange's theorem,

$$\|n\alpha\| \geqq |q_{k-1}\alpha-p_{k-1}| \geqq (q_{k-1}+q_k)^{-1}.$$

Hence

$$n\|n\alpha\| \geqq q_{k-1}/(q_{k-1}+q_k) > (A+2)^{-1} = C, \tag{2.5}$$

where $A=\max_i a_i$.

If $2^{r-1}\leqq n<2^r$, then by (2.5),

$$\|n\alpha\| > C/n > C/2^r.$$

But, for any given integer s, there can be at most two values of n, say n_1 and n_2 in $[2^{r-1}, 2^r)$ satisfying

$$sC\cdot 2^{-r} \leqq \|n_i\alpha\| < (s+1)C\cdot 2^{-r} \qquad (i=1,2). \tag{2.6}$$

Indeed, if there were a third one, we would have an n^* with $|n^*|<2^r$ and

$$\|n^*\alpha\| < C\cdot 2^{-r} < C/|n^*|,$$

which contradicts (2.5).

The two values of n in $[2^{r-1}, 2^r)$ satisfy

$$\|n\alpha\|^{-2} < C^{-2}s^{-2}2^r,$$

and according to (2.4), we find

$$\begin{aligned} I_2 &\leqq \frac{1}{\pi^2}\sum_{r=1}^{\infty} 2^{2-2r}\sum_{s=1}^{\infty}\min\left([mx]^2, C^{-2}s^{-2}2^{2r}\right) \leqq \\ &\leqq \frac{4}{\pi^2 C^2}\sum_{r=1}^{[\log_2 m]}\sum_{1}^{\infty}\frac{1}{s^2}+\frac{1}{\pi^2}\sum_{r>[\log_2 m]}\sum_{s=1}^{\infty}[mx]^{\frac{1}{2}}C^{-\frac{3}{2}}2^{-\frac{r}{2}}s^{-\frac{3}{2}}. \end{aligned} \tag{2.7}$$

Since the first sum is $O(\log m)$ and the second is $O(1)$, we obtain

$$I_2 = O(\log m). \tag{2.8}$$

If α is rational, we denote by d its denominator, and we put

$$n = kd+l \quad \text{with} \quad 0\leqq l<d.$$

We have to single out the terms of (2.2) which correspond to values of n being multiples of d. The sum of these terms does not exceed

$$\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{[mx]^2}{k^2d^2} \leqq \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{12}.$$

Concerning the other terms of (2.2), we note that $\|n\alpha\|=\|l\alpha\|$, while, as in the case of α irrational, $l\|l\alpha\|>C$, and all the more $n\|n\alpha\|>C$, or

$$\|n\alpha\| > \frac{C}{n}.$$

Argueing as in the case of an irrational α, we find that the sum of the terms of (2.2) which correspond to values of n other than multiples of d is smaller than the right-hand side of (2.7), and therefore is $O(\log m)$. Thus (2.8) holds both for rational and irrational values of α.

Concerning I_1, instead of (2.2) we have

$$I_1 = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{0 \leqq j < mx} (\sin 2\pi k\alpha j)^2.$$

The treatment is exactly the same as that of I_2, the only difference being that in the case of α rational, all the terms corresponding to values of k divisible by d vanish. Thus in both cases

$$I_1 = O(\log m). \tag{2.9}$$

Both I_1 and I_2 have to be integrated with respect to x in [0, 1]. Since the upper bounds obtained for them do not depend on x, the double integrals are also $O(\log m)$.

I_3 is quite different. Since the square of the last term in the right-hand side of (2.1) is independent of y, it is equal to I_3. Now it has to be integrated with respect to x in [0, 1]; since it is a step function, in view of the definition of ψ, its integral reduces to the sum (1.4).

Thus, apart from a term which in any event is of a lower order of magnitude,

$$D^{(2)}(\mathbf{Z})^2 = \int_0^1 \int_0^1 G(x, y)^2 \, dx \, dy$$

is the sum of two terms which were found to be $O(\log m)$ and of the sum (1.4). This proves the lemma.

§ 3. Further lemmas about continued fractions

Let α be fixed and a_k, q_k, p_k ($k=1, \ldots$) have the same meaning as before.

