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ON UNAVOIDABLE SUBGRAPHS OF TOURNAMENTS

M. SAKS — V. SÓS

ABSTRACT

A directed graph is said to be *n-unavoidable* if it is contained as a subgraph of every tournament on n vertices. A number of theorems have been proven showing that certain graphs are n-unavoidable, the first being Rédei's result that every tournament has a Hamiltonian path. In this paper, recent results in this area are summarized and some new problems are considered. Some classes of rooted directed trees that are or are not unavoidable are identified. In particular we consider the class of *claws,* rooted digraphs in which each branch is a path. We also produce, for each n, a spanning rooted digraph of small depth that is n-unavoidable. Some additional constructions are presented.

1. INTRODUCTION

Perhaps the first theorem about tournaments is the fact proved by Rédei [9] that every tournament has a Hamiltonian path. Erdős and Moon [2] showed that every tournament on n vertices contains a transitive tournament of $\lceil \log_2 n \rceil + 1$ vertices. These two results give the first

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known examples of *unavoidable subgraphs* of tournaments. A directed graph G is said to be *n-unavoidable* if every n-tournament contains it as a subgraph. It is natural to ask what other digraphs are n-unavoidable. In this note we will review some known results about n-unavoidable graphs and present some new examples and techniques for showing that graphs are n-unavoidable.

We denote by $U(n)$ the set of all n-unavoidable digraphs.

While a complete characterization of $U(n)$ appears impossible, it is possible to identify classes of graphs which are in $U(n)$. We have largely restricted ourselves to the problem of identifying spanning rooted directed trees which are n-unavoidable.

In Section 2 we consider the class of rooted trees in which each branch is a path (called claws).

In Section 3 we consider the problem of how small can the depth of a rooted tree in $U(n)$ be and give a construction to show that $U(n)$ contains rooted trees of depth 3.

In Section 4 we give some simple miscellaneous constructions for producing unavoidable trees.

We use the following notation. Let $G = (V, E)$ be a digraph. If there is an edge from v to w we say that v *points to* w, w is a successor of v, v is a predecessor of w, or $v \to w$. The edge is denoted by $\langle v, w \rangle$. $G^+(v)$ and $G^-(v)$ denote, respectively, the sets of vertices which v points to and which point to v.

Let G and H be digraphs and $v \in V(G)$, $w \in V(H)$. The concatenation of G and H at v and w is the graph obtained by taking disjoint copies of G and H and identifying the vertices v and w.

A directed tree is a *rooted tree* if there is one vertex of in degree 0 (the root) and every other vertex has in degree 1. The depth of a rooted tree is the maximum length of a path from the root.

A *rooted star* of size k, is a tree consisting of one vertex pointing to $k - 1$ other vertices.

If W and X are disjoint subsets of V we write $B[W \leftarrow X]$ to denote the bipartite subgraph of G with vertex set $W \cup X$ and edge set consisting of all edges of G pointing from X to W.

We will often use the König matching theorem, which we restate in a form that is more convenient for our purposes.

Lemma 1.1 [5]. *Let T be a tournament and let W and Z be disjoint sets of vertices. Let $m = \max\{|X| + |Y|: X \subseteq W, \ Y \subseteq Z$ and $B[X \rightarrow Y]$ is a complete bipartite graph$\}$. Then $B[W \leftarrow Z]$ has a matching of size $|W| + |Z| - m$.*

Linial, Saks and Sós [6] estimated the maximum number of edges in a graph in $U(n)$. Let $f(n)$ (resp. $g(n)$) be the largest m such that $U(n)$ contains a digraph (resp. spanning weakly connected digraph) with m edges. They proved:

Theorem 1.2 [6]. *There exist positive constants c_1 and c_2 such that for all positive integers n,*

$$n \log_2 n - c_1 n \geqslant f(n) \geqslant g(n) \geqslant n \log_2 n - c_2 n \operatorname{loglog} n.$$

2. UNAVOIDABLE CLAWS

Let $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a sequence of nonnegative integers. A *claw* $C(\underline{\lambda})$ is a rooted directed tree obtained by concatenating the roots of dipaths of sizes $\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1$. $C(\underline{\lambda})$ has $1 + \sum_{i=1}^{k} \lambda_i$ vertices. We consider the question of which claws on n vertices are in U_n.

