Studies in Pure Mathematics
To the Memory of Paul Turán

On strong irregularities of the distribution of $\{n\alpha\}$ sequences

by

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Abstract

Let $U=\{u_n\}$ be a sequence in $[0, 1]^k$ and $\Delta_N^U = \sup_I |\Delta_N(I)| = \sup_I \left| \sum_{\substack{u_n \in I \\ 1 \leq n \leq N}} 1 - N|I| \right|$ (I a subinterval of $[0, 1]^k$). By Schmidt's theorem $\Delta_N > c_1 \log N$ for any N if $k=2$ while for $k=1$ only $\overline{\lim} \frac{\Delta_N}{\log N} > c_2 > 0$ holds and we have sequences (e.g. $\{n\alpha\}$ sequences) for which $\Delta_N \leq 1$ for infinitely many N. Inspite of this fact we have the following Theorem: Let $u_n = \{n\alpha\}$. With a suitable $\delta \in (0, 1)$ and for every $N > N_0$

$$\Delta_n > c_3 \log N$$

holds for all but at most N^δ values of n, $1 \leq n \leq N$. (Here $c_3 > 0$ is an absolute constant.)

Introduction

Let $E^k = \{(x_1, \ldots, x_k) \in R^k,\ 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq k\}$ be the unit cube in R^k and for $\mathbf{x} \in E^k$.

$I(\mathbf{x}) = \{(t_1, \ldots, t_k): 0 \leq t_i \leq x_i \text{ for } 1 \leq i \leq k\}$; $|I| = \prod_{i=1}^{k} x_i$. For a sequence $\{\mathbf{u}_n\}$, $\mathbf{u}_n \in E^k$ put

$$\Delta_N(I) = \sum_{\substack{\mathbf{u}_n \in I \\ 1 \leq n \leq N}} 1 - N|I| \tag{1}$$

and

$$\|\Delta_N\|_p = \Big(\int_{E^k} |\Delta_N(I(\mathbf{x}))|^p \, dx \Big)^{\frac{1}{p}} ; \tag{2}$$

$$\Delta_N = \|\Delta_N\|_\infty . \tag{3}$$

The infinite sequence $\{\mathbf{u}_n\}$ is uniformly distributed in E^k if $\lim_{N \to \infty} \frac{1}{N} \Delta_N = 0$. $D_N =: \frac{1}{N} \Delta_N$ is called the discrepancy function of the sequence $\{\mathbf{u}_n\}$. Δ_N resp. $\|\Delta_N\|_p$ measures in certain respect the irregularity of the finite sequence $\mathbf{u}_1, \ldots, \mathbf{u}_N$, their behaviour for $N \to \infty$ describes the irregularity of the infinite sequence $\{\mathbf{u}_n\}$.

It was conjectured by VAN DER COPRUT and proved first by VAN AARDENNE–EHRENFEST [22], [23] that for any infinite sequence $\{\mathbf{u}_n\}$ we have

$$\overline{\lim_{N \to \infty}} \Delta_N = \infty$$

i.e. there is no "too well" distributed sequence.

We recall some results which show how the situation changes with increasing dimension.

K. F. ROTH [13] proved, that for all $k \geqq 1$.

(A) for *any infinite* sequence $\{\mathbf{u}_n\}$ in E^k, for any $N > N_k$ there exists an n, $1 \leqq n \leqq N$ such that

$$\|\Delta_n\|_2 > c_k \log^{\frac{k}{2}} N$$

and consequently also

$$\Delta_n > c_k \log^{\frac{k}{2}} N .$$

(B) for *any N points* $\mathbf{u}_1, \ldots, \mathbf{u}_N$ in E^k with $N > N'_k$

$$\|\Delta_N\|_2 > c'_k \log^{\frac{k-1}{2}} N$$

and consequently also

$$\Delta_N > c'_k \log^{\frac{k-1}{2}} N .$$

Here N_k, N'_k, $c_k > 0$, $c'_k > 0$ depend only on k and are absolute constants.

ROTH also proved directly that the case (A) for k-dimension is equivalent to the case (B) for $k-1$ dimension.

Best possible results concerning the order of magnitudes of Δ_N are known only for $k=1$ and for finite sequences also for $k=2$. Namely, W. G. SCHMIDT [17] proved

(A^+) for any infinite sequence $(\mathbf{u}_n)$ in E^1 and for any $N > N_1$ there exists an n, $1 \leqq n \leqq N$ such that

$$\Delta_n > c \log N \tag{4}$$

and

(B^+) for any N points $\mathbf{u}_1, \ldots, \mathbf{u}_N$ in E^2 with $N > N_2$

$$\Delta_N > c' \log N . \tag{5}$$

(Here $c > 0$, $c' > 0$ are effective constants; the best possible constants are not known.)