DEFINITION 3.1. A finite sequence $\langle b_r, \ldots, b_s \rangle$, with $s<n'$ if $\alpha = p_{n'}/q_{n'}$ will be described as *admissible* (with respect to α) when

$$0 \leqq b_r < a_r \quad \text{and} \quad 0 \leqq b_i \leqq a_i,$$
$$\text{but} \quad b_{i-1} = 0 \quad \text{whenever} \quad b_i = a_i \quad (i = r+1, \ldots, s).$$

The two lemmas and two corollaries which follow are well-known (they were exactly implied in OSTROWSKI [6]) and in any event are easy to prove.

LEMMA 3.2. *If $\langle b_1, \ldots, b_{n-1} \rangle$ is an admissible sequence,*

$$b_1 q_1 + \ldots + b_{n-1} q_{n-1} < q_n.$$

LEMMA 3.3. *Assuming that $n \leqq n'$ if $\alpha = p_{n'}/q_{n'}$, any nonnegative integer $p < q_n$ can be uniquely represented in the form*

$$q = b_1 q_1 + \ldots + b_{n-1} q_{n-1} \tag{3.1}$$

where $\langle b_1, \ldots, b_n \rangle$ is an admissible sequence.

COROLLARY 3.4. Assuming that $n \leqq n'$ if $\alpha = p_{n'}/q_{n'}$ there is a one-to-one correspondence between integers $0, 1, \ldots, q_n - 1$ and admissible sequences $\langle b_1, \ldots, b_{n-1} \rangle$ determined by (3.1).

COROLLARY 3.5. Under the same assumptions, the number of admissible sequences $\langle b_1, \ldots, b_{n-1} \rangle$ is equal to q_n.

It is well-known from the theory of continued fractions that for any $i \geqq 1$

$$\alpha = \frac{p_i}{q_i} + \frac{\Theta_i}{q_i q_{i+1}} \tag{3.2}$$

where $|\Theta_i| \leqq 1$. We consider now the various sums

$$S_q = \sum_{j=0}^{q} \left(\{j\alpha\} - \frac{1}{2} \right)$$

with $q < p_{n'}$ if $\alpha = p_{n'}/q_{n'}$. According to Lemma 3.3, q admits a unique representation (3.1) where $\langle b_i \rangle$ is an admissible sequence. Hence S_q can be represented uniquely in the form

$$S_q = \sum_{i=1}^{n-1} \sigma_i \tag{3.3}$$

where

$$\sigma_i = \sum_{\nu=0}^{b_i q_i - 1} \left(\{(\nu + t_i)\alpha\} - \frac{1}{2} \right)$$

when $b_i > 0$, and $\sigma_i = 0$ when $b_i = 0$, while

$$t_i = \sum_{k=1}^{i-1} b_k q_k \qquad (i = 2, \ldots, n-1); \; t_1 = 0.$$

According to OSTROWSKI [6] we have

$$\sigma_i = b_i \left(\frac{(-1)^i}{2} + \Theta_i \frac{b_i q_i + 2 \sum_{k=1}^{i-1} t_k q_k - 1}{2 q_{i+1}} \right), \quad i = 1, \ldots, k-1. \tag{3.4}$$

We consider now the special case when the number α has a finite or infinite continued-fraction expansion whose all partial quotients are equal to a positive integer a. The convergents of the expansion of α are easily found to be

$$p_i/q_i = v_{i-1}/v_i$$

where

$$v_j = (\beta^j + (-1)^{j+1} \beta^{-j})(a^2+4)^{-1/2} \qquad (j = 1, 2, \ldots) \tag{3.5}$$

and

$$\beta = \frac{1}{2}(a + (a^2+4)^{1/2}).$$

If $a = 1$, the sequence is that of the Fibonacci numbers, and (3.5) is nothing else but the Binet formula. Thus either

$$\alpha = v_{n'-1}/v_{n'} \tag{3.6}$$

for some integer n', or

$$\alpha = \lim_{n\to\infty} v_{n-1}/v_n = \beta^{-1} = \frac{1}{2}\left((a^2+4)^{1/2}-a\right). \tag{3.7}$$

In our case, (3.2) becomes

$$\alpha = \frac{v_{i-1}}{v_i} + \frac{\Theta_i}{v_i v_{i+1}} \tag{3.8}$$

with $i<n$ if α is given by (3.6).