For our purposes the order of the numbers λ_i is irrelevant so we will assume that $\lambda_i \geqslant \lambda_{i+1}$ for all i. The sequence $\underline{\lambda}$ is a *partition* of the integer $m = \sum_{i=1}^{k} \lambda_i$. Values of λ_i which are 0 are irrelevant. It will sometimes be convenient to think of $\underline{\lambda}$ as an infinite sequence by appending a string of zeros to it.

Let $\underline{\lambda}$ and $\underline{\mu}$ be partitions with the same sum. We say that $\underline{\lambda}$ dominates $\underline{\mu}$, written $\underline{\lambda} \succ \underline{\mu}$ if $\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i$ for each $j \geqslant 1$.

Lemma 2.1. *Suppose* $\underline{\lambda}$ *and* $\underline{\mu}$ *have the same sum and* $\underline{\lambda} \succ \underline{\mu}$. *If* T *is a tournament and* $C(\underline{\mu})$ *is a subgraph of* T *then so is* $C(\underline{\lambda})$.

Proof. By induction on $d = \sum |\lambda_i - \mu_i|$. If $d = 0$, the result is trivial. So suppose $d > 1$. Let j be the first index such that $\lambda_j \neq \mu_j$; since $\underline{\lambda} \succ \underline{\mu}$ it must be that $\lambda_j > \mu_j$. There must be some last index $k > j$ such that $\mu_k > \lambda_k$. Let $\underline{\nu}$ be the partition given by $\nu_j = \mu_j + 1$, $\nu_k = \mu_k - 1$ and $\nu_i = \mu_i$ for $i \neq j$. If we show that $C(\underline{\nu})$ is a subgraph of T then since $\underline{\lambda} \succ \underline{\nu}$ and $\sum |\lambda_i - \nu_i| < \sum |\lambda_i - \mu_i|$ we are done by induction.

Let C be the copy of $C(\underline{\mu})$ in T and let x be the last vertex in the k-th path and let y be its predecessor. Label the vertices in the j-th path (from the leaf) by $v_1, v_2, \ldots, v_{\mu_j+1}$. Let $q = \mu_j + 1 - \mu_k$. Since $\mu_j \geqslant \mu_k \geqslant 1$, q is between 1 and μ_j.

If $x \to v_q$ in 1 then deleting $\langle v_{q+1}, v_q \rangle$ and adding $\langle x, v_q \rangle$ in C yields a copy of $C(\underline{\nu})$. Otherwise let r be the smallest index for which $v_r \to x$ and delete the edges $\langle y, x \rangle$ and $\langle v_r, v_{r-1} \rangle$ (if $r \neq 1$) and add the edges $\langle v_r, x \rangle$ and $\langle x, v_{r-1} \rangle$ (if $r \neq 1$). This gives a copy of $C(\underline{\nu})$. ∎

Thus to determine which claws are n-unavoidable it suffices to characterize the partitions $\underline{\lambda}$ which are minimal with respect to the domination ordering such that $C(\underline{\lambda})$ is in $U(n)$.

If $\underline{\lambda}$ is a partition of the integer m, the conjugate partition of $\underline{\lambda}$, written $\underline{\lambda}^*$, is the partition of m such that λ_i^* equals the number of vertices in $C(\underline{\lambda})$ which are at distance i from the root. It is well known that $\underline{\lambda} \succ \underline{\mu}$ if and only if $\underline{\mu}^* \succ \underline{\lambda}^*$ and that $\underline{\lambda}^{**} = \underline{\lambda}$.

The following is an obvious necessary condition for $C(\underline{\lambda})$ to be n-unavoidable.

