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ON THE DISCREPANCY OF $(n\alpha)$ SEQUENCES

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Let (u_n) be a sequence of numbers in $[0,1)$, I a subinterval of $[0,1)$ and $|I|$ the Lebesgue-measure of I. Put

$$\Delta_N(I;(u_n)) = \sum_{\substack{u_n \in I \\ 1 \le n \le N}} 1 - N|I|$$

We consider the discrepancy function D_N^* resp. D_N defined by

$$ND_N((u_n)) = \sup_I |\Delta_N(I;(u_n))|$$

resp.

$$ND_N^*((u_n)) = \sup_\beta |\Delta_N([0,\beta);(u_n))|.$$

(Obviously $D_N^*\le D_N\le 2D_N^*$). The behaviour of D_N^* resp. D_N as $N\to\infty$ measures in supremum norm, how well distributed is the sequence (u_n) mod 1. Since (u_n) is uniformly distributed mod 1 if and only if $D_N^*=o(1)$, the question is, how fast D^* can tend to 0 . Van der Corput conjectured that for any sequence (u_n) $\sup_N ND_N^*=+\infty$ and AARDENNE-EHRENFEST [15] proved it in the sharper form $ND_N^*=\Omega(\log\log N/\log\log\log N)$. K.F. ROTH [11] showed the stronger result $ND_N=\Omega(\log^{1/2} N)$. Finally, W.M. SCHMIDT [12] proved the best possible Ω-theorem:

$$ND_N^* = \Omega(\log N)$$

where

$$\overline{\lim}\ \frac{ND_N^*}{\log N} \ge c_0 > 0$$

holds with a universal constant c_0 . (The best known constant given by L. KUIPERS-H. NIEDERREITER [8] is $c_0=$ $=(64\log 4)^{-1}$.)

Let $s^*((u_n))\doteq\sup_N \frac{ND_N^*((u_n))}{\log N}$, $s((u_n))\doteq\sup_N \frac{ND_N((u_n))}{\log N}$

and $s^*(\alpha)\doteq s^*(\{n\alpha\})$, $s(\alpha)\doteq s(\{n\alpha\})$. As it is known, for any $(n\alpha)$-sequence where α is irrational and has bounded partial quotients we have $s^*(\alpha)<\infty$.

For the Van der Corput sequence (c_n). S. HABER [7] proved

$$s^*((c_n)) = (3 \log 2)^{-1} = 0.48 \ldots \quad .$$

R. BEJIRAN [1] constructed a sequence (derived from the Van der Corput's one) for which $s^*=(6 \log 2)^{-1} = 0.24\ldots$ and recently D. FAURE [6] constructed a sequence for which $s^*=0.22\ldots$.

Here we consider only $(n\alpha)$-sequences. For $\alpha=\frac{\sqrt{5}-1}{2}$ A.Gilet and T.Sós respectively proved

$$0{,}15\left(\log \frac{\sqrt{5}+1}{2}\right)^{-1} \leq s^*\left(\frac{\sqrt{5}-1}{2}\right) \leq$$

$$\leq (\sqrt{5}+1)\left(3\sqrt{5} \log\left(\frac{\sqrt{5}+1}{2}\right)\right)^{-1}$$

and

$$s\left(\frac{\sqrt{5}-1}{2}\right) \leq 1 \ .$$

Finally. I. DUPAIN [4] proved the exact result

$$s^*\left(\frac{\sqrt{5}-1}{2}\right) = \frac{3}{20}\left(\log \frac{\sqrt{5}+1}{2}\right)^{-1} \ .$$

We know how the discrepancy of the $(\{n\alpha\})$ sequence depends on its partial quotients $a_1, a_2, \ldots, a_k, \ldots$. It is "small" or "large" depending on how small or large $a_1, \ldots, a_k \ldots$ are. (E.g. for $q_k < N < q_{k+1}$ it is between $c_1 \sum_1^k a_i$ and $c_2 \sum_1^{k+1} a_i$ with universal positive constants c_1, c_2.) Therefore one could expect that it takes the smallest value for $\frac{\sqrt{5}+1}{2} = [1,1,\ldots]$. Surprisingly enough this is not the case.

We will prove that the best $(n\alpha)$-sequence from the point of view of D_N^* norm is obtained not by $\alpha = \frac{\sqrt{5}-1}{2}$ but by $\alpha = \sqrt{2}-1 = [2,2,\ldots]$. More exactly we prove the

THEOREM. *With the notation*

$$s^*(\alpha) = \overline{\lim_N} \frac{N D_N^*(\alpha)}{\log N}$$

we have

(1) $$\inf_\alpha s^*(\alpha) = s^*(\sqrt{2}-1)$$

Before turning to the proof we give the discrepancy -formula for the sequence $(\{n\alpha\})$ and the value of $s^*(\sqrt{2}-1)$ in Part A. These will be used in the proof of the theorem given in Part B.

PART A.

The discrepancy-formula for the sequence $(\{n\alpha\})$.