LEMMA 3.6. *If α is given by* (3.6) *or* (3.7), *the sums $\Theta_r+\ldots+\Theta_s$ with $1\leqq r<s$ and $s<n'$ in the former case are bounded by a number depending only on a.*

PROOF. Since $|\Theta_i|\leqq 1$ for all i, it suffices to show that the sums

$$\sum_{k=1}^{\infty} |\Theta_{2k}+\Theta_{2k+1}|,$$

when α is irrational, and

$$\sum_{k=1}^{\left[\frac{n'-2}{2}\right]} |\Theta_{2k}+\Theta_{2k+1}|$$

when α is rational have an upper bound depending only on a.

By (3.2) with $q_i=v_i$, $q_{i+1}=v_{i+1}$ and $p_i=v_{i-1}$

$$\Theta_i = v_{i+1}(\alpha v_i - v_{i-1}), \tag{3.9}$$

and if α is irrational, we find

$$\Theta_i = \frac{(-1)^{i+1}}{a^2+4}\left(1+\beta^2+(-1)^i\beta^{-2i-2}+(-1)^i\beta^{-2i}\right).$$

Similarly

$$\Theta_{i+1} = \frac{(-1)^{i}}{a^2+4}\left(1+\beta^2+(-1)^{i+1}\beta^{-2i-4}+(-1)^{i+1}\beta^{-2i-2}\right).$$

Hence

$$|\Theta_i+\Theta_{i+1}| = \frac{(1+\beta^{-2})^2}{a^2+4}\beta^{-2i},$$

and further

$$\sum_{i=1}^{\infty} |\Theta_{2i}+\Theta_{2i+1}| = \frac{(1+\beta^{-2})^2}{a^2+4}\,\frac{1}{\beta^4-1}.$$

The case when α is rational, i.e., is given by (3.2), is slightly more complicated. Substituting (3.5) and (3.6) in (3.9), we find

$$\Theta_i =$$

$$= \frac{(\beta^{i+1}+(-1)^i\beta^{-i-1})\left((-1)^{n'}\beta^{i-n'+1}+(-1)^{n'}\beta^{i-n'-1}+(-1)^{i+1}\beta^{n'-i-1}+(-1)^{i+1}\beta^{n'-i+1}\right)}{(a^2+4)^{3/2}v_{n'}}$$

and a similar expression for Θ_{i+1}, with $i+1$ substituted for i everywhere. After some simplifications, we obtain

$$|\Theta_i+\Theta_{i+1}| = \frac{(1+\beta^2)^2\beta^{2i-n'}+(1+\beta^{-2})\beta^{n'-2i}}{(a^2+4)^{3/2}v_{n'}}.$$

Since $(a^2+4)^{3/2}v_{n'}/\beta^{n'}$ is bounded, to prove that $\Theta_r+\ldots+\Theta_s$ is bounded, it suffices to show the boundedness of

$$\sum_{\nu=1}^{\left[\frac{n'-2}{2}\right]} \left((1+\beta^2)^2\beta^{4\nu-2n'}+(1+\beta^{-2})\beta^{-4\nu}\right),$$

which is trivial.

§ 4. A probabilistic interpretation of the problem

The sum (1.4) with $m=q_n$, i.e., the sum

$$\frac{1}{q_n}\sum_{q=1}^{q_n-1} S_q^2,$$

where $n<n'$ if $\alpha=p_{n'}/q_{n'}$, can be regarded as the expectation $\mathsf{E}(S_q^2)$, q being a random variable taking each of the values $0, 1, \ldots, q_n-1$ with the same probability $\frac{1}{q_n}$. To compute

$$\mathsf{E}(S_q^2) = \mathsf{E}(S_q)^2+\operatorname{var} S_q,$$

we need the first and second order moments of the joint probability distribution of $\sigma_1, \ldots, \sigma_{n-1}$. Owing to (3.4), this will be deduced from the relevant moments of $b_1, \ldots, b_{n-1}$.

