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Problems and Results on Additive Properties of General Sequences, V

By

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Dedicated to our friend Professor E. Hlawka on the occasion of his seventieth birthday

Abstract. A very special case of one of the theorems of the authors states as follows: Let $1 \leqslant a_1 \leqslant a_2 \leqslant \ldots$ be an infinite sequence of integers for which all the sums $a_i + a_j$, $1 \leqslant i \leqslant j$, are distinct. Then there are infinitely many integers k for which $2k$ can be represented in the form $a_i + a_j$ but $2k + 1$ cannot be represented in this form. Several unsolved problems are stated.

1. Let $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \ldots)$ be an infinite sequence of positive integers. We denote the complement of A by $\bar{A}$:

$$\bar{A} = \{0, 1, 2, \ldots\} - A .$$

Put

$$A(n) = \sum_{\substack{a \leqslant n \\ a \in A}} 1, \quad \bar{A}(n) = \sum_{\substack{a \leqslant n \\ a \notin A}} 1,$$

and for $n = 0, 1, 2, \ldots$ let $R_1(n), R_2(n), R_3(n)$ denote the number of solutions of

$$a_x + a_y = n, \quad a_x \in A, \ a_y \in A \tag{1}$$

$$a_x + a_y = n, \quad x < y, \ a_x \in A, \ a_y \in A \tag{2}$$

and

$$a_x + a_y = n, \quad x \leqslant y, \ a_x \in A, \ a_y \in A, \tag{3}$$

respectively.

In the first four parts of this series (see [3], [4], [5] and [6]) we studied the regularity properties of the functions $R_1(n)$, $R_2(n)$ and $R_3(n)$. In particular, in Part IV, we studied the monotonicity properties of these functions. We proved that the function $R_1(n)$ is monotone increasing from a certain point on, i. e., there exists an integer n_0 with

$$R_1(n+1) \geqslant R_1(n) \quad \text{for} \quad n \geqslant n_0$$

if and only if the sequence A contains all the integers from a certain point on, i.e., there exists an integer n_1 with

$$A \cap \{n_1, n_1+1, n_1+2, \ldots\} = \{n_1, n_1+1, n_1+2, \ldots\}\,.$$

Furthermore, we proved that the function $R_2(n)$ can be monotone increasing also in a nontrivial way: namely, there exists a sequence A such that

$$A(n) < n - c\,n^{1/3}$$

(so that $\bar{A}(n) > c\,n^{1/3}$) and $R_2(n)$ is monotone increasing from a certain point on. Finally, we showed that if $A(n) = o\left(\frac{n}{\log n}\right)$, then the functions $R_2(n)$ and $R_3(n)$ cannot be monotone increasing. (See [1], [2] and [7] for other related problems and results.)

The purpose of this paper is to prove a result of independent interest on the connection between $R_3(2k)$ and $R_3(2k+1)$ (see Theorem 1 below) which will enable us to improve on our earlier estimates concerning the monotonicity of $R_3(n)$ (see Corollary 1 below).

Theorem 1. *If $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \ldots)$ is an infinite sequence of positive integers such that*

$$\lim_{n\to+\infty} \frac{\bar{A}(n)}{\log n} = \lim_{n\to+\infty} \frac{n - A(n)}{\log n} = +\infty\,, \tag{4}$$

then we have

$$\lim_{N\to+\infty} \sup \sum_{k=1}^{N} (R_3(2k) - R_3(2k+1)) = +\infty\,. \tag{5}$$

(So that, roughly speaking, $a_x + a_y$ assumes more even values than odd ones.) Clearly, this theorem implies that

Corollary 1.[1] *If $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \ldots)$ is an infinite sequence of positive integers such that* (4) *holds, then the function* $R_3(n)$

[1] Corollary 1 has been obtained independently by R. BALASUBRAMANIAN. His

cannot be monotone increasing from a certain point on, i.e., there does not exist an integer n_2 with

$$R_3(n+1) \geqslant R_3(n) \quad \textit{for} \quad n \geqslant n_2 .$$

We recall that in [6] we proved this with the much stronger assumption $A(n) = o\left(\frac{n}{\log n}\right)$ in place of (4). This result seems to suggest that, contrary to our earlier conjecture, also $R_3(n)$ can be monotone increasing only in the trivial way but unfortunately we have not been able to prove this.