As to the sharpness of these results, it is well known that there exist sequences in E^1 for which $\Delta_N = O(\log N)$. ROTH [13] constructed finite sequences in E^2 for which $\Delta_N = O(\log N)$. The best possible result concerning the order of magnitude of Δ_N is not known for $k>2$. However for $\|\Delta_N\|_2$ DAVENPORT [3] constructed finite sequences in E^2 for which $\|\Delta_N\|_2 = O(\log^{\frac{1}{2}} N)$ and quite recently for any $k>2$ ROTH [14], [15] constructed finite sequences in E^k for which

$$\|\Delta_N\|_2 = O(\log N^{\frac{k-1}{2}}).$$

The above results show, that the irregularities of a sequence increase with increasing dimension, which can be expressed in a quantitative form. Moreover, from $k=1$ to $k=2$ this phenomenon has also a qualitative feature.

Namely, for $k=1$ for *any* N we have sequences with $\Delta_N \leqq 1$, for example for the equipartition of E^1: $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}$ we have $\Delta_N \leqq 1$, while for $k \geqq 1$

$$\Delta_N \geqq c_k (\log N)^{\frac{k-1}{2}}$$

and consequently for infinite sequences for $k \geqq 2$

$$\lim_{N\to\infty} \Delta_N = \infty$$

while for $k=1$ we have only

$$\overline{\lim_{N\to\infty}}\, \Delta_N = \infty .$$

(For $k \geqq 2$ a "good equipartition" does not exist.) There are sequences in E^1 for which

$$\Delta_N \leqq 1$$

for infinitely many N. Now the question we are interested in is, the following: for a given sequence $\{\mathbf{u}_n\}$ in E^1 and for a fixed C how often may e.g.

$$\Delta_N \leqq C \tag{7}$$

hold, how often must

$$\Delta_N > c \log N \tag{8}$$

hold. The above theorem leaves open the possibility that for some sequences (8) holds with any $c>0$ only for a sequence of integers of 0 density. The theorems we are going to prove show that at least for $(\{n\alpha\})$ sequences this is not possible.

As to $(\{n\alpha\})$ sequences it was proved already by Hardy–Littlewood [6] and Ostrowski [11] that for any $(\{n\alpha\})$ sequence

$$\Delta_N > c \log N \tag{9}$$

holds for infinitely many N, (where c is a positive absolute constant). This is a "best possible" result concerning the order of magnitude since for any α with bounded partial quotients

$$\Delta_N = O(\log N). \tag{10}$$

(10) means that concerning the order of magnitude of Δ_N the $(\{n\alpha\})$ sequences for α with bounded partial quotients are among the "best" sequences.[1]

We also know that for any α

$$\Delta_n \leqq 1$$

holds for infinitely many n, e.g. for $n_i = q_i$ $(i=1, \dots)$ where q_i's are the denominators of the convergents of α.

The above results suggest that the behaviour of $(\{n\alpha\})$ sequences is quite characteristic for the general situation. Probably results analogous to the ones formulated below, hold for arbitrary sequences too.* See also Remark 2.

Here we are going to prove

Theorem. *Let α be irrational and Δ_n be defined by* (3) *belonging to the sequence $(\{k\alpha\})$. With a suitable $\vartheta \in (0, 1)$, for $N > N_0$*

$$\Delta_n > c \log N \tag{11}$$

holds for all but at most N^ϑ values of n; $1 \leqq n \leqq N$. Here $c > 0$ is an absolute constant.

Without proof we mention the following

Proposition 1. *Let α be irrational, $\alpha = [a_1, a_2, \dots]$ be the continued fraction expansion of α, q_i $(i=1, \dots)$ the denominators of the convergents of α. Then for every N*

$$\frac{1}{N} \sum_{n=1}^{N} \Delta_n > c \left(\sum_{k=1}^{\nu} (a_k + 1) + \frac{N}{q_{\nu+1}} \right) \tag{12}$$

where ν is determined by $q_{\nu+1} \leqq N < q_{\nu+2}$. Here $c > 0$ is an absolute constant.