We consider the case $a_i=a$, $i=1, 2, \ldots$. We begin with the probability $\mathsf{P}[b_i=k]$ with $0<k<a_j$; it is, of course, equal to 0 when $a=1$. If $a>1$, $b_1, \ldots, b_{i-1}$ can form any admissible sequence and according to Corollary (3.5) there are v_i such sequences. Independently of them, $b_{i+1}, \ldots, b_{n-1}$ can be any admissible sequence, which gives v_{n-i} possibilities. Thus the total number of admissible sequences featuring $b_i=k$ is v_iv_{n-i}; since $v_1=1$, this is true, in particular for $i=1$ and for $i=n-1$. Since each of the sequences in question has probability $1/v_n$, we have

(4.1) $$\mathsf{P}(b_i = k) = v_iv_{n-i}/v_n \quad 0<k<a; \quad i = 1, \ldots, n-1.$$

If $b_i=a$ with $i>1$, we must have $b_{i-1}=0$. Hence the factor v_i in (4.1) has to be replaced by v_{i-1} and so

(4.2) $$\mathsf{P}(b_i = a) = v_{i-1}v_{n-i}/v_n \quad (i = 2, \ldots, n-1)$$

while necessarily $\mathsf{P}(b_1=a)=0$.

From (4.1) and (4.2) we obtain

(4.3) $$\mathsf{E}(b_i) = \frac{v_{n-i}}{v_n}\left(\frac{a(a-1)}{2}v_i+av_{i-1}\right).$$

This is still valid for $i=1$ since $v_0=0$. Substituting here, for v_i, v_{i-1}, v_{n-i} and v_n their expressions in terms of β, we find

$$\mathsf{E}(b_i) = A+O(\beta^{-2i})+O(\beta^{2i-2n}), \tag{4.4}$$

where

$$A = \frac{a(a-1)+2a\beta^{-1}}{2(a^2+4)^{1/2}}.$$

Similarly, we deduce from (4.1) and (4.2)

$$\mathsf{E}(b_i^2) = \frac{v_{n-1}}{v_n}\left(\frac{2a^3-3a^2+a}{6}v_i+a^2v_{i-1}\right),$$

and eventually, in terms of β,

$$\mathsf{E}(b_i^2) = B+O(\beta^{-2i})+O(\beta^{2i-2n}), \tag{4.5}$$

where

$$B = \frac{2a^3-3a^2+a+6a^2\beta^{-1}}{6(a^2+4)^{1/2}}.$$

In view of (3.4), we also need $\mathsf{E}(b_h b_i)$. Assuming $h<i$, and argueing as before, we find

$$\mathsf{P}(b_h = k,\ b_i = l) = v_h v_{i-h} v_{n-i}/v_n \qquad 0<k<a;\ \ 0<l<a, \tag{4.6}$$

and similarly

$$\mathsf{P}(b_h = a,\ b_i = l) = v_{h-1} v_{i-h} v_{n-i}/v_n \qquad (0<l<a) \tag{4.7}$$

$$\mathsf{P}(b_h = k,\ b_i = a) = v_h v_{i-h-1} v_{n-i}/v_n \qquad (0<k<a) \tag{4.8}$$

$$\mathsf{P}(b_h = b_i = a) = v_{h-1} v_{i-h-1} v_{n-i}/v_n. \tag{4.9}$$

Consequently,

$$\mathsf{E}(b_h b_i) = \frac{a^2(a-1)^2 v_h v_{i-h} v_{n-i}}{4v_n}+ \\ +\frac{a^2(a-1)(v_h v_{i-h-1}+v_{h-1}v_{i-h})v_{n-i}}{2v_n}+\frac{a^2 v_{h-1}v_{i-h-1}v_{n-i}}{v_n} \tag{4.10}$$

and eventually, in terms of β,

$$\mathsf{E}(b_h b_i) = A^2+(-1)^{i-h+1}C\beta^{2h-2i}+O(\beta^{-2h})+O(\beta^{2i-2n}), \tag{4.11}$$

where

$$C = \frac{a^2((a-1)^2-2(a-1)(\beta-\beta^{-1})-4)}{4a^2+4}.$$

Now we can prove the following proposition:

LEMMA 4.1. *With the previous notations,* $\mathsf{E}(S_q)$ *has an upper bound which depends only on a.*

PROOF. According to (3.3) and (3.4)

$$S_q = \sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)}+\sigma^{(4)},$$

where

$$\sigma^{(1)} = \sum_{i=1}^{n-1} \frac{(-1)^i}{2} b_i; \qquad \sigma^{(2)} = \sum_{i=1}^{n-1} \frac{\Theta_i v_i b_i^2}{2v_{i+1}};$$

$$\sigma^{(3)} = \sum_{i=1}^{n-1} \frac{\Theta_i}{v_{i+1}} \sum_{h=1}^{i-1} v_h b_h b_i; \quad \sigma^{(4)} = \sum_{i=1}^{n-1} \frac{\Theta_i b_i}{2v_{i+1}}.$$

Now, according to (4.4),

$$\mathsf{E}(\sigma^{(1)}) = A \sum_{i=1}^{n-1} \frac{(-1)^i}{2} + \sum_{i=1}^{n-1} O(\beta^{-2i}) + \sum_{i=1}^{n-1} O(\beta^{2i-2n}),$$

which is obviously bounded.

