Studies in Pure Mathematics
To the Memory of Paul Turán

On a generalization of Turán's graph-theorem

by

P. ERDŐS and V. T. SÓS (Budapest)

The so-called extremal graph theory started with the well-known theorem of TURÁN [12], [13]. He determined the smallest integer $f(n; k)$ so that every graph $G(n; e)$ of n vertices and $e > f(n; k)$ edges contains a complete graph of k vertices. A general problem in extremal graph theory can be formulated as follows: Let L be a fixed graph and $f(n; L)$ the smallest integer so that every graph of n vertices having more than $f(n; L)$ edges contains a graph isomorphic to L as a subgraph. One of the general theorems in this theory is the ERDŐS–SIMONOVITS [5], see also ERDŐS–STONE [7] theorem which states as follows:

Let the chromatic number of L be $\chi(L)$ and $\chi(L) = r \geqq 3$. Then

$$f(n; L) = (1 + o(1))\left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}.$$

For the case $r = 2$ the theorem only states $f(n; L) = o(n^2)$ but some sharper theorems are also known [1], [4], [8], [9]. The exact value of $f(n; L)$ is known only for very few graphs [2], [3], [10], [11] and if $\chi(L) = 2$ even asymptotic formulas are rarely known.

Now we state Turán's theorem in the exact form:

We use—as above—the notation $f(n; k)$ instead of $f(n; K_k)$.

Theorem (TURÁN). *Let $n = (k-1)t + r$; $0 \leqq r < k-1$. Then*

$$f(n; k) = \frac{1}{2}\frac{k-2}{k-2}n^2 - \frac{1}{2}r\left(1 - \frac{r}{k-1}\right).$$

The only graph (up to isomorphism) of n vertices and $f(n; k)$ edges which does not contain a K_k is the complete $(k-1)$-chromatic graph $K_{k-1}(n_1, \ldots, n_{k-1})$ with n_i vertices in its i'th class where

$$n = n_1 + \ldots + n_k \quad \text{and} \quad |n_i - n_j| \leqq 1 \quad \text{for} \quad 1 \leqq i < j \leqq n.$$

In this note we first of all prove a generalization of this theorem.

We need the following definition and notations:

Let $G=\langle V;E\rangle$ be a graph with vertex set $V=\{v_1,\ldots,v_n\}$ an edge-set E. The star $S_G^i=\langle V_i;E_i\rangle$ of a vertex $v_i\in V$ is the graph spanned by those vertices of G which are joined by an edge to v_i. (*I.e.* $E_i=\{(v_\nu,v_\mu):(v_\nu,v_\mu)\in E, v_\nu, v_\mu\in V_i\}$.) Let $d_i=|V_i|$, $e_i=|E_i|$ for $i=1,\ldots,n$.

The above described $K_{k-1}(n_1,\ldots,n_{k-1})$ will be denoted by $T_{k-1,n}$.

Theorem. *Let $G=\langle V;E\rangle$ be a graph with $|V|=n$ and $e=|E|>f(n;k)$. Then for at least one vertex $v_i\in V$ we have*

$$e_i>f(d_i;k-1).$$

The only graph G with $|V|=n$, $|E|=f(n;k)$ and $e_i\leqq f(d_i;k-1)$ for $i=1,\ldots,n$ is the "Turán-graph" $T_{k-1,n}$.

This theorem clearly implies Turán's theorem. At first sight it seems to be deeper but it turns out that the proof is very simple.

Proof. Let

$$d_i\equiv r_i \bmod (k-2)\,;\quad 0\leqq r_i<k-2\,.$$

Supposing

$$e_i\leqq f(d_i;k-1)\quad\text{for}\quad 1\leqq i\leqq n \tag{1}$$

we shall show

$$e\leqq f(n;k)\,. \tag{2}$$

Since $\sum_1^n e_i=3T$ where T is the number of triangles in G and

$$3T\geqq \sum_{(v_i,v_j)\in E}((d_i+d_j)-n)=\sum_1^n d_i^2-en\,, \tag{3}$$

we have

$$\begin{gathered}\sum_{i=1}^n d_i^2-en\leqq\sum_{i=1}^n e_i\leqq\sum_{i=1}^n f(d_i;k-1)\\ \sum_{i=1}^n d_i^2-\sum_{i=1}^n f(d_i;k-1)\leqq en\end{gathered} \tag{4}$$

Using $\sum_{i=1}^n d_i=2e$ as the first simple result we get

$$\sum_{i=1}^n d_i^2-\sum_{i=1}^n f(d_i;k-1)=\frac{1}{2}\frac{k-1}{k-2}\sum_{i=1}^n d_i^2+\sum_{i=1}^n\frac{r_i}{2}\left(1-\frac{r_i}{k-2}\right)\geqq\frac{1}{2}\frac{k-1}{k-2}\left(\frac{2e}{n}\right)^2 n\,. \tag{5}$$