[1] As to the best possible constant in (9) with Y. Dupain we proved in [2] that for

$$c(\alpha) = \overline{\lim_N} \frac{\Delta_N}{\log N}$$

we have

$$\inf_\alpha c(\alpha) = \min_\alpha c(\alpha) = c(\sqrt{2} - 1) = \frac{1}{4 \log(\sqrt{2} + 1)} \sim 0.2836.$$

* See "Added in proof".

Remark 1. Proposition 1 asserts more than our theorem in the case when α has "large" partial quotients; $\overline{\lim} \frac{\sum a_k}{\log N} = \infty$. The result in KUIPERS–NIEDERREITER [9], that for every n; $1 \leqq n \leqq N$

$$\Delta_n < c'\left(\sum_{k=1}^{\nu} (a_k + 1) + \frac{N}{q_{\nu+1}} \right)$$

holds with an absolute constant c', shows that in certain sense (12) is best possible.

Proposition 2. *Let α be irrational and $\Delta_n(\beta) = : \Delta_n([0, \beta])$ defined by* (1) *belonging to the sequence* ($\{k\alpha\}$). *Then for almost all $\beta \in (0, 1)$ we have*

$$\overline{\lim_{N \to \infty}} \frac{|\Delta_N(\beta)|}{\log N} > c\,. \tag{13}$$

Here $c > 0$ is an absolute constant.

Moreover, the exeptional set- the set of β's, for which (13) *does not hold — has Hausdorff-dimension* 0.

Remark 2. KESTEN [8] proved that for $\{n\alpha\}$ sequences $\Delta_n(I)$ is bounded only if $|I| = \{k\alpha\}$ for some integer k (and it is bounded for $|I| = \{k\alpha\}$ according to a theorem of HECKE [7])[2]

SCHMIDT [17] proved, that for any sequence the lengths of all intervals for which $\Delta_N(I)$ is bounded form a countable set. Moreover, a recent result of SCHMIDT [17] states that for any sequence

$$\lim_{N \to \infty} \frac{|\Delta_N(\beta)|}{\log \log N} > c$$

holds for almost every β, were $c > 0$ is an absolute constant. In [17] SCHMIDT asks whether the analogous result holds with $\log N$ instead of $\log \log N$. So Proposition 2 gives an affirmative answer in the case of $\{n\alpha\}$ sequences.

For the proofs of Proposition 1 and Proposition 2, see V. T. Sós [21].

[2] For ergodic-theoretical generalizations and proofs of KESTEN's theorem see e.g. FÜRSTENBERG–KEYNES–SHAPIRO [4], HALÁSZ [5], PETERSEN [12].

Notation

Let $\alpha=[a_1, \ldots, a_n, \ldots]$ be the continued fractions expansion of α. We shall use the notations and consequences

$$\frac{p_n}{q_n}=[a_1, \ldots, a_{n-1}]\,; \quad q_{n+1}=a_n q_n+q_{n-1}, \quad p_{n+1}=a_n p_n+p_{n-1},$$

$$\Theta_n=q_n\alpha-p_n\,; \quad \Theta_{n+1}=a_n\Theta_n+\Theta_{n-1},$$

$$\lambda_n=|\Theta_n|=(-1)^{n+1}\Theta_n\,;$$

$$\sum_{\nu=0}^{\infty} a_{k+2\nu}\Theta_{k+2\nu}=-\Theta_{k-1}\,; \quad k=1, \ldots \quad (\Theta_0=-1)$$

$$\sum_{\nu=0}^{n} a_{k+2\nu}q_{k+2\nu}=q_{k+2n+1}-q_{k-1}\,; \quad k=1, \ldots \quad (q_0=0).$$

We shall say that the sequence of integers $(b_1, \ldots, b_\nu)$ is a "permitted" sequence if it satisfies

(14) $$0\leqq b_1\leqq a_1-1, \quad 0\leqq b_k\leqq a_k \quad \text{if} \quad 2\leqq k\leqq \nu$$

and

(15) $$b_k=0 \quad \text{if} \quad b_{k+1}=a_{k+1} \quad (1\leqq k\leqq \nu-1).$$

It is well known that every positive integer $N<q_{\nu+1}$ can be uniquely represented in the form

(16) $$N = \sum_{k=1}^{\nu} b_k q_k$$

where $(b_1, \ldots, b_\nu)$ is a "permitted" sequence (and conversely, for every "permitted" sequence $(b_1, \ldots, b_\nu)$

$$N= \sum_{\nu=1}^{\nu} b_i q_i<q_{\nu+1}).$$

It is also known (Descombes [1], Sós [20], Lesca [10]) that each β with $-\alpha\leqq\beta<$ $<1-\alpha$ can be represented in the form

(17) $$\beta = \sum_{k=1}^{\infty} d_k\Theta_k$$

where (d_k) is a "permitted" infinite sequence which satisfies

(18) $$0\leqq d_1\leqq a_1-1, \quad 0\leqq d_k\leqq a_k \quad \text{if} \quad k\geqq 2,$$

$$d_k=0 \quad \text{if} \quad d_{k+1}=a_{k+1}\,, \tag{19}$$

$$d_{2k+1}\neq a_{2k+1} \tag{20}$$

for infinitely many positive integer k.