Concerning $\sigma^{(2)}$, we observe that

$$\frac{v_i}{v_{i+1}} = \frac{\beta^i+(-1)^{i+1}\beta^{-i}}{\beta^{i+1}+(-1)^i\beta^{-i-1}} = \beta^{-1}+O(\beta^{-2i}).$$

Consequently, since $b_i^2 \leqq a^2$ and $|\Theta_i| \leqq 1$, we find

$$\mathsf{E}(\sigma^{(2)}) = \frac{1}{2\beta} \sum_{i=1}^{n-1} \Theta_i \mathsf{E}(b_i^2) + \sum_{i=1}^{n-1} O(\beta^{-2i}),$$

and further, by (4.5),

$$\mathsf{E}(\sigma^{(2)}) = \frac{B}{2\beta} \sum_{i=1}^{n-1} \Theta_i + \sum_{i=1}^{n-1} O(\beta^{-2i}) + \sum_{i=1}^{n-1} O(\beta^{2i-2n}).$$

Here, the last two terms are immediately seen to be bounded, and so is the first by Lemma 3.6. Thus $\mathsf{E}(\sigma^{(2)})$ is bounded.

Passing to $\sigma^{(3)}$, we observe that, with $h<i$,

$$\frac{v_h}{v_{i+1}} = \frac{\beta^h+(-1)^{h+1}\beta^{-h}}{\beta^{i+1}+(-1)^i\beta^{-i-1}} = \beta^{h-i-1}+O(\beta^{-h-i}).$$

Consequently

$$\mathsf{E}(\sigma^{(3)}) = \sum_{i=1}^{n-1} \Theta_i \sum_{h=1}^{i-1} \beta^{h-i-1} \mathsf{E}(b_h b_i) + \sum_{i=1}^{n-1} \Theta_i \sum_{h=1}^{i-1} \mathsf{E}(b_h b_i) O(\beta^{-h-i}).$$

Here, Θ_i and $\mathsf{E}(b_h b_i)$ being bounded, the second term is easily seen to be bounded. Substituting (4.11) in the first term, we find

$$A^2 \sum_{i=1}^{n-1} \Theta_i \sum_{h=1}^{i-1} \beta^{h-i-1} + C \sum_{i=1}^{n-1} \Theta_i \sum_{h=1}^{i-1} (-1)^{i-h+1} \beta^{3h-3i-1} +$$

$$+ \sum_{i=1}^{n-1} \Theta_i \sum_{h=1}^{i-1} O(\beta^{-h-i-1}) + \sum_{i=1}^{n-1} \Theta_i \sum_{n=1}^{i-1} O(\beta^{h+i-2n}).$$

The first term of this expression is easily seen to be bounded in view of Lemma 3.6. A similar argument shows that the second term is also bounded. Since Θ_i is bounded, there is no difficulty over the boundedness of the last two terms. The boundedness of $\mathsf{E}(\sigma^{(4)})$ is trivial, since $\sigma^{(4)}$ itself is bounded.

§ 5. The variance of S_q

Throughout this section we assume that all the partial quotients in the continued fraction expansion of α are equal to an integer a. Some variances and covariances have to be computed before attacking $\operatorname{var} S_q$. There would be no difficulty in writing down an exact expression for $\operatorname{var} b_i$ on the basis of (4.4) and (4.5). However, it suffices for our purpose to note that $\operatorname{var} b_i$ is obviously bounded, say

$$\operatorname{var} b_i \leqq V \qquad (i = 1, \ldots, n-1). \tag{5.1}$$

Similarly, it suffices to know that for some W

$$\operatorname{var} b_i^2 \leqq W \qquad (i = 1, \ldots, n-1). \tag{5.2}$$

We need to know more about $\operatorname{cov}(b_h, b_i) = \mathsf{E}(b_h b_i) - \mathsf{E}(b_h)\mathsf{E}(b_i)$. We can rewrite (4.10) in the following form:

$$\mathsf{E}(b_h b_i) = a^2\left(v_h\frac{a-1}{2}+v_{h-1}\right)\left(v_{i-h}\frac{a-1}{2}+v_{i-h-1}\right)v_{n-i}/v_n. \tag{5.3}$$