Let $\alpha=[a_1,a_2,\ldots]$ be the continued fraction expansion of the irrational $\alpha\in[0,1)$. Define

$$p_{-1}=0,\quad p_0=1,\quad q_0=0,\quad q_{-1}=1,\quad a_0=0,\quad \theta_0=-1$$

$$\frac{p_n}{q_n}=[a_1,\ldots,a_{n-1}],\quad \theta_n=q_n\alpha-p_n\ ,\quad \lambda_n=|\theta_n|=(-1)^{n+1}\theta_n\ .$$

We have the following formulas.

$$q_{n+1}=a_nq_n+q_{n-1}\ ,\quad p_{n+1}=a_np_n+p_{n-1}$$

$$\theta_{n+1}=a_n\theta_n+\theta_{n-1}\ ,\quad \lambda_{n+1}q_n+\lambda_nq_{n+1}=1$$

$$\sum_{\nu=0}^{\infty} a_{k+2\nu}\theta_{k+2\nu}=-\theta_{k-1}\ ,\quad k=1,\ldots$$

$$\sum_{\nu=0}^{n} a_{k+2\nu}\ q_{k+2\nu}=q_{k+2n+1}-q_{k-1}\ ,\quad k=1,\ldots$$

A sequence of integers $(b_1,\ldots,b_\nu)$ is called a *permitted sequence*, if it satisfies

(2) $$0\le b_1\le a_1-1\ ,\quad 0\le b_k\le a_k\quad \text{if}\quad k\ge 2$$

and

$$(3) \qquad b_k=0 \text{ if } b_{k+1}=a_{k+1} \text{ for } k\geq 1 .$$

It is well known that every positive integer $N<q_{\nu+1}$ can be uniquely represented in the form

$$(4) \qquad N = \sum_{k=1}^{\nu} b_k q_k$$

where $(b_1,\ldots,b_\nu)$ is a permitted sequence and conversely, for every permitted sequence $(b_1,\ldots,b_\nu)$

$$N = \sum_{i=1}^{\nu} b_i q_i < q_{\nu+1} .$$

It is also known (DESCOMBES [2], T. SÓS [13], LESCA [9]) that every $\beta\in[-\alpha,1-\alpha)$ can be uniquely represented in the form

$$(5) \qquad \beta = \sum_{k=1}^{\infty} d_k \Theta_k$$

where (d_k) is a permitted sequence and satisfies

$$(6) \qquad d_{2k+1} \neq a_{2k+1} \text{ for infinitely many } k .$$

Conversely, every permitted sequence which satisfies (6) determines a $\beta \in [-\alpha, 1-\alpha)$ by (5).

Set

$$\Delta_N(\beta) = \Delta_N([0,\beta);\alpha) \qquad 0<\beta$$

and

$$\Delta_N(\beta) = \Delta_N([0,1+\beta);\alpha) \text{ for } -\alpha<\beta<0$$

The proof of the theorem will be based on the following explicit formula for $\Delta_N(\beta)$. (In this form see T.SÓS [14]).

$$\Delta_N(\beta) = \sum_{k=1}^{\nu} ((-1)^{k+1} \min(b_k, d_k) - \tag{7}$$
$$- d_k(q_k \sum_{i=k+1}^{\infty} b_i\Theta_i + \Theta_k \sum_{i=1}^{k} b_i q_i) + \sum_{k=1}^{\nu} \delta_k$$

where

$$\delta_k = \begin{cases} 1, & \text{if } k \text{ is odd, } d_k>b_k \text{ and } \sum_{i=1}^{k-1} b_i q_i > \sum_{i=1}^{k-1} d_i q_i \\ -1, & \text{if } k \text{ is even, } d_k<b_k \text{ and } \sum_{i=1}^{k-1} b_i q_i \le \sum_{i=1}^{k-1} d_i q_i \\ 0 & \text{otherwise} \end{cases}$$

We remark in advance that in the cases we shall use this formula $\delta_k=0$ for every k. So it will be easier to handle this expression for $\Delta_N(\beta)$.

The value of $s^*(\sqrt{2}-1)$

We shall use

$$(8) \qquad s^*(\sqrt{2}-1) = (4\log(\sqrt{2}+1))^{-1} \quad 0{,}2836 .$$

For the proof see RAMSHAW [10] and Y. DUPAIN-V.T. SOS [5].

Here we just sketch the proof of (8) given in [5]. By the discrepancy-formula (7) it is easy to see that for $\alpha=\sqrt{2}-1=[2,2,\dots]$ the choice

$$N_r = \sum_{2i+1\le r} q_{2i+1} , \quad \beta_r = \sum_{2i+1\le r} \theta_{2i+r}$$

gives

$$\Delta_{N_r}(\beta) \sim \frac{r}{2} , \quad \text{if} \quad r\to\infty .$$

Since $\log N_r \sim \log q_r \sim r \log(\sqrt{2}+1)$, we get

$$s^*(\sqrt{2}-1) \ge (4\log(\sqrt{2}+1))^{-1} .$$

We need a much more involved proof to show

$$s^*(\sqrt{2}-1) \le (4\log(\sqrt{2}+1))^{-1}.$$

The idea of the proof is the following.

Let

$$N = \sum_{i\le r} d_i q_i \,, \quad \beta = \sum_{i=1}^{\infty} b_i q_i \,,$$

Consider the sequences $(b_1,\ldots,b_r)$, $(d_1,\ldots,d_r)$. Our aim is to prove, that the maximum of $|\Delta_N(\beta)|$ is taken when both sequences are the 1,0,1,0,... sequence. To prove this we give an improving algorithm for the case when at least one of the sequences is different from 1,0,1,0,... . We show that changing some of the values of b_i and d_i appropriately, $|\Delta_N(\beta)|$ increases.

PART B.

The main idea in the proof is the construction of numbers $\beta^{-1},\beta^{+}\in[0,1)$ and sequences N_r^-, N_r^+ so that

$$N_r^- < \log q_{r+1} \,, \quad N_r^+ < \log q_{r+1}$$

and

$$\Delta^+_{N_r}(\beta^+) - \Delta^-_{N_r}(\beta^-) > 2s^*(\sqrt{2}-1)\log q_{r+1} + o(r), \tag{9}$$

hold.