Conversely, every sequence which satisfies (18)–(20) by (17) determines a $\beta\in(-\alpha,\,1-\alpha)$.

The expansions above turned out to be useful for different types of investigations in diophantine approximation. Our proof will be based on the result that it is possible to handle Δ_N by these expansions.

Let

$$\Delta_N(\beta)=\Delta_N([0,\beta)) \quad \text{for} \quad 0<\beta<1$$

and

$$\Delta_N(\beta)=\Delta_N([0,1+\beta)) \quad \text{for} \quad -\alpha<\beta<0\,.$$

We shall use with the notation of (16) and (17) the "explicit" formula for $\Delta_N(\beta)$ (in this form see T. Sós [19]).

$$\Delta_N(\beta) = \sum_{k=1}^{\infty}((-1)^{k+1}\min(b_k,d_k)-d_k(q_k\sum_{i=k+1}^{\infty}b_i\Theta_i+\Theta_k\sum_{i=1}^{k}b_iq_i))+\sum_{k=1}^{\nu}\delta_k\,, \tag{21}$$

where

$$\delta_k=\begin{cases} 1, \text{ if } k \text{ is odd},\ d_k>b_k & \text{and } \sum_{i=1}^{k-1}b_iq_i>\sum_{i=1}^{k-1}d_iq_i \\ -1 \text{ if } k \text{ is even},\ d_k<b_k & \text{and } \sum_{i=1}^{k-1}b_iq_i\leqq\sum_{i=1}^{k-1}d_iq_i \\ 0 \text{ otherwise.} & \end{cases} \tag{22}$$

Proof of the Theorem

Let α be fixed. Without loss of generality we may assume that $N=c_{\nu+1}q_{\nu+1}$ for some ν with $0<c_{\nu+1}\leqq a_{\nu+1}$, $c_{\nu+1}$ integer.

For any n, $1\leqq n\leqq N$ we consider the expansion

$$n=\sum_{k=1}^{\nu+1}b_k(n)q_k$$

where b_i satisfies (14)–(15) if $1\leqq i\leqq\nu$ and $0\leqq b_{\nu+1}<c_{\nu+1}$. We shall write b_k instead of $b_k(n)$ when it is not misunderstandable. Instead of determining the number of values of n, $1\leqq n\leqq N$ with certain conditions on $b_k(n)$; $1\leqq k\leqq\nu+1$, we shall determine the permitted sequences with the given properties.

In order to obtain a good lower bound for Δ_n we shall define a β_n^+ resp. a β_n^- for which $\Delta_n(\beta_n^+)$ or $-\Delta_n(\beta_n^-)$ is large and we use

$$\Delta_n \geqq \frac{1}{2}(\Delta_n(\beta_n^+) - \Delta_n(\beta_n^-)).$$

Let $1 \leqq n \leqq N$, $n = \sum_{i=1}^{\nu+1} b_i q_i$. Let β_n^+, β_n^- be defined by

$$\beta_n^+ = \sum_{2k+1 \leqq \nu+1} b_{2k+1}\lambda_{2k+1}$$

$$\beta_n^- = - \sum_{2k \leqq \nu+1} b_{2k}\lambda_{2k}.$$

First for the values of δ_k in (22) we remark the following: For β_n^+ we have $\delta_{2k+1}=0$ since $d_{2k+1}=b_{2k+1}$ $(k=1, \dots)$. Since $d_{2i}=0$, $d_{2i+1}=b_{2i+1}$; we have $\sum_{i=1}^{k-1} b_i q_i \leqq \sum_{i=1}^{k-1} d_i q_i$ iff $b_{2i}=0$ for $i<k$. This means that $\delta_{2k}=-1$ for at most one value of k. For β_n^- we have $\delta_{2k+1}=0$, since $d_{2k+1}=0$ and $\delta_{2k}=0$, since $d_{2k}=b_{2k}$.