A sequence $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \ldots)$ of positive integers is said to be a Sidon sequence if $R_3(n) \leqslant 1$ for all n, i.e., if

$$a_x + a_y = a_u + a_v, \quad x \leqslant y, \ u \leqslant v$$

implies that $x = u$, $y = v$. (We remark that very little is known on the properties of Sidon sequences; see eg. [7].) Theorem 1 implies trivially that

Corollary 2. *If A is an infinite Sidon sequence, then there exist infinitely many integers k such that $R_3(2k) = 1$ and $R_3(2k+1) = 0$, i.e., $2k$ can be represented in the form*

$$a_i + a_j = 2k$$

but

$$a_x + a_y = 2k + 1$$

is not solvable.

(In fact, it can be shown by analyzing the proof of Theorem 1 that there exist infinitely many positive integers N such that the assertion of Corollary 2 holds for $\gg A(N)$ integers k with $k \leqslant N$.)

Theorem 1 is near the best possible as the following results shows:

Theorem 2. *There exists a sequence $A = \{a_1, a_2, \ldots\}$ $(a_1 < a_2 < \ldots)$ of positive integers such that for some positive real numbers c, n_3 we have*

$$\bar{A}(n) > c \log n \quad (\textit{for } n > n_3) \tag{6}$$

paper contains several other related results of independent interest. His paper will appear in Acta Arithmetica.

and

$$\limsup_{N\to+\infty} \sum_{k=1}^{N} (R_3(2k) - R_3(2k+1)) < +\infty . \tag{7}$$

2. The proof of Theorem 1 will be based on the following idea: If A is a *finite* sequence of positive integers, and we denote the number of even elements and odd elements of it by A_0 and A_1, respectively, then the sum in (5) can be estimated in the following way:

$$\sum_{k=1}^{+\infty} (R_3(2k) - R_3(2k+1)) = \sum_{k=1}^{+\infty} R_3(2k) - \sum_{k=1}^{+\infty} R_3(2k+1) =$$

$$= \sum_{\substack{a\in A, a'\in A \\ a\leqslant a' \\ a+a'\equiv 0 \,(\mathrm{mod}\, 2)}} 1 - \sum_{\substack{a\in A, a'\in A \\ a< a' \\ a+a'\equiv 1 \,(\mathrm{mod}\, 2)}} 1 = \frac{1}{2} \sum_{\substack{a\in A, a'\in A \\ a+a'\equiv 0 \,(\mathrm{mod}\, 2)}} 1 + \frac{1}{2}\sum_{a\in A} 1 -$$

$$- \frac{1}{2} \sum_{\substack{a\in A, a'\in A \\ a+a'\equiv 1 \,(\mathrm{mod}\, 2)}} 1 =$$

$$= \frac{1}{2}(A_0^2 + A_1^2) + \frac{1}{2}(A_0 + A_1) - \frac{1}{2}(A_0 A_1 + A_1 A_0) =$$

$$= \frac{1}{2}(A_0 - A_1)^2 + \frac{1}{2}(A_0 + A_1) \geqslant \frac{1}{2}(A_0 + A_1)$$

which tends to infinity if the cardinality $(= A_0 + A_1)$ of the sequence A tends to infinity. However, of course, the situation is much more complicated for infinite sequences.

For $-1 < r < +1$, put

$$f(r) = \sum_{a\in A} r^a$$

so that

$$f^2(r) = \left(\sum_{a\in A} r^a\right)\left(\sum_{a'\in A} r^{a'}\right) = \sum_{a\in A, a'\in A} r^{a+a'} \quad \left(= \sum_{n=1}^{+\infty} R_1(n) r^n\right)$$

and hence

$$\sum_{n=1}^{+\infty} R_3(n) r^n = \sum_{\substack{a\in A, a'\in A \\ a\leqslant a'}} r^{a+a'} =$$

$$= \frac{1}{2} \sum_{a\in A, a'\in A} r^{a+a'} + \frac{1}{2} \sum_{a\in A} r^{2a} = \frac{1}{2}(f^2(r) + f(r^2)) .$$

(Note that here and in what follows all the infinite power series are absolutely convergent trivially for $-1 < r < +1$.)