The construction of $\beta^-, \beta^+, N_r^-, N_r^+$.

First we define a sequence (ε_k).

$$\varepsilon_{2n+1} = \begin{cases} 0, \text{ if } a_{2n+1} \text{ is even} \\ 1, \text{ if } a_{2n+1} \text{ is odd and } \sum\limits_{k \le n} a_{2k+1} \text{ is even} \\ -1, \text{ if } a_{2n+1} \text{ is odd and } \sum\limits_{k \le n} a_{2k+1} \text{ is even.} \end{cases}$$

(i.e. for the subsequence a_{2n+1} odd $\varepsilon_{2n+1} = \pm 1$ alternatively).

Similarly let

$$\varepsilon_{2n} = \begin{cases} 0, \text{ if } a_{2n} \text{ is even} \\ 1, \text{ if } a_{2n} \text{ is odd and } \sum\limits_{k \le n} a_{2k} \text{ is odd} \\ -1, \text{ if } a_{2n} \text{ is odd and } \sum\limits_{k \le n} a_{2k} \text{ is even.} \end{cases}$$

Now we define the sequences (b_n), (b_n'), N_r^+ and N_r^- .

Let

$$b_{2n} = 0, \quad b_{2n}' = \frac{a_{2n} + \varepsilon_{2n}}{2} \qquad n = 1, 2, \ldots$$

$$d_{2n} = 0, \quad d_{2n}' = \frac{a_{2n} - \varepsilon_{2n}}{2} \qquad n = 1, 2, \ldots$$

$$b_{2n+1} = \frac{a_{2n+1} + \varepsilon_{2n+1}}{2}, \quad b_{2n+1}' = 0 \qquad n = 0, 1, \ldots$$

$$d_{2n+1} = \frac{a_{2n+1}-\varepsilon_{2n+1}}{2}, \quad d'_{2n+1}=0 \qquad n=0,1,\ldots$$

Consequently let

$$\beta^+ = \sum_{n=1}^{\infty} b_n \Theta_n , \quad N_r^+ = \sum_{n=1}^{r} d_n q_n$$

$$\beta^- = \sum_{n=1}^{\infty} b_n \Theta_n , \quad N_r^- = \sum_{n=1}^{r} d'_n q_n .$$

This choice of the digits of $\beta^+, \beta^-, N^+, N^-$ is motivated by the following.

We can see from the discrepancy-formula that for k odd the contribution of a_k to the value of $\Delta_N(\beta)$ is

$$\min(b_k, d_k) - b_k d_k q_k \lambda_k \geq 0 ,$$

The maximum of it in b_k, d_k is about $\max_b \frac{b(a_k-b)}{a_k}$. This suggests, that in case a_k even $d_k=b_k=\frac{a_k}{2}$ gives the maximum. If a_k is odd one of the choices $d_k=\frac{a_k\pm 1}{2}$, $b_k=\frac{a_k\pm 1}{2}$ will be the best.

A similar reasoning works for k even. Now we determine the values of N and β so that $|\Delta_N(\beta)|$ is large. It is necessary that the contribution of the positive terms belonging to odd indices and the contribution of the negative terms belonging to even indices should not compensate each other. The contribution of the terms belonging to odd resp. even indices can be about $\frac{1}{4}\sum a_{2i+1}$ resp. $\frac{1}{4}\sum_{2i\le r} a_{2i}$.

As a technical simplification we consider the difference of the $\Delta_N(\beta)$'s belonging to the two different choices.

LEMMA 1.

$$\Delta_{N_r^+}(\beta+) - \Delta_{N_r^-}(\beta^-)$$

$$= \frac{1}{4}\sum_{n=1}^{r} a_n + \frac{1}{4}\sum_{n=1}^{r} \varepsilon_n^2 \Theta_n q_n +$$

$$+ \frac{1}{2}\sum_{n=1}^{r} \sum_{\substack{k<n \\ k\equiv n(2)}} \varepsilon_n \varepsilon_k \Theta_n q_k + o(1)$$

PROOF. We determine first the value of $\Delta_{N_r^+}(\beta^+)$. Here we apply the discrepancy-formula (7). For the sequence δ_n we have

$$\delta_{2n}=0 \quad \text{since} \quad d_{2n}=0$$

$$\delta_{2n+1}=0 \quad \text{if} \quad \varepsilon_{2n+1}=0 \quad (b_{2n+1}=d_{2n+1})$$

$$\delta_{2n+1}=1 \quad \text{if} \quad \varepsilon_{2n+1}=1 .$$

By this

$$\min(b_{2n+1}, d_{2n+1}) + \delta_{2n+1} = \frac{a_{2n+1}+\varepsilon_{2n+1}}{2} .$$

To estimate the remaining terms in (7) for n odd let

$$c_n = -\frac{a_n-\varepsilon_n}{2}(\lambda_n \sum_{\substack{k\le n\\ k\equiv 1(2)}} \frac{a_k+\varepsilon_k}{2} q_k +$$

$$+ q_n \sum_{\substack{k>n\\ k\equiv 1(2)}} \frac{a_k+\varepsilon_k}{2} \lambda_k).$$