Hence using the discrepancy-formula (21) we get

$$\Delta_n \geqq \frac{1}{2}(\Delta_n(\beta_n^+) - \Delta_n(\beta_n^-)) =$$

(23)
$$= \frac{1}{2}\sum_{k=1}^{\nu+1} b_k(1 - b_k q_k \lambda_k - \lambda_k \sum_{i<k} b_i q_i + q_k \sum_{i>k} (-1)^{i+1-k} b_i \lambda_i) - 1.$$

Now we consider the k'th term:

$$s_k =: b_k(1 - b_k q_k \lambda_k - \lambda_k \sum_{i<k} b_i q_i + q_i \sum_{i>k} (-1)^{i+1-k} b_i \lambda_i).$$

Using

$$1 = q_{k+1}\lambda_k + q_k\lambda_{k+1} = a_k q_k \lambda_k + \lambda_k q_{k-1} + q_k \lambda_{k+1},$$

$$\sum_{i>k} (-1)^{i+1-k} b_i \lambda_i > -\lambda_{k+1},$$

$$\sum_{i<k} b_i q_i < q_k,$$

and in case $b_k = a_k$ $(b_{k-1}=0)$

$$\sum_{i<k} b_i q_i = \sum_{i<k-1} b_i q_i < q_{k-1},$$

we get

$$s_k \geqq 0 \quad k=1, \ldots, v+1\,. \tag{24}$$

We also have

$$s_k \geqq b_k\left((a_k - b_k)\lambda_k q_k + \lambda_k q_{k-1} - \lambda_k \sum_{i=1}^{k-1} b_i q_i\right).$$

By this we have the following positive lower bounds:

$$s_k > b_k(a_k - b_k - 1)\lambda_k q_k > \frac{b_k(a_k - b_k - 1)}{a_k + 2} \quad \text{if} \quad 0 < b_k < a_k - 1 \tag{25}$$

$$s_k > b_k \lambda_k q_{k-1} > \frac{1}{(a_k+2)(a_{k-1}+1)} \quad \text{if} \quad 0 < b_k = a_k - 1 \tag{26}$$

$$s_k > \lambda_k(q_{k-1} - q_{k-2}) > \left(\prod_{i=0}^{3} a_{k-i} + 2\right)^{-1} \quad \text{if} \quad b_k = a_k,\, b_{k-1} = b_{k-2} = 0\,. \tag{27}$$

In order to prove the Theorem we shall show that for all but at most N^9 values of n, $1 \leqq n \leqq N$ at least one of the three cases holds for many values of k, and moreover in such cases these terms give an essential contribution to Δ_n. To prove this we need the following lemmas

Lemma 1. *Let* $N = c_{v+1} q_{v+1}$ *for an integer* $c_{v+1} \in [1, a_{v+1}]$, $1 < k_1 < \ldots < k_l \leqq \leqq v+1$, $\dfrac{3}{a_{k_i}+3} < t_{k_i} < 1$ *for* $i=1, \ldots, l$ *and* $S(t_1, \ldots, t_l)$ *be the number of "permitted" sequences* $(b_1, \ldots, b_{v+1})$ *which satisfy also* $b_{v+1} < c_{v+1}$ *and*

$$\min(b_{k_i}, a_{k_i} - b_{k_i} - 1) < t_{k_i} a_{k_i}, \quad i = 1, \quad \ldots, \quad l,$$

$$\min(b_{v+1}, c_{v+1} - b_{v+1}) < t_{v+1} c_{v+1} \quad \text{if} \quad k_l = v+1\,.$$

Then

$$S(t_1, \ldots, t_l) < \prod_{i=1}^{l} (4 t_{k_i})\, c_{v+1} q_{v+1}\,. \tag{28}$$

Proof. Note that the total number of "permitted" sequences $(b_1, \ldots, b_{v+1})$ for which $b_{v+1} < c_{v+1}$ holds is just $c_{v+1} q_{v+1}$.