For $-1 < r < +1$, put

$$g(r) = \sum_{n=1}^{+\infty} R_3(n) r^n = \tfrac{1}{2}(f^2(r) + f(r^2)) \tag{8}$$

and

$$h(r) = \sum_{k=1}^{+\infty} (R_3(2k) - R_3(2k+1)) r^{2k+1} .$$

Then for $0 < r < 1$ we have

$$\begin{aligned} h(r) &= r \sum_{k=1}^{+\infty} (R_3(2k) r^{2k} - \sum_{k=1}^{+\infty} R_3(2k+1) r^{2k+1} = \\ &= r \sum_{n=1}^{+\infty} \tfrac{1}{2} R_3(n)(r^n + (-r)^n) - \sum_{n=1}^{+\infty} \tfrac{1}{2} R_3(n)(r^n - (-r)^n) = \\ &= -\tfrac{1}{2}(1-r) \sum_{n=1}^{+\infty} R_3(n) r^n + \tfrac{1}{2}(1+r) \sum_{n=1}^{+\infty} R_3(n)(-r)^n = \\ &= -\tfrac{1}{2}(1-r) g(r) + \tfrac{1}{2}(1+r) g(-r) . \end{aligned} \tag{9}$$

To prove (5), it is enough to show that

$$\lim_{r \to 1-0} \sup h(r) = +\infty . \tag{10}$$

In fact, if we start from the indirect assumption that (5) does not hold, then there exists a positive real number B such that

$$\sum_{k=1}^{N} (R_3(2k) - R_3(2k+1)) \leqslant B \quad \text{for} \quad N = 1, 2, \dots ,$$

and hence for all $0 < r < 1$,

$$\begin{aligned} \frac{1}{1-r} h(r) &= \sum_{i=0}^{+\infty} r^i \sum_{k=1}^{+\infty} (R_3(2k) - R_3(2k+1)) r^{2k+1} = \\ &= \sum_{n=0}^{+\infty} \sum_{k=1}^{[(n-1)/2]} (R_3(2k) - R_3(2k+1)) r^n \leqslant \\ &\leqslant \sum_{n=0}^{+\infty} B r^n = B \sum_{n=0}^{+\infty} r^n = \frac{B}{1-r} \end{aligned}$$

so that

$$h(r) \leqslant B$$

which contradicts (10).

In view of (8) and (9), clearly we have

$$\begin{aligned} 4h(r) &= -2(1-r)g(r) + 2(1+r)g(-r) = \\ &= -(1-r)(f^2(r) + f(r^2)) + (1+r)(f^2(-r) + f(r^2)) = \qquad (11)\\ &= -(1-r)f^2(r) + 2rf(r^2) + (1+r)f^2(-r) \geqslant \\ &\geqslant -(1-r)f^2(r) + 2rf(r^2)\,. \end{aligned}$$

For $k = 1, 2, \ldots$, put $r_k = \exp(-1/2^k)$, so that $r_1 < r_2 < \ldots < 1$, $\lim_{k\to+\infty} r_k = 1$,

$$r_{k-1} = r_k^2 \quad (\text{for } k = 2, 3, \ldots) \tag{12}$$

and

$$\frac{1}{2^{k+1}} < 1 - r_k = 1 - \exp(-1/2^k) < \frac{1}{2^k} \quad \text{for} \quad k = 1, 2, \ldots, \tag{13}$$

since

$$\frac{x}{2} < x\left(1 - \frac{x}{2}\right) = x - \frac{x^2}{2} < 1 - e^{-x} < x \quad \text{for} \quad 0 < x < 1\,.$$

For $k = 1, 2, \ldots$ we write

$$H(k) = h(r_k) \quad \text{and} \quad F(k) = f(r_k)\,.$$

Furthermore, we put

$$\gamma = \lim_{k\to+\infty} \sup\,(1 - r_k)F(k) \quad \text{and} \quad \delta = \lim_{k\to+\infty} \inf(1 - r_k)F(k)\,.$$