By the recursive formulas for λ_n and q_n we have

$$c_n = -\frac{a_n-\varepsilon_n}{2}(\lambda_n q_{n+1}+\lambda_{n+1}q_n+\lambda_n \sum_{\substack{k\le n\\ k\equiv 1(2)}} \varepsilon_k q_k +$$

$$+ q_n \sum_{\substack{k>n\\ k\equiv 1(2)}} \varepsilon_k\lambda_k) = -\frac{a_n}{4} + \frac{\varepsilon_n}{4} -$$

$$-\frac{1}{4}(\lambda_{n-1}-\lambda_{n+1} \sum_{\substack{k\le n\\ k\equiv 1(2)}} \varepsilon_k q_k + (q_{n+1}-q_{n-1}) \sum_{\substack{k>n\\ k\equiv 1(2)}} \varepsilon_k\lambda_k) +$$

$$+ \frac{\varepsilon_n}{4}\left(\lambda_n \sum_{\substack{k\le n\\ k\equiv 1(2)}} \varepsilon_k q_k + q_n \sum_{\substack{k>n\\ k\equiv 1(2)}} \varepsilon_k \lambda_k\right).$$

In order to estimate $\sum_{n\equiv 1(2)} c_n$ first we prove

$$\Sigma_1 = \sum_{\substack{n\le r\\ n\equiv 1(2)}} (\lambda_{n-1}-\lambda_{n+1}) \sum_{\substack{k\le n\\ k\equiv 1(2)}} \varepsilon_k q_k +$$

$$+ \sum_{\substack{n\le r\\ n\equiv 1(2)}} (q_{n+1}-q_{n-1}) \sum_{\substack{k>n\\ k\equiv 1(2)}} \varepsilon_k \lambda_k = o(1).$$

By rearranging the terms we get

$$\Sigma_1 = \sum_{\substack{k\le r\\ k\equiv 1(2)}} \varepsilon_k q_k \sum_{\substack{r\ge n\ge k\\ n\equiv 1(2)}} (\lambda_{n-1}-\lambda_{n+1}) +$$

$$+ \sum_{\substack{n\le r\\ n\equiv 1(2)}} \varepsilon_k \lambda_k \sum_{\substack{k\le n\\ k\equiv 1(2)}} (q_{n+1}-q_{n-1})$$

$$= \sum_{\substack{k\le r\\ k\equiv 1(2)}} \varepsilon_k (q_k \lambda_{k-1} \lambda_k) - \lambda_{r'} \sum_{\substack{k\le r\\ k\equiv 1(2)}} \varepsilon_k q_k -$$

$$- q_0 \sum_{\substack{1\le k\le r\\ k\equiv 1(2)}} \varepsilon_k \lambda_k + q_{r+1} \sum_{\substack{k>r\\ k\equiv 1(2)}} \varepsilon_k \lambda_k$$

where $r'=r+1$ if r is odd, $r'=r$ if r is even. Using $\lambda_{k-1}q_k+\lambda_k q_{k-1}=1$ and that the nonzero values of ε_n are alternatively ± 1, we get

$$\Sigma_1 = o(1).$$

Now consider the sum

$$\Sigma_2 = \frac{1}{4} \sum_{\substack{n\le r\\ n\equiv 1(2)}} \varepsilon_n (\lambda_n \sum_{\substack{k\le n\\ k\equiv 1(2)}} \varepsilon_k q_k + q_n \sum_{\substack{k>n\\ k\equiv 1(2)}} \varepsilon_k \lambda_k).$$

We find that

$$\Sigma_2 = \frac{1}{4} \sum_{\substack{n\le r\\ n\equiv 1(2)}} \varepsilon_n^2 \lambda_n q_n + \frac{1}{2} \sum_{\substack{n\le r\\ n\equiv 1(2)}} \sum_{\substack{k<n\\ k\equiv 1(2)}} \varepsilon_n \varepsilon_k \lambda_n q_k +$$

$$+ \frac{1}{2} \sum_{\substack{n\le r\\ n\equiv 1(2)}} \varepsilon_n q_n \sum_{\substack{k>r\\ k\equiv 1(2)}} \varepsilon_k \lambda_k =$$

$$= \frac{1}{4} \sum_{\substack{n\le r\\ n\equiv 1(2)}} \varepsilon_n^2 \lambda_n q_n + \frac{1}{2} \sum_{\substack{n\le r\\ n\equiv 1(2)}} \sum_{\substack{k<n\\ k\equiv 1(2)}} \varepsilon_n \varepsilon_k \lambda_n q_k +$$

$$+ o(1) .$$

Considering the terms of even indices only we get a similar formula for $\Delta_{N_r^-}(\beta^-)$.

These proves Lemma 1.

Lower bound for $\Delta_{N_r^+}(\beta^+)-\Delta_{N_r^-}(\beta^-)$.

Set

$$(10) \qquad s_n = a_n + \varepsilon_n^2\lambda_n q_n + 2\varepsilon_n \sum_{\substack{k<n \\ k\equiv n(2)}} \varepsilon_k\lambda_n q_k .$$

We start by estimating this in the special case $\alpha=\frac{\sqrt{5}-1}{2}$. In this case

$$s_n = a_n + \lambda_n q_n - 2\lambda_n \sum_{2\le 2\nu\le n} q_{n-2\nu}$$

and

$$\frac{\sum_{n=1}^{r} s_n}{\log q_{r+1}} \to \frac{6}{5}\log(\frac{\sqrt{5}+1}{2}^{-1}).$$

By this and by Lemma 1. for $\alpha=\frac{\sqrt{5}-1}{2}$ we get

$$\overline{\lim_n}\frac{\Delta_n}{\log n} \ge \frac{3}{20}(\log(\frac{\sqrt{5}+1}{2})^{-1} \approx 0{,}31\ldots \ge$$

$$\ge (4\log(\sqrt{2}+1))^{-1} .$$

Now, let $\alpha \neq \frac{\sqrt{5}-1}{2}$. For $r>k_1(\alpha)=k_1$ we construct a sequence of indices $k_1<k_2<\ldots<k_\nu=r+1$ so that

$$(11) \qquad \sum_{n=k_j}^{k_{j+1}-1} s_n \geq 8s^*(\sqrt{2}-1)\cdot\log\frac{q_{k_j+1}}{q_{k_j}} \quad \text{if}$$

$$j=1,\ldots,\nu-1 .$$

The proof of the theorem will follow by Lemma 1 and by (11).

