FURTHER RESULTS ON MAXIMAL ANTI-RAMSEY GRAPHS

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ABSTRACT

A well-studied question in graph theory asks the following: Given a graph L and an integer* $r > 0$, *which graphs G have the property that no matter how the edges of G are r-colored, a monochromatic copy of L must always occur in G? (More precisely, G has a subgraph isomorphic to L in which all edges have the same color. We will typically use this type of informal description when the meaning is clear**.) Indeed, the forthcoming book [BFRS] will list several hundred papers which deal with various aspects of this subject. In [BEGS], we recently initiated a study of a related problem which in a certain sense goes in the opposite direction.*

* for undefined terminology in graph theory, see [H].
** to us.

1. Introduction

Our motivation actually originated from a question of Berkowitz (see [BEGS]) concerning time-space tradeoffs for models of computation (which still is unresolved). Basically, we investigated the following. Given a graph L and integers n and e, what is the *smallest* number r so that for *some* graph $G = G(n,e)$ with n vertices and e edges, the edges of G can be r-colored so that in every copy of L in G, all edges have *different* colors. We say that such a copy of L is *totally multicolored* (abbreviated TMC), and we denote this least value r by the expression $\chi_S(n,e,L)$. This notation comes from the fact that the value we seek is also the strong chromatic number of the hypergraph which has as its set of points the edges of G, with (hyper)edges consisting of the sets of edges of G which form copies of L.

In [BEGS] a substantial body of results are given concerning $\chi_S(n,e,L)$, as well as an unexpectedly large number of open problems. Some of these we will mention at the end. Several of the nicest results, however, could not be included there because of space limitations. Our purpose in this note is to fulfill the promise made in [BEGS] and present the details of these results here.

2. Preliminaries

We will call two edges of a graph *strongly independent* if they are disjoint and their four endpoints span no other edges. Our two main results will contrast the different behavior of $\chi_S(n,e,L)$ depending on whether or not L has two strongly independent edges.

Lemma 1 If a bipartite graph contains at least $(m-1)^2+1$ edges, then it either contains m independent edges or else a vertex of degree at least m. This result is sharp.

Proof This follows immediately from the well-known result of König: a bipartite graph with maximum degree Δ is the union of Δ matchings. If G is a bipartite graph with $q \geq (m-1)^2+1$ edges and $\Delta(G) \leq m-1$, then some matching contains at least $\lceil q/\Delta \rceil \geq m$ edges. The sharpness is shown by any bipartite graph in which one part has exactly $m-1$ vertices, each of degree $m-1$.

Lemma 2 Let $0 < \delta < 1$ be a fixed real number and let p be a fixed positive integer. There exists a positive number $\alpha = \alpha(\delta, p)$ such that for all sufficiently large n, every graph with n vertices and average degree at least δn contains the complete bipartite graph $K(p, \lceil \alpha n \rceil)$.

Proof Just use the standard argument: if the average degree d of a graph satisfies

$$n\binom{d}{p} > (q-1)\binom{n}{p}.$$

then the graph contains $K(p,q)$. With $\alpha = \frac{1}{2}\delta^p$ and $q = \lceil \alpha n \rceil$, the above inequality holds for all sufficiently large n.

Notation If A and B are disjoint subsets of $V(G)$, the bipartite subgraph of G with vertex partition (A, B) and edge set $\{xy | x \in A, y \in B, xy \in E(G)\}$ will be denoted $< A, B >_G$ or, if G is understood, simply $< A, B >$.

Lemma 3 Let $0 < \epsilon < 1$ be fixed and let p be a fixed positive integer. There are then numbers $\alpha, \beta, \gamma \in (0, 1)$ so that for all sufficiently large n, every graph G with n vertices and at least $e(n) = \lceil \frac{1}{2}\epsilon n^2 \rceil$ edges contains a bipartite subgraph $< A, B >$ such that

(i) $|A| = \lceil \alpha n \rceil$ and $|B| = \lceil \beta n \rceil$,

(ii) every vertex in B is adjacent to at least $\lceil \gamma n \rceil$ vertices of A, and

(iii) there exists $X \subset B$ with $|X| = p$ such that $< A, X >$ is complete.

Proof Since G has average degree at least ϵn, it contains a subgraph G' with $|V(G')| > \epsilon n$ in which every vertex has degree at least $\epsilon n/2$. [Just remove vertices of degree less than $\epsilon n/2$ until there are none left; for each such removal, the average degree does not decrease. Since the remaining graph has average degree at least ϵn, it has more than ϵn vertices.] By Lemma 2 (with $\delta = \epsilon/2$ and n replaced by $\epsilon n)G'$ contains a bipartite subgraph

$< X, A >\cong K(p, \lceil \alpha n \rceil)$, where $\alpha = (\epsilon/2)^{p+1}$. Set $R = V(G')\backslash\{A \cup X\}$. For simplicity of notation, let $|A| = a, |R| = r$.