3. In order to derive (10) from (11), we have to distinguish four cases.

Case 1. Assume first that

$$\delta < 1 \tag{14}$$

and

$$\gamma > 0\,. \tag{15}$$

Put $\varrho = \dfrac{\delta + \gamma}{2}$ so that

$$0 < \varrho < 1 \tag{16}$$

and

$$\varrho = \delta = \gamma \quad \text{if} \quad \delta = \gamma, \tag{17}$$

$$\delta < \varrho < \gamma \quad \text{if} \quad \delta < \gamma\,. \tag{18}$$

If (17) holds, then

$$\lim_{k \to +\infty} (1 - r_k) F(k) = \varrho\,,$$

hence in view of (14), for all $\varepsilon > 0$ and $k > k_0(\varepsilon)$ we have

$$(1 + \varepsilon)^{1/2} (1 - r_{k-1}) F(k - 1) > \varrho \tag{19}$$

and

$$(1 - r_k) F(k) < (1 + \varepsilon)^{1/2} \varrho\,. \tag{20}$$

(19) and (20) imply that

$$(1 - r_k) F(k) < (1 + \varepsilon)^{1/2} \varrho < (1 + \varepsilon)(1 - r_{k-1}) F(k - 1)\,. \tag{21}$$

If (18) holds, then by the definition of δ and γ, there exists an infinite sequence $k_1 < k_2 < \dots$ of positive integers such that for $i = 1, 2, \dots$,

$$(1 - r_{k_{2i-1}}) F(k_{2i-1}) > \varrho > (1 - r_{k_{2i}}) F(k_{2i})\,.$$

Then for all i, there exists an integer k with $k_{2i-1} > k \geqslant k_{2i}$ and

$$(1 - r_{k-1}) F(k - 1) \geqslant \varrho > (1 - r_k) F(k) \tag{22}$$

so that (22) holds for infinitely many positive integers k.

Either (21) holds for $k > k_0(\varepsilon)$ or (22) holds for infinitely many k, there exist infinitely many positive integers k with

$$(1 - r_k) F(k) < (1 + \varepsilon)(1 - r_{k-1}) F(k - 1)\,.$$

Hence, in view of (12), $(1 - r_k) F(k) < (1 + \varepsilon)(1 - r_k^2) F(k - 1)$ and

$$F(k) < (1 + \varepsilon)(1 + r_k) F(k - 1)\,. \tag{23}$$

In view of (11), (12), (20), (22) and (23), for sufficiently large k we have

$$4\,h(r_k) = 4\,H(k) \geqslant -(1 - r_k) f^2(r_k) + 2\,r_k f(r_k^2) =$$

$$= -(1 - r_k) f^2(r_k) + 2\,r_k f(r_{k-1}) = -(1 - r_k) F^2(k) + 2\,r_k F(k - 1) >$$

$$> -(1 - r_k) F^2(k) + \frac{2\,r_k}{(1 + \varepsilon)(1 + r_k)} F(k) > \tag{24}$$

$$> -(1-r_k)F^2(k) + \frac{1}{1+2\varepsilon}F(k) = F(k)\left(\frac{1}{1+2\varepsilon} - (1-r_k)F(k)\right) >$$

$$> F(k)\left(\frac{1}{1+2\varepsilon} - (1+\varepsilon)^{1/2}\varrho\right).$$

If ε is sufficiently small in terms of ϱ, then in view of (16) we have

$$\frac{1}{1+2\varepsilon} - (1+\varepsilon)^{1/2}\varrho > \frac{1-\varrho}{2}. \tag{25}$$

It follows from (24) and (25) that for infinitely many positive integers k we have

$$4h(r_k) > \frac{1-\varrho}{2}F(k)$$

which tends to $+\infty$ as $k \to +\infty$ since clearly, for infinite sequences A we have

$$\lim_{r \to 1-0} f(r) = +\infty,$$

and this completes the proof of (10) in Case 1.