First we define the blocks of (a_n) of type $B_1,\ldots$ $\ldots,B_6$ in the following way:

Type B_1: If $a_n\geq 4$, then a_n forms a block of type B_1 .

Type B_2: If $a_n=3$, then a_n forms a block of type B_2 .

Type B_3: If $a_n=\ldots=a_{n+k}=2$ and $a_{n-1}\geq 3$, then $a_n,\ldots,a_{n+k}$ form a block of type B_3 .

Type B_4: If $a_n=\ldots=a_{n+k}=1$ and $a_{n-1}\neq 1$, then $a_n,\ldots,a_{n+k}$ form a block of type B_4 .

Type B_5: If $a_n=\ldots=a_{n+k}=1$, $a_{n+k+1}=\ldots=a_{n+k+l}=2$, and $a_{n-1}\geq 3$ then $a_n,\ldots,a_{n+k+l}$ form a block of type B_5 .

Type B_6: If $a_{n-1}=2$, $a_n=\ldots=a_{n+k}=1$, $a_{n+k+1}=\ldots=a_{n+k+l}=2$ and $a_{n-2}\geq 3$ or $n=2$ then $a_n,\ldots,a_{n+k+l}$ form a block of type B_6 .

In case B_3, B_4, B_5 we may have $k=0$. If $\alpha \neq \frac{\sqrt{5}-1}{2}$ then for $r > k_1(\alpha)$ we can partition $a_{k_1}, \ldots, a_r$ into blocks of type $B_1, \ldots, B_6$.

Now we define the sequence $k_1 < \ldots < k_\nu = r+1$. Let k_ν be the first index in the ν's block; for any $\nu < m$ $a_{k_\nu}, \ldots, a_{k_{\nu+1}} - 1$ form a block one of the type $B_1, \ldots, B_6$.

To prove (11) we need

Lemma 2. Suppose

$$a_n = \ldots = a_{n+k} = 1, \quad a_{n+k+1} = \ldots = a_{n+k+\ell} = 2$$

and $a_{n-1} \neq 1$. Then

$$\frac{q_{n+k+\ell+1}}{q_n} < \left(\frac{\sqrt{5}+1}{2}\right)^{k+1} (\sqrt{2}+1)^{\ell} \quad \text{if} \quad a_{n-1} \geq 3 \tag{12}$$

$$\text{or} \quad a_{n-2} \leq 2 \quad \text{or} \quad n=1$$

$$\frac{q_{n+k+\ell+1}}{q_n} < \left(\frac{\sqrt{5}+1}{2}\right)^{k+1} (\sqrt{2}+1)^{\ell+1} \quad \text{if} \quad a_{n-1} = 2 \tag{13}$$

$$\text{and} \quad a_{n-2} \geq 3 \quad \text{or} \quad n=2 .$$

PROOF. Denote (q_i^*) resp. (q_i^{**}) the sequence of partial quotients of

$$\frac{\sqrt{5}-1}{2} = [1,1,\ldots] \quad \text{resp.} \quad \sqrt{2}-1 = [2,2,\ldots]$$

$$(q_0^* \doteq 0, \quad q_1^* = 1, \quad q_2^* = 1, \quad q_3^* = 2, \ldots$$

$$q_0^{**} \doteq 0, \quad q_1^{**} = 1, \quad q_2^{**} = 2, \quad q_3^{**} = 5, \ldots)$$

We can easily prove the following relations:

$$q_{n+1} = q_n + q_{n-1} = q_2^* q_n + q_1^* q_{n-1}$$

$$q_{n+2} = 2q_n + q_{n-1} = q_3^* q_n + q_2^* q_{n-1}$$

$$\vdots$$

$$q_{n+k+1} = q_{k+2}^* q_n + q_{k+1}^* q_{n-1}$$

$$q_{n+k+2} = 2q_{n+k+1} = q_{n+k}$$

$$\vdots$$

$$q_{n+k+\ell+1} = q_{\ell+1}^{**} q_{n+k+1} + q_\ell^* q_{n+k} =$$

$$= q_{\ell+1}^{**}(q_{k+2}^* q_n + q_{k+1}^* q_{n-1}) + q_\ell^{**}(q_{k+1}^* q_n + q_k^* q_{n-1}) =$$

$$= (q_{\ell+1}^{**} q_{k+2}^* + q_\ell^{**} q_{k+1}) q_n + (q_{\ell+1}^{**} q_{k+1}^* + q_\ell^{**} q_k^*) q_{n-1} \; .$$