Proposition 2.2. *Let* $\underline{\lambda}$ *be a partition of* $n - 1$ *such that* $C(\underline{\lambda})$ *is in* $U(n)$. *Then the partition* $\left(\left\lceil \frac{n-1}{2} \right\rceil, \left\lfloor \frac{n-1}{2} \right\rfloor\right)$ *dominates* $\underline{\lambda}^*$.

Proof. In a regular n-tournament the maximum out degree of any vertex is $\left\lceil \frac{n-1}{2} \right\rceil$. Thus if $C(\underline{\lambda})$ is n-unavoidable $\left\lceil \frac{n-1}{2} \right\rceil \geqslant \lambda^*$, which suffices to show that $\left(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor\right)$ dominates $\underline{\lambda}^*$. ∎

We are led to the following:

Conjecture 2.3. *For any* n, *the claw* $C(\underline{\mu})$ *is n-unavoidable where* $\underline{\mu} = \left(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor\right)^*$.

If this conjecture is true then by Lemma 2.1 and Proposition 2.2 we would have that $C(\underline{\lambda})$ is in $U(n)$ if and only if $\left(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor\right) \succ \underline{\lambda}^*$, completely characterizing unavoidable claws. We have proved the conjecture up to $n = 9$ but the general problem remains open. The best general result we have is:

Theorem 2.4. *For a given* $n \geqslant 2$, *let* $\underline{\mu}(n)$ *be a partition of* $n - 1$ *defined as follows:*

$$\mu_1 = \left\lceil \frac{n-1}{2} \right\rceil \quad \textit{and} \quad \mu_k = \left\lfloor \frac{n - \sum_{i=1}^{k-1} \mu_i}{2} \right\rfloor \quad \textit{for } k > 1.$$

Then $C(\underline{\mu}^*)$ *is in* $U(n)$.

Before proving the theorem, it is useful to give a better picture of the claw obtained in this way, by considering the case when $n = 2^q$. Then $\mu_i(n) = 2^{q-i}$ for $1 \leqslant i \leqslant q$ and $\mu_i^*(n) = q - \lceil \log_2 i \rceil$.

Proof of Theorem 2.4. Let T be an n-tournament. We will construct a sequence of claws in T, $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_k$ where the final claw spans the vertices of T. The maximum depth of a vertex in C_i is i and $D_i = C_i - C_{i-1}$ consists only of vertices at depth i.

We proceed as follows. The claw C_0 is a single vertex v of maximum out degree in T and C_1 is a rooted star consisting of the vertex v and all of $T^+(v)$. Given C_{i-1}, let L_{i-1} denote its leaf set, and let W_{i-1} be the vertices in $T - C_{i-1}$. Find the largest matching in $B[W_{i-1} \leftarrow L_{i-1}]$. C_i then consists of C_{i-1} together with the vertices of W_{i-1} and edges contained in this matching.

Observe that, by the construction the only vertices in L_{i-1} which are matched to vertices in W_{i-1} are vertices in L_{i-1} which were not in L_{i-2} (otherwise the matching used to construct C_{i-1} was not maximum). The vertices in $L_{i-1} - L_{i-2}$ are precisely the vertices of D_{i-1}. Therefore, by induction, if all vertices in D_{i-1} are at depth $i-1$ then all vertices in D_i are at depth i.

Lemma 2.5. $|D_1| \geqslant \lceil \frac{n-1}{2} \rceil$ *and for* $i > 1$,

$$|D_i| \geqslant \lfloor \frac{n+1-|C_{i-1}|}{2} \rfloor .$$

This lemma implies (i) the construction eventually produces a spanning tree of T, and (ii) the partition $\underline{\lambda} = (|D_1|, |D_2|, \ldots)$ dominates the partition $\underline{\mu}(n)$. Since $C(\underline{\lambda}^*)$ is constructed as a subgraph of T, Lemma 3.1 implies that $C(\underline{\mu}^*)$ is also a subgraph of T, proving the theorem.