In view of (4.3), we have therefore

$$\operatorname{cov}(b_h, b_i) = a^2\frac{\left(v_h\dfrac{a-1}{2}+v_{h-1}\right)v_{n-i}}{v_n}\times$$
$$\times\frac{\left(v_{i-h}\dfrac{a-1}{2}+v_{i-h-1}\right)v_n - v_{n-h}\left(v_i\dfrac{a-1}{2}+v_{i-1}\right)}{v_n}. \tag{5.4}$$

It is easily seen that, here, the first fraction is $O(\beta^{h-i})$. In the numerator of the second fraction, if we express it in term of β, we find a linear combination of β^{h-i+n}, $\beta^{h-i+n+1}$, β^{i-h-n}, β^{n-i}, $\beta^{i-h-n-1}$, $\beta^{h-i-n+1}$, β^{n-i-h}, $\beta^{n-i-h+1}$, β^{i+h-n}, β^{h-i-n}, $\beta^{h+i-n-1}$, and $\beta^{h-i-n+1}$. Taking into account the denominator v_n, which is exactly of the order of β^n, it can be seen that

$$\operatorname{cov}(b_h, b_i) = O(\beta^{2h-2i}) \quad \text{when} \quad h < i.$$

In fact

$$|\operatorname{cov}(b_h, b_i)| \leqq C\beta^{2h-2i} \quad \text{when} \quad h < i, \tag{5.5}$$

where C has the same value as in (4.11), but the precise value of this coefficient is irrelevant from our viewpoint.

In an exactly similar way, we evaluate $\operatorname{cov}(b_h^2, b_i^2)$, obtaining, for a D depending only on a,

(5.6) $$|\operatorname{cov}(b_h^2, b_i^2)| \leqq D\beta^{2h-2i} \quad \text{when} \quad h < i.$$

Now we proceed to compute $\mathsf{E}(b_h b_i b_k b_j)$ for $0<h<i\leqq k<j\leqq n$. We have for K, L, R and S in $(0, a)$

$$\mathsf{P}(b_h = K,\ b_i = L,\ b_k = R,\ b_j = S) = v_h v_{i-h} v_{k-i} v_{j-k} v_{n-j}/v_n,$$

$$\mathsf{P}(b_h = K,\ b_i = L,\ b_k = R,\ b_j = a) = v_h v_{i-h} v_{k-i} v_{j-k-1} v_{n-j}/v_n,$$

and soon. Eventually we find

$$\mathsf{E}(b_h b_i b_k b_j) = a^4\left(v_h\frac{a-1}{2}+v_{h-1}\right)\left(v_{i-h}\frac{a-1}{2}+v_{i-h-1}\right)\cdot$$

$$\cdot\left(v_{k-i}\frac{a-1}{2}+v_{k-i-1}\right)\left(v_{j-k}\frac{a-1}{2}+v_{j-k-1}\right)v_{n-j}/v_n,$$

and, in view of (5.3),

$$\operatorname{cov}(b_h b_i, b_k b_j) =$$

$$= a^3\left(v_h\frac{a-1}{2}+v_{h-1}\right)\left(v_{i-h}\frac{a-1}{2}+v_{i-h-1}\right)\left(v_{j-k}\frac{a-1}{2}+v_{j-k-1}\right)v_{n-j}v_n^{-1}\times$$

$$\times a\left(\left(v_{k-i}\frac{a-1}{2}+v_{k-i-1}\right)v_n - v_{n-i}\left(v_k\frac{a-1}{2}+v_{k-1}\right)\right)v_n^{-1}.$$

The first line above is easily seen to be $O(\beta^{i-k})$. In the second line, if we express the v's in terms of β, we find, after crucial simplifications, a linear combination of β^{i-k}, β^{k-i-2n}, β^{-i-k}, β^{k+i-2n} and β^{i-k-2n}. Under our assumption, $i-k$ is the biggest exponent of β; hence the second line is also $O(\beta^{i-k})$. Thus there exists a constant M depending only on a such that

(5.7) $$\operatorname{cov}(b_h b_i, b_k b_j) \leqq M\beta^{2(i-k)} \quad \text{when} \quad 0 < h < i \leqq k < j < n.$$

Obviously, there exists also a number N such that

(5.8) $$\operatorname{cov}(b_h b_i, b_k b_j) \leqq N \quad \text{for} \quad h, i, k, j \quad \text{between } 0 \text{ and } n.$$

LEMMA 5.1. *With our previous notations*

$$\operatorname{var} S_q = O(n) = O(\log m),$$

where $m=v_n$.