For a fixed q_n and q_{n-1} put

$$A(i,j) =: (q^{**}_{j+1}q^{*}_{i+2}+q^{*}_{j}q^{**}_{i+1})q_n +$$

$$+ (q^{**}_{j+1}q^{*}_{i+1}+q^{*}_{j}q^{*}_{i}\ q_{n-1}).$$

Observe that

$$A(k,\ell) = q_{n+k+\ell+1}\ .$$

For $A(i,j)$ we have the following recursive formulas:

$$A(i+1,j) = A(i,j) + A(i-1,j)$$

$$A(i,j+1) = 2A(i,j) + A(i,j-1).$$

Hence, if

$$A(i-1,j)\le x \quad \text{and} \quad A(i,j) \le x\,\frac{\sqrt{5}+1}{2}\ ,$$

then

(14) $$A(i+1,j) \le x\left(\frac{\sqrt{5}+1}{2}\right)^2\ .$$

If

$$A(i,j-1)\le x \quad \text{and} \quad A(i,j) \le x(\sqrt{2}+1)$$

then

$$(15) \qquad A(i,j+1) \le x(\sqrt{2}+1)^2 .$$

Since $a_{n-1} \neq 1$, $a_n = 1$, we get

$$q_{n+1} = q_n + q_{n-1} < (1+\tfrac{1}{2})q_n < \frac{\sqrt{5}+1}{2} q_n .$$

By this

$$(16) \qquad A(0,0) = q_{n+1} < \frac{\sqrt{5}+1}{2} q_n .$$

If $a_{n-1} \ge 3$ or $a_{n-2} \le 2$ or $n=1$, then

$$(17) \qquad A(1,0) = q_n + q_{n-1} < (\frac{\sqrt{5}+1}{2})^2 q_n$$

$$(18) \qquad A(0,1) = 3q_n + 2q_{n-1} = (3+2\frac{q_{n-1}}{q_n})q_n \le$$

$$\le (3+2\frac{1}{2+\frac{1}{3}})q_n < \frac{\sqrt{5}+1}{2}(\sqrt{2}+1)q_n .$$

By this and by (13)-(14) we get

$$q_{n+k+\ell+1} = A(k,\ell) \le (\frac{\sqrt{5}+1}{2})^{k+1}(\sqrt{2}+1)^{\ell} q_n .$$

This proves (12). In cases $a_{n-1}=2$, $a_{n-2}\geq 3$ or $n=2$ (16) and (17) also hold. Instead of (18) now we have

$$A(0,1)=3q_n+2q_{n-1}=8q_{n-1}+3q_{n-2}<9q_{n-1} ,$$

(19) $$A(0,1)<\frac{\sqrt{5}+1}{2}(\sqrt{2}+1)^2 q_{n-1} .$$

By (16),(17) and (18)

(20) $$q_{n+k+\ell+1}=A(k,\ell)\leq\left(\frac{\sqrt{5}+1}{2}\right)^{k+1}(\sqrt{2}+1)^{\ell+1}q_{n-1} .$$

This proves (13)

LEMMA 3. If $a_n=\ldots=a_{n+k}=2$ and $a_{n-1}\geq 3$, then

$$q_{n+k+1}<(\sqrt{2}+1)^{k+1}q_n .$$

PROOF. This follows simply by (15) and by

$$q_{n+1}=2q_n+q_{n-1}<(\sqrt{2}+1)q_n .$$

LEMMA 4. If $a_n=\ldots=a_{n+k}=1$ and $a_{n-1}\neq 1$, then

$$q_{n+k+1}<\left(\frac{\sqrt{5}+1}{2}\right)^{k+1}q_n .$$

PROOF. It follows by (14) and (15).

Now we prove that in each of the 6 cases, $B_1, B_2, \ldots$ $\ldots, B_6$ we have

$$\sum_{n=k_j}^{k_{j+1}-1} s_n \geq 8c \log \frac{q_{k_{j+1}}}{q_{k_j}}$$

where $8c=8(4\log(\sqrt{2}+1))^{-1}=0{,}2691853\ldots$

First we remark that

(21) $$s_n \geq a_n .$$

Case of B_1. By $a_n \geq 4$, and

$$\log \frac{q_{n+1}}{q_n} = \log(a_n + \frac{q_{n-1}}{q_n}) < \log(a_n+1),$$

we obtain

$$\frac{s_n}{\log \frac{q_{n+1}}{q_n}} > \frac{a_n}{\log(a_n+1)} > \frac{4}{\log 5} > 8c .$$

Case of B_2. ($a_n=3$). First we consider the case, when

(22) $$a_{n-1} \geq 2 \quad \text{or} \quad a_{n-2} \leq 2 .$$

Then

$$\frac{q_{n+1}}{q_n} = 3 + \frac{1}{a_{n-1}+\frac{1}{a_{n-2}+\ldots}} \leq 3 + \frac{1}{1+\frac{1}{2+1}}$$

and by this

$$\frac{s_n}{\log\frac{q_{n+1}}{q_n}} > \frac{3}{\log 3,55} = 2,26 \ldots > 8c .$$

In the other case, when

(23) $\quad a_{n-1}=1$ and $a_{n-2}\geq 3$

we will consider also the additional terms in (10)

Since ε_n has alternative sign (if different from 0)

$$\varepsilon_n^2\lambda_n q_n + 2 \sum_{\substack{k<n \\ k\equiv n(2)}} \varepsilon_n\varepsilon_k\lambda_n q_k \geq$$

$$\geq \begin{cases} \lambda_n q_n - 2\lambda_n q_{n-2} & \text{if } a_{n-2} \text{ is odd} \\ \\ \lambda_n q_n - 2\lambda_n q_{n-4} & \text{if } a_{n-2} \text{ is even, } a_{n-2}\geq 4 , \end{cases}$$