Proof of Lemma 2.5. Since v is chosen to be of maximum out degree in T, $|D_1| = |T^+(v)| \geqslant \lceil \frac{n-1}{2} \rceil$. Now, $|D_i|$ is equal to the size of the largest matching in $B[W_{i-1} \leftarrow L_{i-1}]$. We apply Lemma 1.1 to obtain the desired bound on D_i. Let $X \subseteq W_{i-1}$ and $Y \subseteq L_{i-1}$ be chosen so that $|X| + |Y|$ is maximum and $B[X \rightarrow Y]$ is a complete bipartite graph. We want to bound the average out degree of vertices in X. Each vertex in X point to v and to every vertex in Y, and the average out degree in the tournament spanned by $|X|$ is $\frac{|X|-1}{2}$. Thus the average out degree in T of a vertex in X is at least $|Y| + 1 + \frac{|X|-1}{2}$. Now this is at most $|T^+(v)| = |L_{i-1}|$, so $|Y| + 1 + \frac{|X|-1}{2} \leqslant |L_{i-1}|$. By Lemma 1.1, $B[W_{i-1} \leftarrow L_{i-1}]$ has a matching of size $|L_{i-1}| + |W_{i-1}| - |X| - |Y|$ which is at least

$$\lceil |W_{i-1}| + \tfrac{1}{2} - \tfrac{|X|}{2} \rceil \geqslant \lceil \frac{|W_{i-1}|+1}{2} \rceil = \lceil \frac{n+1-|C_{i-1}|}{2} \rceil . \blacksquare$$

This completes the proof of Theorem 2.4. ∎

3. SPANNING UNAVOIDABLE TREES OF SMALL DEPTH

In the last section we produced an n-unavoidable claw on n vertices of depth $\log_2 n$. It is natural to ask how small the depth of a spanning n-unavoidable tree can be. If Conjecture 2.3 is true, this would give a tree of depth 2 for all n, which is clearly best possible. In this section we produce, for each n, an n-unavoidable tree of depth 3.

Let n and $s_1, s_2, \dots, s_q$ be positive integers such that $n \geqslant 1 + q + \sum_{i=1}^{q} s_i$. Define the rooted tree $H(n; s_1, s_2, \dots, s_q)$ as follows. The root has $n - 1 - \sum_{i=1}^{q} s_i$ successors. For $1 \leqslant i \leqslant q$, the i-th successor of the root has exactly one successor; this successor is the root of a star of size s_i. The remaining successors of the root are leaves. Observe that $H(n; s_1, \dots, s_q)$ has n vertices and depth at most 3, and that H does not depend on the order of the s_i. The main result of this section is:

Theorem 3.1. *Let k and m be the unique integers such that*

$$\left\lfloor \frac{n-1}{2} \right\rfloor = \frac{k(k+1)}{2} + m \quad \text{and} \quad 0 < m \leqslant k - 1.$$

Then $H(n; 1, 2, 3, \dots, k, m)$ is in U_n.

The proof of the theorem makes use of the following lemmas.

Lemma 3.2. *Let T be a tournament on n vertices and let $\{s_1, \dots, s_{q+1}\}$ be a multiset of positive integers such that $n \geqslant 1 + q + \sum_{i=1}^{q} s_i + 2s_{q+1}$. Then if $H(n; s_1, \dots, s_q)$ is a subgraph of T, so is $H(n; s_1, \dots, s_q, s_{q+1})$.*

Proof. Let v be the root of the copy of $H = H(n; s_1, \dots, s_q)$ in T and let L be the set of successors of v that are leaves. Then $|L| = n - 1 - q - \sum_{i=1}^{q} s_i$, which is at least $2s_{q+1}$ by hypothesis. We now claim that the subtournament of T induced on L contains a copy of a star on $s_{q+1} + 1$ vertices in which exactly one leaf y points to the center of the star. This is obtained by choosing a vertex of maximum out degree having in degree at least one. Since T has $2s_{q+1}$ vertices this vertex has out degree at least $s_{q+1} - 1$. Transform H by deleting the edges from v to the vertices of this star, except $\langle v, y \rangle$, and adding the edges of the star. This produces a copy of $H(n; s_1, \ldots, s_{q+1})$ in T. ∎