From (4) and (5) we get

$$e \leqq \frac{1}{2}\frac{k-2}{k-1}n^2 . \tag{6}$$

This gives the desired result in the case $k-1/n$. To get (2) for the general case write

$$e = \frac{1}{2}\frac{k-2}{k-1}n^2 - \frac{r}{2}\left(1 - \frac{r}{k-1}\right) + \Delta . \tag{7}$$

From (6) we know already that

$$\Delta \leqq \frac{r}{2}\left(1 - \frac{r}{k-1}\right)$$

and we may suppose $\Delta > 0$.

As one can see easily, under the condition

$$\sum_{i=1}^{n} d_i = 2e$$

the minimal value of $\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} f(d_i; k-2)$ is taken for the system (d_i^*) where

$$|d_i^* - d_j^*| \leqq 1 ; \quad 1 \leqq i < j \leqq n .$$

For this (d_i^*) system we get easily—for a suitable choice of the indices—

$$d_i^* = \begin{cases} n-(t+1) = (k-2)t + r & \text{if } \ 1 \leqq i \leqq rt - 2\Delta \\ n-t = (k-2)t + r - 1 & \text{if } \ rt - 2\Delta < i \leqq n = (k-1)t + r \end{cases} \tag{9}$$

(from $\Delta \leqq \frac{r}{2}\left(1 - \frac{r}{k-2}\right)$ we have $2\Delta \leqq r$), and consequently

$$r_i^* = \begin{cases} r, & \text{if } \ 1 \leqq i \leqq rt - 2 \\ r-1, & \text{if } \ \mathrm{rt} - 2\Delta < r \leqq (k-1)t + r . \end{cases} \tag{10}$$

Hence, as an easy computation shows

$$\sum d_i^2 - \sum f(d_i, k-2) \geqq f(n; k) + \frac{k-1}{k-2}\Delta(2(n-t-1) + (k-2) - 2r + 1) . \tag{11}$$

From (4) and this

$$\frac{k-1}{k-2}\Delta(2(n-t-1) + k - 2 - 2r + 1) \leqq \Delta n$$

follows which contradicts to $\Delta > 0$.

From the above reasoning it also follows easily that

$$e = f(n; k) \quad \text{and}$$

(12)

$$e_i \leqq f(d_i; k-2) \quad \text{for} \quad i = 1, \ldots, n$$

holds if and only if G is the "Turán-graph" $T_{k-1,n}$.

Namely (12) holds only if we have equality in (11), our system (d_i) is the system (d_i^*) and we have equality in (3).

Problems and remarks

One can try to find other extremal problems for which an analogous method applies.

(a) Consider the case of a pyramid. A pyramid P is a graph where $V = \{x, y_1, \ldots, y_k\}$ and $E = \{(x, y_i), (y_i, y_{i+1}); 1 \leqq i \leqq k\}$ with $y_{k+1} = y_k$. It is known [6] that any $G(V; E)$ with $|V| = n$ and $|E| \geqq \frac{n^2}{4} + \frac{n}{2}$ contains a pyramid. By a similar method we used one can show that a graph with $e \geqq \frac{n^2}{4} + \frac{n}{2}$ contains a vertex where S_G^i contains more than d_i edges.

This phenomenon does not remain valid if we fix the size of the pyramid.

In the above examples the L graphs under consideration were such that they had a vertex which is joined to all other vertices and in this case we had to consider only the S_G^i star graphs; the existence of a proper subgraph in S_G^i assured the existence of a subgraph in G isomorphic to L. Now an analogous phenomenon may occur for complete bipartite graphs.

(b) Let us consider the case of a $K_{2,2,2}$. In Erdős and Simonovits [6] proved that

$$f(n; K_{2,2,2}) = \frac{n^2}{4} + cn^{3/2} + O(n^{3/2})$$

for a certain c which value can be determined. A relatively simple computation proves that every G with $|V| = n$, $|E| = \frac{n^2}{4} + c^* n^{3/2}$ contains two vertices for whose the intersection of their stars contains enough edges to ensure a c_4. But we have not proved that $c^* = c$.

Added in proof. The same result was also proved by BOLLOBÁS and ELDRIDGE and recently a very simple proof was found by BONDY which will appear soon.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H-1053 BUDAPEST, REÁLTANODA U. 13—15.
HUNGARY

EÖTVÖS LORÁND UNIVERSITY
H-1088 BUDAPEST, MÚZEUM KRT. 6—8.
HUNGARY