Case 2. Assume now that

$$\delta = \gamma = \lim_{k \to +\infty} (1-r_k)F(k) = 0. \tag{26}$$

We are going to show that there exist infinitely many positive integers k with

$$F(k) < 4F(k-1). \tag{27}$$

In fact, let us start from the indirect assumption that there exists a positive integer K such that for $k \geqslant K$ we have $F(k) \geqslant 4F(k-1)$ (for $k \geqslant K$).

This implies by straight induction that for $j = 0, 1, 2, \dots$ we have

$$F(K+j) \geqslant 4^j F(K). \tag{28}$$

On the other hand, for all $0 < r < 1$,

$$f(r) = \sum_{a \in A} r^a < \sum_{n=0}^{+\infty} r^n = \frac{1}{1-r}$$

so that in view of (12),

$$F(K+j) = f(r_{K+j}) = f(r_K^{1/2^j}) < \frac{1}{1-r_K^{1/2^j}} =$$
$$= \frac{1}{1-r_K}\cdot\frac{1-r_k}{1-r_K^{1/2^j}} = \frac{1}{1-r_K}\sum_{i=0}^{2^j-1} r_K^{i/2^j} < \frac{1}{1-r_K}\sum_{i=0}^{2^j-1} 1 = \frac{2^j}{1-r_K}. \tag{29}$$

It follows from (28) and (29) that

$$\frac{2^j}{1-r_K} > 4^j F(K) = 4^j f(r_K)$$

but if j is sufficiently large in terms of r_K, then this inequality cannot hold (note that $0 < r_K < 1$ and that $f(r) > 0$ for all $0 < r < 1$), and this contradiction proves the existence of infinitely many positive integers k satisfying (27).

Then in view of (12) and (26), we obtain from (11) that if k satisfies (27) and is sufficiently large,

$$\begin{aligned} 4h(r_k) = 4H(k) &\geqslant -(1-r_k)f^2(r_k) + 2r_k f(r_k^2) = \\ &= -(1-r_k)f^2(r_k) + 2r_k f(r_{k-1}) = \\ &= -(1-r_k)F^2(k) + 2r_k F(k-1) = \\ &= -(1-r_k)F(k)\cdot 4F(k-1) + 2r_k F(k-1) = \\ &= F(k-1)(-4(1-r_k)F(k) + 2r_k) > \\ &> F(k-1)(-\tfrac{1}{2}+1) > \tfrac{1}{2}F(k-1) \end{aligned}$$

which tends to $+\infty$ as $k \to +\infty$ (since A is infinite) and this completes the proof of (10) in Case 2.

4. In order to study the cases with $\delta = 1$, we introduce the following notation: we put

$$p(r) = \frac{1}{1-r} - f(r) = \sum_{n=0}^{+\infty} r^n - \sum_{a\in A} r^a = \sum_{n\in\bar{A}} r^n \tag{30}$$

and

$$P(k) = p(r_k) \quad (k = 1, 2, \ldots)$$

so that

$$\begin{aligned} \lim_{k\to+\infty} \sup(1-r_k)p(r_k) &= \lim_{k\to+\infty} \sup(1-(1-r_k)f(r_k)) = \\ &= 1 - \lim_{k\to+\infty} \inf(1-r_k)F(k) = 1-\delta = 0 \quad \text{for} \quad \delta = 1\,, \end{aligned} \tag{31}$$

and in view of (4), for arbitrary large positive number L and for $r \to 1-0$ we have

$$p(r) = (1-r)\left(\frac{1}{1-r}\sum_{n \in \bar{A}} r^n\right) =$$

$$= (1-r)\left(\sum_{i=0}^{+\infty} r^i \sum_{n \in \bar{A}} r^n\right) = (1-r)\sum_{n=0}^{+\infty} \bar{A}(n) r^n >$$

$$> (1-r)(O(1) + \sum_{n=1}^{+\infty} L(\log n) r^n) =$$

$$= o(1) + \sum_{n=1}^{+\infty} L(\log n)(r^n - r^{n+1}) =$$

$$= o(1) + L\sum_{n=2}^{+\infty} (\log n - \log(n-1)) r^n =$$

$$= o(1) + L\sum_{n=2}^{+\infty}\left(\log\left(1 + \frac{1}{n-1}\right)\right) r^n >$$

$$> o(1) + cL\sum_{n=1}^{+\infty}\frac{r^n}{n} = o(1) + cL\log\frac{1}{1-r}$$

(where c is a positive absolute constant). This holds for all $L > 0$ whence

$$\lim_{r \to 1-0} p(r)\left(\log\frac{1}{1-r}\right)^{-1} = +\infty\,. \tag{32}$$