Lemma 3.3. *Let T be a tournament on n vertices and let $s_1, \ldots, s_k$ be a sequence of positive integers such that $\sum_{i=1}^{k} s_i \leqslant \frac{n-1}{2}$ and let $s_1, \ldots, s_q$ be a subsequence. If $H(n; s_1, \ldots, s_q)$ is in T then so is $H(n; s_1, \ldots, s_k)$.*

Proof. $(s_1, \ldots, s_{q+1})$ satisfies the conditions of the previous lemma. Applying the lemma and induction on q yields the desired result. ∎

Proof of Theorem 3.1. Let T be an n-tournament and let v be a vertex of maximum out degree. We will first partition the set $T^-(v)$ into rooted stars $\{S_1, S_2, \ldots, S_q\}$ whose sizes $\{s_1, \ldots, s_q\}$ comprise a sub-multiset of $(1, 2, 3, \ldots, k, m)$. To do this we construct the numbers $(s_1, \ldots, s_q)$ as follows:

If $|T^-(v)| = \lfloor \frac{n-1}{2} \rfloor$ then

$$(s_1, \ldots, s_q) = (1, 2, 3, \ldots, k, m).$$

If $|T^-(v)| = \lfloor \frac{n-1}{2} \rfloor - i$, where $i \leqslant k$ then

$$(s_1, \ldots, s_q) = (1, 2, \ldots, i-1, i+1, \ldots, k, m).$$

If $|T^-(v)| = \frac{h(h+1)}{2} - i$, where $0 \leqslant i < h \leqslant k$ then

$$(s_1, \ldots, s_q) = (1, 2, \ldots, i-1, i+1, \ldots, h),$$

By a simple inductive argument it is easy to see, that $T^-(v)$ can be partitioned as required.

Let $W = \{w_1, \ldots, w_q\}$ be the roots of $S_1, \ldots, S_q$. We claim that each w_i can be assigned a unique vertex u_i in $T^+(v)$ so that $u_i \to w_i$, i.e. the bipartite graph $B = B[W \leftarrow T^+(v)]$ has a matching of size $|W|$. Let $X \subseteq W$ and $Y \subseteq T^+(v)$ maximize $|X| + |Y|$ subject to the condition that $B[X \to Y]$ is a complete bipartite graph. By Lemma 1.1, it suffices to show that $|T^+(v)| \geq |X| + |Y|$. If X is empty the result is trivial. So suppose $X \neq \phi$ and sum the out degrees of the vertices in X. Each such vertex points to v and to all vertices in Y, and X spans a total of $\binom{|X|}{2}$ arcs. Also each vertex in X is the root of a star. Since the j-th smallest star among $S_1, S_2, \ldots, S_q$ has at least $j-2$ leaves, this adds $\sum_{2}^{|X|} j - 2 = \binom{|X|-1}{2}$ to the out degree sum of X. The total sum of the out degrees of X is at least $|X|(1 + |Y|) + \binom{|X|}{2} + \binom{|X|-1}{2}$, so the average out degree is at least $|Y| + |X| - 1 + \frac{1}{|X|}$. Since v has maximum out degree in T, this quantity is at most $|T^+(v)|$ so $|T^+(v)| \geq |Y| + |X|$ as required for the existence of a matching of W in B. This shows that the graph $H(n; s_1, \ldots, s_q)$ is a subgraph of T. Lemma 3.3 therefore implies that $H(n; 1, 2, \ldots, k, m)$ is a subgraph of T. ∎

4. MISCELLANEOUS CONSTRUCTIONS

In trying to identify classes of graphs which belong or do not belong to $U(n)$, it is useful to find some general constructions and transformations for producing new unavoidable graphs from old ones. In this section we describe a few such constructions. While quite simple, they have some interesting applications.