Now first let $l=1$, $k_1 = k \leqq v$, $t_1 = t$. Assumption (18) means, that b_k can take only the values

$$0, \ldots, [t a_k],\, a_k - [t a_k] - 1, \ldots, a_k\,. \tag{29}$$

Put $a_k' = 2[t a_k] + 2$ and $q_k' = q_k$. Now the number of "permitted" sequences under the restriction that b_k can take only the $a_k' + 1$ different values in (29) is $c_{v+1} q_{v+1}'$ where q_{v+1}'

is determined by the recursive formulae

$$q'_{k+1}=a'_k q_k+q_{k-1},\ q'_k=q_k$$

$$q'_{k+j+1}=a_{k+j}q'_{k+j}+q'_{k+j-1} \quad \text{if} \quad 1\leqq j\leqq v-k\,.$$

Since

$$q'_{k+1}=\frac{a'_k q_k+q_{k-1}}{a_k q_k+q_{k-1}}q_{k+1}<\frac{a'_k+1}{a_k+1}q_{k+1}<3tq_{k+1}$$

and

$$q'_{k+2}=\frac{a_{k+1}q'_{k+1}+q_k}{a_{k+1}q_{k+1}+q_k}q_{k+2}<\frac{3ta_{k+1}+1}{a_{k+1}+1}q_{k+2}<4tq_{k+2}$$

we get (28) for $l=1$ by induction on j.

A similar argument holds in case $k=v+1$. Now by induction on l we get (28) for the general cases.

Lemma 2. *Let* $\delta>0$, $M>M_0(\delta)$

$$K_1=\{k: a_k\geqq M;\ 1\leqq k\leqq v+1\}$$

$$a'_k=\begin{cases} a_k, & \text{if} \quad k\leqq r\\ c_{r+1}, & \text{if} \quad k=r+1\end{cases}$$

and

$$B_1(n)=\{k: \min(b_k(n),\, a'_k-b_k(n)-1)<\frac{1}{4}\log a'_k,\ k\in K_1\}\,,$$

$$N_1=\{n: \sum_{k\in B_1(n)}\log a'_k>\delta\log N, \quad 1\leqq n\leqq N\}\,.$$

Then with a suitable $\vartheta\in(0,1)$

$$|N_1|<N^{1-\vartheta}\,.$$

Proof. By the definition of N_1 we have $\prod\limits_{k\in B_1(n)} a'_k>N^\delta$ if $n\in N_1$.

Let $A\subseteq K_1$ and

$$N_1(A)=\{n: B_1(n)=A,\ n\in N_1\}\,.$$

Let us fix a $\Theta\in(0,1)$. Then for $M>M_0$

$$\prod_{k\in A}(a_k'^{-1}\log a'_k)<\prod_{k\in A}a_k'^{-1+\Theta}<N^{-\delta(1-\Theta)}\,.$$

By Lemma 1

$$|N_1(A)| < N \cdot N^{-\delta(1-\Theta)}$$

and consequently by summation on A we obtain

$$|N_1| = \sum_A |N_1(A)| < N \cdot 2^{v_1} N^{-\delta(1-\Theta)}.$$

Now taking into consideration, that

$$N > \prod_{a'_k \geqq M} a'_k \geqq M^{v_1} \quad \text{and} \quad 2 < M^{\frac{\delta(1-\Theta)}{2}}, \quad \text{if} \quad M > M_0,$$

we obtain

$$|N_1| < N \cdot N^{\frac{\delta(1-\Theta)}{2}} \cdot N^{-\delta(1-\Theta)} = N^{1-\frac{1}{2}\delta(1-\Theta)}$$

Lemma 3. *Let* $K_2 = \{k: 2 \leqq a_k < M,\ 1 \leqq k \leqq v\}$ *and* $K'_2 \subset K_2$, $v'_2 = |K'_2|$,

$$B_2(n) = \{k: b_k \in \{0, a_k\}, k \in K'_2\},$$

$$N_2' = \{n: |B_2(n)| > (1-\delta)v'_2;\ 1 \leqq n \leqq N\}.$$

If $v'_2 > v_0$, $\delta > \delta_0$, *then* $|N_2| < \Theta^{v'_2} N$ *with a suitable* $\Theta = \Theta(\delta) \in (0, 1)$.

Proof. First let $1 \leqq k_1 < \ldots < k_l \leqq v$, $k_i \in K'_2$ $(i = 1, \ldots, l)$ and $S_l(k_1, \ldots, k_l)$ be the number of "permitted" sequences $b_1, \ldots, b_{v+1}$ satisfying

$$b_{k_i} \in \{0, a_{k_i}\} \quad i = 1, \ldots,$$

We shall prove

$$S_l \leqq \left(\frac{5}{6}\right)^l N. \tag{30}$$

First let $l = 1$, $k = k_1 \leqq v$. Similarly to the proof of Lemma 1 put

$$a'_k = 1, q'_k = q_k \quad \text{and} \quad q'_{k+1} = q_k + q_{k-1}$$

$$q'_{k+j+1} = a_{k+1} q'_{k+j} + q'_{k+j-1} \quad \text{for} \quad 1 \leqq j \leqq v - k.$$