PROOF. Owing to Schwarz inequality, we only need to show that

$$\operatorname{var}\sigma^{(k)} = O(n) \qquad (k = 1, 2, 3, 4).$$

According to (5.1) and (5.5),

$$\operatorname{var}\sigma^{(1)} \leqq \frac{n-1}{4}V+\frac{C}{2}\sum_{i=2}^{n-1}\sum_{h=1}^{i-1}\beta^{2h-2i} < \frac{n-1}{4}V+\frac{C(n-2)}{2(\beta^2-1)} = O(n).$$

Since $\beta v_i/(2v_{i+1})$ is bounded in view of (5.6), $\sigma^{(2)}$ can be treated exactly like $\sigma^{(1)}$, yielding

$$\operatorname{var}\sigma^{(2)} = O(n).$$

Now

$$\sigma^{(3)} = \sum_{i=1}^{n-1} Q_i$$

where

$$Q_i = \Theta_i \sum_{h=1}^{i-1} v_h v_{i+1}^{-1} b_h b_i.$$

Assuming $i \leqq j$, we have, according to (5.7) and (5.8),

$$|\operatorname{cov}(Q_i, Q_j)| \leqq \frac{1}{v_{i+1}v_{j+1}}\sum_{h=1}^{i-1}\sum_{k=1}^{j-1} v_h v_k |\operatorname{cov}(b_h b_i, b_k b_j)| \leqq$$

$$\leqq \frac{M}{v_{i+1}v_{j+1}}\sum_{h=1}^{i-1} v_h \sum_{k=1}^{j-1} v_k \beta^{2i-2k} + \frac{N}{v_{i+1}v_{j+1}}\sum_{h=1}^{i-1} v_h \sum_{k=1}^{i-1} v_k,$$

it being understood that when $i=j$, 0 should be substituted for the first term of the last expression. When $i<j$, this term is of order of

$$\beta^{-i-j}\sum_{h=1}^{i-1}\beta^h \sum_{k=i}^{j-1}\beta^{2i-k} = \beta^{i-j}\sum_{h=1}^{i-1}\beta^h \sum_{k=1}^{j-1}\beta^{-k} = O(\beta^{i-j}).$$

The second term is of the order of

$$\beta^{-i-j}\sum_{h=1}^{i-1}\beta^h \sum_{k=1}^{i-1}\beta^k = O(\beta^{i-j}).$$

Thus there exists a constant R, depending only on a, such that

$$\operatorname{cov}(Q_i, Q_j) \leqq R\beta^{i-j} \quad \text{when} \quad i \leqq j.$$

Eventually,

$$\operatorname{var}\sigma^{(3)} = \sum_{i,j=1}^{n-1}\operatorname{cov}(Q_i, Q_j) \leqq (n-1)R + 2R\sum_{j=2}^{n-1}\sum_{i=1}^{j-1}\beta^{i-j} = O(n).$$

Finally, $\sigma^{(4)}$ being obviously bounded, so is $\operatorname{var}\sigma^{(4)}$.

THEOREM 5.2. *Let all the partial quotients in the continued-fraction expansion of α be equal to a positive integer a. Then there exists a constant Λ depending only on a, and such that*

$$\frac{1}{m}\sum_{q=0}^{m-1} S_q^2 \leqq \Lambda \log m$$

for any positive integer $m>1$ and not exceeding its denominator if α is rational.

PROOF. As an immediate consequence of Lemmas 4.1 and 5.1, there exists a constant λ depending only on a and satisfying

$$\frac{1}{v_n}\sum_{q=0}^{v_n-1} S_q^2 \leqq \lambda \log v_n$$

for all n if α is irrational and for all $n \leqq n'$ if $\alpha = v_{n'-1}/v_{n'}$. If $v_{n-1} < m < v_n$, then

$$\frac{1}{m}\sum_{q=0}^{m-1} S_q^2 < \frac{1}{v_{n-1}}\sum_{q=0}^{v_n-1} S_q^2 < \frac{a+1}{v_n}\sum_{q=0}^{v_n-1} S_q^2 \leqq (a+1)\lambda \log v_n,$$

and if we put, for instance

$$\Lambda = \lambda(a+1)\log\big(a(a+1)\big)(\log a)^{-1},$$

we have (5.9), at least when $m \geqq v_2 = a$.