It follows from (13) and (32) that

$$\lim_{k \to +\infty}\frac{P(k)}{k} \geqslant \lim_{r \to 1-0} p(r_k)\log 2\left(\log\frac{1}{1-r_k}\right)^{-1} = +\infty. \tag{33}$$

Finally, in view of (12), it follows from (11) and (30) that

$$4H(k) = 4h(r_k) \geqslant -(1-r_k)f^2(r_k) + 2r_k f(r_k^2) =$$

$$= -(1-r_k)\left(\frac{1}{1-r_k} - p(r_k)\right)^2 + 2r_k\left(\frac{1}{1-r_k^2} - p(r_k^2)\right) =$$

$$= -\frac{1}{1-r_k} + 2P(k) - (1-r_k)P^2(k) + \frac{2r_k}{1-r_k^2} - 2r_kP(k-1) = \tag{34}$$

$$= -\frac{1}{1+r_k} + 2P(k) - (1-r_k)P^2(k) - 2r_k P(k-1) >$$

$$> -1 + 2P(k) - (1-r_k)P^2(k) - 2P(k-1).$$

Case 3. Assume that

$$\delta = 1 \tag{35}$$

and

$$\lim_{k\to+\infty} \sup P(k)(1-r_k)^{1/2} > 0\,. \tag{36}$$

It follows from (13) and (36) that

$$\begin{aligned} 0 < \lim_{k\to+\infty} \sup P(k)(1-r_k)^{1/2} &< \lim_{k\to+\infty} \sup P(k)2^{-k/2} < \\ &< \lim_{k\to+\infty} \sup P(k)e^{-k/4}\,. \end{aligned} \tag{37}$$

We are going to show that there exist infinitely many integers k with

$$P(k) > e^{1/8}P(k-1)\,. \tag{38}$$

In fact, let us start from the indirect assumption that there exists a positive integer K such that for $k \geqslant K$ we have

$$P(k) \leqslant e^{1/8}P(k-1) \quad (\text{for } k \geqslant K)\,.$$

This implies by straight induction that for $j = 0, 1, 2, \ldots$ we have

$$P(K+j) \leqslant e^{j/8}P(K)\,,$$

i.e.,

$$P(k) \leqslant e^{-K/8}e^{k/8}P(K) \quad \text{for} \quad k \geqslant K$$

hence

$$\begin{aligned} \lim_{k\to+\infty} \sup P(k)e^{-k/4} &\leqslant \lim_{k\to+\infty} \sup e^{-K/8}e^{k/8}P(K)e^{-k/4} = \\ &= \lim_{k\to+\infty} \sup e^{-K/8}P(K)e^{-k/8} = 0 \end{aligned}$$

which cannot hold by (37) and this contradiction proves the existence of infinitely many integers k satisfying (38).

Then in view of (31) and (33), we obtain from (34) that if k satisfies (38) and is sufficiently large,

$$\begin{aligned} 4H(k) &> -1 + 2P(k) - (1-r_k)P^2(k) - 2P(k-1) > \\ &> -1 + 2P(k) - (1-r_k)P^2(k) - 2e^{-1/8}P(k) = \end{aligned}$$

$$= P(k)\left(-\frac{1}{P(k)} + 2 - (1 - r_k)P(k) - 2e^{-1/8}\right)$$

$$> P(k)\left(-\frac{1}{k} + 2 - o(1) - 2e^{-1/8}\right) =$$

$$= P(k)(2(1 - e^{-1/8}) - o(1)) > (1 - e^{-1/8})P(k)$$

which, by (33) and $1 - e^{-1/8} > 0$, tends to $+\infty$ as $k \to +\infty$ and this completes the proof of (10) in Case 3.