Now we have

$$q'_{k+1} = \frac{q_k + q_{k-1}}{a_k q_k + q_{k-1}} q_{k+1} < \frac{2}{3} q_{k+1}$$

$$q'_{k+2} = \frac{a_{k+1} q'_{k+1} + q_k}{a_{k+1} q_{k+1} + q_k} q_{k+2} < \frac{\frac{2}{3} a_{k+1} + 1}{a_{k+1}} q_{k+2} < \frac{5}{6} q_{k+2}.$$

By induction on j we get

$$S_1(k) = q'_{\nu+1} < \left(\frac{5}{6}\right) q_{\nu+1} .$$

Hence by induction on l we obtain (30).

By (30) we have

$$|N_2| < \sum_{t > (1-\delta)\nu'_2} \binom{\nu'_2}{t} \left(\frac{5}{6}\right)^t N < \Theta^{\nu'_2} N$$

if $\nu'_2 > \nu_0(\delta)$.

Lemma 4. *Let*

$$K_3 = \{k : a_k = 1, 1 \leqq k \leqq \nu\}, \nu_3 = |K_3| > \frac{97}{100}\nu ,$$

$$B_3(n) = \{k : b_{k-2} = b_{k-1} = 0, b_k = 1, a_{k-i} = 1 \quad \text{for} \quad 0 \leqq i \leqq 3, \quad 1 \leqq k \leqq \nu\}$$

$$N_3 = \{n : |B_3(n)| < \delta\nu_3, 1 \leqq n \leqq N\} .$$

Then, with a suitable $\Theta \in (0, 1)$. $|N_3| < \Theta^\nu N$, *if* $\delta > \delta_0$, $r > r_0$.

Proof. Consider the blocks of indices $I_j = \{10j + i, 1 \leqq i \leqq 10\}$ for $0 \leqq j \leqq \left[\frac{\nu}{10}\right]$ and let

$$J = \{I_j : a_k = 1 \quad \text{if} \quad k \in I_j\}.$$

By the assumption $\nu_3 > \frac{97}{100}\nu$ we have $|J| > \frac{\nu}{20}$. Now we consider only the blocks in J and we shall show that for all but at most N values of n, $1 \leqq n \leqq N$

$$B_3(n) \cap I_j \neq \emptyset$$

for at least $10^{-2}\nu_3$ values of j, $I_j \in J$.

Let $1 \leqq j_1 < \ldots < j_l \leqq \nu$, $I_{j_i} \in J$ and $S_l(j_1, \ldots, j_l) = \{n : B_3(n) \cap I_{j_i} = \emptyset, i = 1, \ldots, l\}$. Then

(31) $$|S_l| < 2^{-l} N .$$

To see this we have to remark only that

(a) the number of permitted 0,1 — sequences $d_1, \ldots, d_{10}$ for which

(32) $$(d_i, d_{i+1}, d_{i+2}) \neq (0, 0, 1) \quad \text{if} \quad 1 \leqq i \leqq 8$$

is 11.

(b) the number of "permitted" 0,1 — sequences $d_1, \ldots, d_{10}$ for which

$$d_1=0, \quad d_i=d_{i+1}=0, \quad d_{i+2}=1, \quad d_{10}=0 \tag{33}$$

with some $1 \leqq i \leqq 6$ is >22.

Consequently to any $n \in S_l(j_1, \ldots, j_l)$ by replacing the blocks $(b_{10_j+i}, 1 \leqq i \leqq 10)$ of type (32) for blocks of type (33) we can order 2 different $n \notin S_l$ on such a way that to $n_1 \neq n_2$ we order different ones. ($d_1 = d_{10} = 0$ in (33) gives the possibility to choose sequences of type (33) for different blocks I_{j_1}, I_{j_2} independently.)

By (31), with the notation $v' = \dfrac{v_3}{20}$

$$|N_3| < \sum_{l>(1-\delta)v} \binom{v'}{t} 2^{-t} N < \Theta^{v'} N$$

if $v_3 > v_0$.