If necessary, an adjustment of the value of Λ will take care of the case $1 < m < v_2$.

As a corollary to Lemma 2.1 and Theorem 5.2 we have the following proposition:

THEOREM 5.3. *If all the partial quotients in the continued-fraction expansion of α are equal to a positive integer a and if $\mathbf{Z}$ is the sequence of points*

$$\langle 0,0\rangle, \left\langle \frac{1}{m}, \{\alpha\}\right\rangle, \left\langle \frac{2}{m}, \{2\alpha\}\right\rangle, \dots, \left\langle \frac{m-1}{m}, \{(m-1)\alpha\}\right\rangle,$$

m being an arbitrary positive integer if α is irrational and not exceeding its denominator if α is rational, then the mean-square discrepancy $D^{(2)}(\mathbf{Z})$ of $\mathbf{Z}$ satisfies

$$D^{(2)}(\mathbf{Z}) = O\big(m^{-1}(\log m)^{1/2}\big),$$

where the constant implied in the right-hand side depends only on a.

It may be worth returning for a moment to the behaviour of S_q.

LEMMA 5.4. *Under the conditions of Theorem 5.2, to any $\varepsilon>0$ there corresponds a number c depending only on a and ε, and such that*

$$S_q < c\sqrt{\log m}$$

holds for all but at most εm values of $q \in [2, m)$, $m \geqq 2$ being arbitrary.

PROOF. We return to the probabilistic interpretation of our problem. According to the Chebyshev inequality, for any positive K, $\mathsf{P}[S_q^2 \geqq K] \leqq K^{-2}\mathsf{E}(S_q^2)$. In view of (5.9), putting $K=\sqrt{\Lambda \log m/\varepsilon}$, we find

$$\mathsf{P}\left[|S_q| \geqq \sqrt{\varepsilon^{-1}\Lambda \log m}\right] \leqq \varepsilon,$$

and the Lemma holds with $c=\sqrt{\Lambda/\varepsilon}$.

THEOREM 5.5. *Under the conditions of Theorem* 5.2, *to any* $\varepsilon>0$ *there corresponds a number C depending only on a and* ε, *and such that*

$$S_q < C\sqrt{\log q}$$

holds for all but at most εm *values of q in the interval* $[2, m)$, $m \geqq 2$ *being still an arbitrary integer.*

PROOF. We divide $[2, m)$ into the intervals $[2, 2^2)$, $[2^2, 2^3)$, ..., $[2^{r-1}, 2^r)$, and $[2^r, m)$, where $2^r < m \leqq 2^{r+1}$. According to the preceding Lemma, to any $\varepsilon'>0$ there corresponds a number c' depending only on a and ε', and such that $S_q \geqq c'\sqrt{\log 2^\nu}$ holds in the interval $[2^{\nu-1}, 2^\nu)$ for not more than $2^\nu\varepsilon'$ values of q. But for these values of q, $\log 2^\nu \leqq 2\log q$, and so we have

$$S_q \geqq C'\sqrt{2\log q} \tag{5.10}$$

for at most $2^\nu\varepsilon'$ values of q in $[2^{\nu-1}, 2^\nu)$. Similarly, (5.10) holds for at most $m\varepsilon'$ values of q in $[2^r, m)$. Thus, in all, the number of values of q in $[2, m)$ for which (5.10) holds is at most $(2^2+2^3+\ldots+2^r+m)\varepsilon' < 3m\varepsilon'$, and if we put $\varepsilon'=\varepsilon/3$ and $C=c'\sqrt{2}$, we obtain the conclusion of the theorem.

REMARK. The above Theorem might be surprising knowing the following:

(∗) *For* S_N *we have the same best possible* Ω*-estimation, as for* D_N:

$$S_N = \Omega(\log N)$$

and

$$D_N = \Omega(\log N).$$

(∗∗) *For* D_N *a much stronger result is true: for an arbitrary* α

$$D > c\log N$$

holds for all but at most N^ε *values of* q; $1 \leqq q \leqq N$ *where* $\varepsilon \to 0$ *with* $c \to 0$. (See V. T. Sós [13].)

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