Case 4. Assume finally that $\delta = 1$ and

$$\lim_{k \to +\infty} P(k)(1 - r_k)^{1/2} = 0 . \tag{39}$$

Then in view of (33), (34) and (39), for sufficiently large N we have

$$4\frac{1}{N}\sum_{k=2}^{N} H(k) \geqslant \frac{1}{N}\sum_{k=2}^{N}(-1 + 2P(k) - (1 - r_k)P^2(k) - 2P(k-1)) >$$

$$> -1 + \frac{2}{N}\sum_{k=2}^{N}(P(k) - P(k-1)) - \frac{1}{N}\sum_{k=2}^{N}(1 - r_k)P^2(k) >$$

$$> -1 + 2P(N)N^{-1} - 2P(1)N^{-1} - N^{-1}\sum_{k=2}^{N}(P(k)(1 - r_k)^{1/2})^2 >$$

$$> -1 + 2P(N)N^{-1} - 1 - N^{-1}(O(1) + \sum_{k=2}^{N} 1) >$$

$$> -1 + 2P(N)N^{-1} - 1 - 2 > P(N)N^{-1}$$

which, by (33), tends to $+\infty$ as $N \to +\infty$ and this proves (10) also in Case 4 which completes the proof of Theorem 1.

5. Proof of Theorem 2. Let $B = \{17, 64, \ldots, 4^{2k} + 1, 4^{2k+1}, \ldots\}$ and define the sequence A by

$$A = \bar{B} - \{0\} = \{1, 2, 3, \ldots, n, \ldots\} - B .$$

This sequence A satisfies (6) trivially. We are going to show that it satisfies also (7).

Let us write

$$\eta(x) = \begin{cases} 1 \text{ if } x \in B \\ 0 \text{ if } x \notin B \end{cases}$$

and

$$B_0(n) = \sum_{\substack{b \leqslant n,\, b \in B \\ b \equiv 0 \pmod 2}} 1 \quad \text{and} \quad B_1(n) = \sum_{\substack{b \leqslant n,\, b \in B \\ b \equiv 1 \pmod 2}} 1$$

so that

$$B_0(n) + B_1(n) = \sum_{\substack{b \in B \\ b \leqslant n}} 1 = B(n),$$

and by the construction of the sequence B,

$$|B_0(n) - B_1(n)| \leqslant 1 \quad \text{for all} \quad n. \tag{40}$$

Clearly we have

$$R_3(n) = \sum_{i \leqslant n/2} (1 - \eta(i))(1 - \eta(n - i)) =$$

$$= \sum_{i \leqslant n/2} 1 - \sum_{i=1}^{n-1} \eta(i) - \eta(n/2) + \sum_{i \leqslant n/2} \eta(i)\eta(n - i) =$$

$$= \sum_{i \leqslant n/2} 1 - B(n - 1) + \sum_{i < n/2} \eta(i)\eta(n - i).$$

Hence

$$R_3(2k) - R_3(2k + 1) =$$

$$= \Big(\sum_{i \leqslant k} 1 - \sum_{i \leqslant k + 1/2} 1\Big) + (B(2k) - B(2k - 1)) +$$

$$+ \sum_{i \leqslant k-1} \eta(i)\eta(2k - i) - \sum_{i \leqslant k} \eta(i)\eta(2k + 1 - i) =$$

$$= \eta(2k) + \sum_{i \leqslant k-1} \eta(i)\eta(2k - i) - \sum_{i \leqslant k} \eta(i)\eta(2k + 1 - i)$$

so that

$$\sum_{k=1}^{N} (R_3(2k) - R_3(2k + 1)) = \tag{41}$$

$$= \sum_{k=1}^{N} \eta(2k) + \sum_{k=1}^{N} \sum_{i \leqslant k-1} \eta(i)\eta(2k - i) - \sum_{k=1}^{N} \sum_{i \leqslant k} \eta(i)\eta(2k + 1 - i) =$$

$$= B_0(2N) + \Sigma_1 - \Sigma_2$$

where

$$\Sigma_1 = \sum_{k=1}^{N} \sum_{i \leqslant k-1} \eta(i)\eta(2k-i) \quad \text{and} \quad \Sigma_2 = \sum_{k=1}^{N} \sum_{i \leqslant k} \eta(i)\eta(2k+1-i)\,.$$