Proposition 4.1. *Let $G \in U(n)$ and $H \in U(k)$ and let x be any vertex of G. For each vertex v of H, let G^v be a copy of G, with x^v denoting the copy of x in G^v. Let C denote the graph obtained by simultaneously concatenating, for each $v \in V(H)$, G^v and H at x^v and v. Then $C \in U(nk)$.*

Proof. Let T be any tournament on nk vertices. Partition the vertex set arbitrarily into k sets of size n. Each of the tournaments induced on a single set of n vertices contains a copy of G. Let X be the set of the k copies of x appearing in the k copies of G. The tournament induced on X contains a copy of H. Combining this with the copies of G yields a copy of C. ∎

Example 4.2. Inductively define the sequence of rooted trees A_i, $i \geqslant 0$ as follows. A_0 is a single vertex. A_i has a root of out degree i. The successors of the root are the roots of copies of $A_0, A_1, \ldots, A_{i-1}$. A_i has 2^i vertices and is in $U(2^i)$. This is easily proved by induction: A_1 is a single edge and is trivially in $U(2)$. The induction step follows by noting if we take $G = A_{i-1}$ and $H = A_1$ in Proposition 4.1. then C is A_i.

Proposition 4.3. *Let G be a digraph on n vertices and suppose x is the only vertex of G with in degree zero. Then $G \in U(n)$ implies $G - x \in U(n-1)$.*

Proof. Let T be any tournament on $n - 1$ vertices and let T' be the tournament obtained by adding a single vertex y pointing to every vertex in T. If G is in $U(n)$ then it is a subgraph of T' and in the copy of G in T' the vertex x corresponds to y since every vertex in G other than x has in degree at least one. Thus $G - x$ is a subgraph of $T = T' - y$ and since T was arbitrary $G - x$ is in $U(n - 1)$. ∎

Example 4.4. Let $G(m, k)$ denote the graph on $m + k - 1$ vertices obtained by concatenating the leaf of a rooted path on k vertices to the root of a rooted star on m vertices. Then $G(m, k)$ is *not* in $U(m + k - 1)$ for $k \geqslant 1$ and $m \geqslant 3$. To prove this by induction note that $G(m, 1)$ is just an m vertex star which is not in $U(m)$ for $n \geqslant 3$. Applying Proposition 4.2 with $G = G(m, k)$ and $G - x$ being $G(m, k - 1)$ proves the induction step.

Remark 4.5. The previous example, together with the results of Section 2 provide an example of a class of trees for which the orientation "matters". If we reverse the orientation of the edges in the path of $G(m, k)$ we obtain the claw $C(k - 1, 1, 1, 1, \ldots, 1)$, with $m - 1$ ones. Theorem 2.4 and Lemma 2.1 imply that if $m \leqslant k$ this is n-unavoidable.

5. OTHER DIRECTIONS

The general problem of characterizing $U(n)$ is wide open. We have dealt with a few limited classes of graphs, but have barely scratched the surface.

1. We would also like to know how many edges can a spanning digraph in $U(n)$ contain given that it has a Hamiltonian path. P. Ungár considered this problem and proved that digraph obtained from the Hamiltonian path by adding the edge from the first to the least node is n-unavoidable.

2. J. Urrutia and V. Neumann-Lara [8] have considered the problem of which strongly connected graphs are contained as subgraphs of all strongly connected n-tournaments.

3. One can also pose analogous problems for complete oriented k-regular hypergraphs. An oriented hypergraph is one where the vertices in each hyperedge are sequenced. A k-hypertournament on n vertices consists of all k-sets of the n vertices, each with an orientation. Are there any non-trivial oriented k-regular hypergraphs which are subhypergraphs of every k-hypertournament?

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M. Saks

Department of Mathematics, Rutgers University, New Brunswick, N.J. 08904, USA.

V. Sós

Eötvös University of Budapest, Department of Anal. I, Múzeum krt. 6–8, H-1088, Hungary.