By this and by (10)

$$s_n \geq 3 + \lambda_n q_n - 2\lambda_n q_{n-2} .$$

Put $x=\frac{q_{n-2}}{q_{n-1}}$. Then

$$q_n = q_{n-1} + q_{n-2} = \frac{x+1}{x} q_{n-2} ,$$

$$\lambda_n q_n = 2\lambda_n q_{n-2} = \frac{1-x}{1+x} \lambda_n q_n .$$

Since

$$\lambda_n q_n = \frac{1}{3 + \frac{q_{n-1}}{q_n} + \frac{\lambda_{n+1}}{\lambda_n}} > \frac{1}{5}$$

we have

(24) $$\frac{s_n}{\log \frac{q_{n+1}}{q_n}} > \frac{3 + \frac{1}{5}\frac{1-x}{1+x}}{\log(3 + \frac{1}{1+x})} =: f(x) .$$

If $a_{n-2}=3$, $\frac{1}{4}<x<\frac{1}{3}$, then $f(x)>\frac{3 + \frac{1}{10}}{\log 3{,}8} = 2{,}3...$

If $a_{n-2}=5$, $\frac{1}{6}<x<\frac{1}{5}$, then $f(x)>\frac{3+2/15}{\log 27/7} = 2{,}3...$

If $a_{n-2}\geq 7$, $x<\frac{1}{7}$, then $f(x)>\frac{3+3/20}{\log 4} = 2{,}27...$

If is even, a_{n-2}, $a_{n-2}\geq 4$, $q_{n-4}<\frac{q_n}{8}$ and $s_n\geq 3+\frac{1}{5}\ \frac{3}{4}$.

Hence

$$\frac{s_n}{\log\frac{q_{n+1}}{q_n}} > \frac{3 + 3/20}{\log 4} = 2.27 \ldots$$

Case B_3 . $(a_n = \ldots = a_{n+k} = 2,\ a_{n-1} \geq 3)$. By (14)

$$\sum_{i=1}^{n+k} s_i = 2(k+1) .$$

By this and by Lemma 3 we obtain

$$\sum_{i=1}^{n+k} s_i > 2(k+1) = 8\log(\sqrt{2}+1)^{k+1}\ 4(\log\sqrt{2}+1)^{-1} >$$

$$> 8c \log \frac{q_{n+k+1}}{q_n} .$$

Case B_4 . Now $a_n = \ldots = a_{n+k} = 1,\ a_{n-1} \neq 1$.

We estimate first s_n and in case $k>0$ s_{n+1} .

If $a_{n-1} > 2$, then

$$q_n > 4q_{n-2}\ ,\quad q_{n+1} > 4q_{n-1} \quad \text{and} \quad q_n > 3q_{n-1}\ ,$$

hence

$$s_n \geq 1 + \lambda_n(q_n - 2q_{n-2}) > 1 + \frac{1}{2}\lambda_n q_n .$$

Now

$$\lambda_n q_n = \frac{1}{1 + \frac{q_{n-1}}{q_n} + \frac{\lambda_{n+1}}{\lambda_n}} > \frac{3}{7} .$$

Thus

$$s_n > 1 + \frac{3}{14} = 1{,}214\ldots . \tag{25}$$

Similarly, if $k>0$

$$s_{n+1} > 1 + \lambda_{n+1}(q_{n+1}-2q_{n-1}) > 1 + \frac{1}{2}\lambda_{n+1}q_{n+1}$$

and

$$s_{n+1} > 1 + \frac{1}{6} = 1{,}166 \ldots . \tag{26}$$

Let us now consider the remaining case $a_{n-1}=2$. As above, we get $q_n>3q_{n-2}$. Using $q_n>2q_{n-1}$ we have

$$s_n \geq 1 + \lambda_n(q_n-2q_{n-2}) > 1 + \frac{1}{3}\lambda_n q_n =$$

$$= 1 + \frac{1}{3}\frac{1}{1 + \frac{q_{n-1}}{q_n} + \frac{\lambda_{n+1}}{\lambda_n}} > 1 + \frac{1}{3}\frac{2}{5} = 1{,}133\ldots .$$

Similarly, since a_{n-2} is odd,

$$s_{n+1} > 1 + \lambda_{n+1}(q_{n-1}-2q_{n-3}) \quad \text{if} \quad k>0 .$$

Now

$$q_{n+1} = 2q_{n-1} + q_{n-2} = 5q_{n-2} + 2q_{n-3} > 7q_{n-3}$$

and

$$\lambda_{n+1}q_{n+1} = \frac{1}{1 + \frac{q_n}{q_{n+1}} + \frac{\lambda_{n+2}}{\lambda_{n+1}}} > \frac{1}{3} .$$

Hence

$$s_{n+1} > 1 + \frac{5}{21} = 1,238 \ldots .$$

Let us estimate now $s_{n+2},\ldots,s_{n+k}$ if $k\geq 2$.