Here Σ_1 is the number of solutions of

$$b + b' < 2N + 1,\ b + b' \equiv 0 \pmod 2,\ b < b',\ b \in B,\ b' \in B, \tag{42}$$

while Σ_2 is the number of solutions of

$$b + b' < 2N + 1,\ b + b' \equiv 1 \pmod 2,\ b < b',\ b \in B,\ b' \in B\,. \tag{43}$$

Let us define j by

$$b_j < 2N + 1 \leqslant b_{j+1}\,,$$

and let us classify the pairs satisfying (42) according to that whether $b' < b_j$ or $b' = b_j$. If $b' < b_j$, then the pair b, b' in (42) can be chosen in $\binom{B_0(b_j-1)}{2}$ ways from the $B_0(b_j - 1)$ integers b with $b \equiv 0 \pmod 2$, $b \leqslant b_j - 1$, $b \in B$, or it can be chosen in $\binom{B_1(b_j-1)}{2}$ ways from the $B_1(b_j - 1)$ integers b with $b \equiv 1 \pmod 2$, $b \leqslant b_j - 1$, $b \in B$. Furthermore, if $b' = b_j$, then b in (42) can be any of the integers b with $b \equiv b_j \pmod 2$, $b \leqslant 2N + 1 - b_j$, $b \in B$, apart from the case $2b_j \leqslant$ $\leqslant 2N + 1$ when $b = b_j$ must not occur. Thus writing

$$\theta_N = \begin{cases} 1 & \text{if } \ 2b_j \leqslant 2N + 1 \\ 0 & \text{if } \ 2b_j > 2N + 1\,, \end{cases}$$

we have

$$\Sigma_1 = \binom{B_0(b_j-1)}{2} + \binom{B_1(b_j-1)}{2} + \sum_{\substack{b \equiv b_j \pmod 2 \\ b \leqslant 2N+1-b_j,\, b \in B}} 1 - \theta_N\,. \tag{44}$$

Similarly, if $b' < b_j$ in (43), then b, b' in (43) can be any of the $B_0(b_j - 1)\, B_1(b_j - 1)$ pairs b, b' with $b \not\equiv b' \pmod 2$, $b \leqslant b_j - 1$, $b' \leqslant b_j - 1$, $b \in B$, $b' \in B$. If $b' = b_j$ in (43), then b can be any integer with $b \not\equiv b_j \pmod 2$, $b \leqslant 2N + 1 - b_j$, $b \in B$ so that

$$\Sigma_2 = B_0(b_j - 1)\, B_1(b_j - 1) - \sum_{\substack{b \not\equiv b_j \pmod 2 \\ b \leqslant 2N+1-b_j,\, b \in B}} 1\,. \tag{45}$$

It follows from (41), (44) and (45) that

$$\sum_{k=1}^{N} (R_3(2k) - R_3(2k+1)) =$$

$$= B_0(2N) + \left(\binom{B_0(b_j-1)}{2} + \binom{B_1(b_j-1)}{2} - B_0(b_j-1)B_1(b_j-1)\right) +$$

$$+ \Bigg(\sum_{\substack{b \equiv b_j (\mathrm{mod}\, 2) \\ b \leqslant 2N+1-b_j,\, b \in B}} 1 - \sum_{\substack{b \not\equiv b_j (\mathrm{mod}\, 2) \\ b \leqslant 2N+1-b_j,\, b \in B}} 1\Bigg) - \theta_N \leqslant$$

$$\leqslant \tfrac{1}{2}(B_0(b_j-1) - B_1(b_j-1))^2 + |B_0(2N) - B_0(b_j-1)| +$$

$$+ \tfrac{1}{2}|B_0(b_j-1) - B_1(b_j-1)|$$

hence, in view of (40),

$$\sum_{k=1}^{N} (R_3(2k) - R_3(2k+1)) \leqslant \tfrac{1}{2} + 1 + \tfrac{1}{2} = 2$$

which completes the proof of Theorem 2.

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