To finish the proof we distinguish three cases:

Case 1. Suppose, with the notation of Lemma 2,

$$\sum_{k \in K_1} \log(a_k+1) > \frac{1}{200} \log N . \tag{33}$$

Let $n \notin N_1$. Then by Lemma 2, (23), (24) and (25) we have

$$\Delta_n \geqq \sum_{k=1}^{v+1} s_k \geqq \sum_{k \in K_1} s_k \geqq \sum_{k \in K_1 \setminus B_1(n)} \log(a_k+1) =$$

$$= \sum_{k \in K_1} \log(a_k+1) - \sum_{k \in B_1(n)} \log(a_k+1) > \frac{1}{5 \cdot 10^3} \log N .$$

Case 2. Let $K_2' = \{k: 2 \leqq a_k < M,\ a_{k-1} \leqq M,\ 1 \leqq k \leqq v\}$. Suppose, with the notation of Lemma 2 and Lemma 3,

$$\sum_{k \in K_1} \log(a_k+1) < \frac{1}{200} \log N$$

and

$$\sum_{k \in K_2} \log(a_k+1) > \frac{1}{100} \log N . \tag{34}$$

By these we have

$$v_1 = |K_1| < \frac{1}{200 \log(M+1)} \log N$$

$$v_2 = |K_2| > \frac{1}{100 \log(M+1)} \log N$$

and consequently

$$v_2' = |K_2'| > \frac{1}{200 \log (M+1)} \log N .$$

Now we apply Lemma 3. Let $n \notin N_2$. Then by (23), (24) and (26) we have

$$\Delta_n > \sum_{k=1}^{v+1} s_k \geqq \sum_{k \in K_2' \setminus B_2(n)} b_k \lambda_k q_{k-1} \geqq \frac{1}{(M+1)^2} |K_2' \setminus B_2(n)| \geqq$$

$$\geqq \frac{1}{(M+1)^2} \cdot \frac{1}{10^3} \cdot \frac{1}{200} \cdot \frac{1}{\log (M+1)} \cdot \log N .$$

By (34) $v_2' > c_M \log N$. By this and by Lemma 2 we get with a suitable $\Theta \in (0, 1)$ and $\vartheta = \vartheta(\Theta, c_M) \in (0, 1)$ that $|N_2| < \Theta^{v_2'} N < \Theta^{c_M \log N} < N^{\vartheta}$.

Case 3. Suppose neither (33) nor (34) holds. In this case

$$v_1 < \frac{1}{200 \log (M+1)} \log N , \tag{35}$$

$$v_2 < \frac{1}{100 \log 3} \log N \tag{36}$$

and consequently

$$v_1 + v_2 < \frac{3}{200} \log N .$$

Since

$$N = c_{v+1} \prod_{k \in K_1}{}_1 \frac{q_{k+1}}{q_k} \prod_{k \in K_2}{}_2 \frac{q_{k+1}}{q_k} \prod_{\substack{k \notin K_1 \cup K_2 \\ k \leqq v}}{}_3 \frac{q_{k+1}}{q_k}$$

$$\log N = \log c_{v+1} + \Sigma_1 \log (a_k + 1) + \Sigma_2 \log (a_k + 1) + \log \Pi_3 ,$$

we obtain

$$\left(1 - \frac{3}{200}\right) \log N < \log \Pi_3 < v \log 2 < v .$$

By (35) and (36) we get

$$v_3 > \frac{98}{100} v .$$

Now using Lemma 4, by (27) we obtain for $n \notin N_3$

$$\Delta_3 \geqq \sum_{k=1}^{v+1} s_k > \sum_{k \notin B_3(n)} \lambda_k q_{k-3} > \frac{1}{10} |B_3(n)| > \frac{1}{10^3} \log N .$$

Now by the assumptions (35)–(36) we have

$$\log N < 2\nu_3, \qquad N < 10^{\nu_3}.$$

Therefore, using Lemma 4, with a suitable $\vartheta = \vartheta(\Theta) \in (0, 1)$ we have

$$|N_3| < \Theta^{\nu_3} N < N^{\vartheta}.$$

This completes the proof.

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Added in proof. This paper was submitted in 1978. I lectured on this topic and formulated the conjecture concerning arbitrary sequences in 1979 in Oberwolfach. On strong irregularities of the distribution of $(\{n\alpha\})$ sequences, *Tagungsbericht Oberwolfach* 23 (1979), 17–18. Since that G. HALÁSZ (On Roth's Method in the Theory of Irregularities of Point distributions, *Recent Progress in Analytic Number Theory*. Acad. Press, 1981, (79–94)) and R. TIJDEMAN, and G. WAGNER (A sequence has almost nowhere small discrepancy. *Monatshefte für Math.* **90** (1980), 315–329) proved the conjecture and more general results.

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