(27) $$s_{n+2} \geq 1 .$$

If $k\geq 3$, since $a_{n-1}\geq 2$, we get that

$$q_{n+4} = 5q_n + 3q_{n-1} < \frac{13}{2} q_n .$$

Hence

(28) $$s_{n+3} \geq 1 + \lambda_{n+3}(q_{n+3}-2q_{n+1}) = 1 + \lambda_{n+3}q_n =$$

$$= 1 + \lambda_{n+3}q_{n+4}\,\frac{q_n}{q_{n+4}} > 1 + \frac{1}{2}\,\frac{13}{2} = 1{,}076 \ldots .$$

If $n+4 \leq i \leq n+k$, by

$$q_i = 2q_{i-2} + q_{i-3} < 3q_{i-2}$$

and

$$\lambda_i q_i > \frac{1}{3}$$

we get that

$$s_i \geq 1 + \lambda_i(q_i-2q_{i-2}+2q_{i-4}-2q_{i-6}) >$$

$$> 1 + \lambda_i q_{i-2} > 1 + \frac{1}{9}$$

(29) $$s_i \geq 1{,}11 \ldots .$$

Now we have to prove that

(30) $$\sum_{i=n}^{n+k} s_i \geq 8c \log \frac{q_{n+k+1}}{q_n} .$$

By Lemma 4

$$q_{n+k+1} < \left(\frac{\sqrt{5}+1}{2}\right)^{k+1} q_n .$$

So it is enough to prove that

$$\sum_{i=1}^{n+k} s_i > 8c(k+1)\log\left(\frac{\sqrt{5}+1}{2}\right). \tag{31}$$

Compare the values obtained in (26)-(30) to the value of $8c(k+1)\log(\frac{\sqrt{5}+1}{2})$. We have the following lower bounds for $\sum_{i=n}^{n+k} s_i$ in cases

k	$a_{n-1}=2$	$a_{n-1}>2$	$8c(k+1)\log(\frac{\sqrt{5}+1}{2})$
0	1,091...	1,133...	1,214...
1	2,183...	2,371...	2,38...
2	3,275...	3,371...	3,38...
3	4,367...	4,447...	4,456...
≥4	4,367...+ +(k−3)1,09	4,447...+ +(k−3)1,1...	4,456...+(k−3)1,09..

This proves (31).

Case B_5. ($a_n=\ldots=a_{n=k}+1$, $a_{n+k+1}=\ldots=a_{n+k+\ell}=2$, $a_{n-1}>2$ or $a_{n-2}\leq 2$ or $n=1$.

By what we proved in case B_4

(32) $$\sum_{i=1}^{n} s_i > 8c(k+1)\log(\frac{\sqrt{5}+1}{2})$$

Using $s^*(\sqrt{2}-1)=(4\log(\sqrt{2}+1))^{-1}=c$

(33) $$\sum_{i=n+k+1}^{n+k+\ell} s_i = 2\ell = 8c\ell\log(\sqrt{2}+1).$$

By (32)-(33)

$$\sum_{i=1}^{n+k+\ell} s_i > 8c[(k+1)\log(\frac{\sqrt{5}+1}{2}) + \ell\log(\sqrt{2}+1)].$$

By this and by Lemma 3 we get

$$\sum_{i=n}^{n+k+\ell} s_i > 8c\log\frac{q_{n+k+\ell+1}}{q_n}.$$

Case B_6. $(a_{n-1}=2,\ a_n=\ldots=a_{n+k+1}=1,\ a_{n+k+1}=\ldots$
$\ldots=a_{n+k+\ell}=2)$ $a_{n-2}\geq 3$ or $n=2$.

An argument similar to that of B_5 gives

$$\sum_{i=n-1}^{n+k+\ell} s_i > 8c((k+1)\log(\frac{\sqrt{5}+1}{2}) + (\ell+1)\log(\sqrt{2}+1)).$$

Hence by Lemma 3 we get

$$\sum_{i=n-1}^{n+k+\ell} s_n > 8c \log \frac{q_{n+k+\ell+1}}{q_{n-1}} .$$

This proves the theorem.

REFERENCES

[1] Bejian, R., Sur certaines suites présentant une faible discrépance à l'origine, C.R.A.S. Paris t. 286 (1978).

[2] Descombes, I.R., Sur la répartition des sommets d'une ligne polygonale réguilère non fermée, *Ann. Sci. de l'Ecole normale Supérieure*, 75 (1956).

[3] Dupain, Y., Intervalles à restes majorés pour la suite $\{n\alpha\}$, *Acta Math. Hung.* XXIX 3-4 (1977).

[4] Dupain, Y., Discrépance de la suite $(\{n \frac{1+\sqrt{5}}{2}\})$, *Annales de l'Institut Fourier* (à paraitre).

[5] Dupain, Y.-Sós V.T., $(\{n\alpha\})$ sequences with least discrepancy. (In Y. Dupain: Repartition et discrepance. These l'Universite de Bordeaux I. N$^{\text{o}}$ 576. p. 173-202.

[6] de Faure, Discrepances de suites associeés á un systéme de numeration, *Bull. Soc. Math. France* 109(1981), no.2, 143-182.

[7] Habe, S.,

[8] Kuipers, L-Niederreiter, H., Uniform distribution of sequences, Wiley, New York, 1974.

[9] Lesca, Sur la répartition modulo 1 des suites $\{n\alpha\}$, Séminaire Delange-Pisot-Poitou (1966-67), f.1 n.2.

[10] Ramshaw, L., On the discrepancy of the sequence formed by the multiples of an irrational number. *Journal of Number Theory* 13 (1981)no.2, 138-175.

[11] Roth, K.F., On the irregularities of distribution, Mathematika I. (1954) 73-79.

[12] Schmidt, V.M., Irregularities of distribution VII. *Acta Arith.* 21 (1972), 45-50.

[13] Sós, V.T., On the theory of diophantine approximation II, *Acta Math. Hung.* IX. 1-2 (1958).

[14] Sós, V.T., On the discrepancy of the sequence $\{n\alpha\}$, *Coll. Math. Soc. J. Bolyai,* vol. 13 (1976).

[15] Van-Aardenne -Ehrenfest: Proof of the impossibility of a just distribution of $(\{n\alpha\})$ sequences. *Indag. Math.* 7 (1945), 71-76.

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