Suppose that each vertex of A is adjacent to at least λr vertices of R. Then there are at least

$$s = \lceil [\frac{\lambda r}{2 - \lambda} [$$

vertices in R which are each adjacent to at least $\lambda a/2$ vertices of A. If not, the number of $A - R$ edges is less than $sa + (r - s)\lambda a/2$ and since

$$sa + (r - s)\lambda a/2 \leq a\lambda r,$$

this contradicts the fact that every vertex in A is adjacent to at least λr vertices of R. Since $a = \lceil \alpha n \rceil$ and each vertex in A adjacent to at least $(\epsilon/2 - p - a + 1)$ vertices of R, it follows that for all sufficiently large n, the above hypothesis is satisfied with $\lambda = \epsilon/2 - \alpha$. Since $r \geq (\epsilon - \alpha)n - 0(1)$ and $s/r > \lambda/2$, the stated conclusion follows by setting $\beta = \lambda(\epsilon - \alpha)/2$ and $\gamma = \lambda\alpha/2$.

The Main Results

Theorem 1 Let L be a bipartite graph satisfying $\Delta(L) \geq 2$ and having at least two strongly independent edges. Let ϵ be a real number in (0,1) and set $e(n) = \lceil \frac{1}{2}\epsilon n^2 \rceil$. Then there is a positive number $\kappa(L, \epsilon)$ such that

$$\chi_S(n, e(n), L) > \kappa n^2$$

for all sufficiently large n.

Two proofs of this result will be given. The first is based on results from extremal graph theory while the second one relies heavily on Szemerédi's regularity lemma.

First Proof Let G be an arbitrary graph with n vertices and $e(n)$ edges. We want to show that there is a positive number κ such that in every coloring of $E(G)$ using Kn^2 or fewer colors, there is a non-TMC copy of L. Choose p

and q so that L is a subgraph of $K(p,q)\backslash 2K_2$. Consider the bipartite subgraph of G which is guaranteed by Lemma 3 and let $C = B\backslash X$. Choose $m \geq q$ so that

$$m\binom{\lceil \gamma n \rceil}{q} > \binom{\lceil \alpha n \rceil}{q}.$$

(If n is large enough, $m > 2(\alpha/\gamma)^q$ will suffice to ensure this inequality.) Then for every set of at least m vertices in C, there will be two vertices which have a common neighbourhood in A of at least q vertices. Implicit in the proof of Lemma 3 is the fact that there is a number ζ such that every vertex in A is adjacent to at least $\lceil \zeta n \rceil$ vertices of C. Suppose that $\kappa < \alpha\zeta/m^2$ and G is colored using κn^2 or fewer colors. Then at least m^2 of the $A - C$ edges have the same color and by Lemma 1 the bipartite subgraph contains either m independent monochromatic edges or else a monochromatic star of degree at least m. We now distinguish three cases.

Case (i) - a matching of m independent monochromatic edges. In view of our choice of m, there are two vertices in C which are incident with edges of the monochromatic matching which have a common neighborhood in A of at least q vertices. Now it is clear that there is an embedding of L into G using the two matching edges and this copy of L is non-TMC.

Case (ii) - monochromatic star of degree m with center in A. Then there are two vertices in C which are end vertices of this star and which have a common neighborhood in A of at least q vertices. Thus we have a $K(p+2, q+1)$ subgraph of G which has two edges of the same color and so a non-TMC copy of L.

Case (iii) - monochromatic star of degree m with center in C. In this case, there is a $K(p+1, m)$ subgraph which has a monochromatic star of degree M and again a non-TMC copy of L.

Before giving the other proof let us recall the regularity lemma.

For a graph G with vertex set $V = V(G)$ and disjoint subsets $A, B \subset V$, define $e(A,B)$ as the number of edges of G with one endpoint in each of A and B. The *density* of the pair (a,B) is simply $e(A,B)/|A|\cdot|B|$ and it is denoted by $d(A,B)$.

For $0<\epsilon<1$, a pair (A,B) is called *ϵ-regular* if $|d(A_0,B_0)-d(A,B)|\leq\epsilon$ holds for all $A_0\subset A, B_0\subset B$ with $|A_0|>\epsilon|A|,|B_0|>\epsilon|B|$.

The regularity lemma (Szemerédi) [Sz]. For every $\epsilon>0$ there exists an integer $M_0=M_0(\epsilon)$ such that the vertices of every graph G can be partitioned into $m+1$ classes $C_0,\cdots,C_m$ for some $m, 1/\epsilon<m<M_0$, so that the following hold: $|C_0|<\epsilon n, |C_1|=\cdots=|C_m|$ and all but $\epsilon\binom{m}{2}$ of the pairs $C_i,C_j)$ are ϵ-regular.

Because we shall often need it, let us state the following immediate consequence of the definition of ϵ-regularity.

Lemma 5 If (C_i,C_j) is a regular pair with density β and $A\subset C_i, |A|>\epsilon|C_i|$, then the number of vertices of C_j which are connected to fewer than $(\beta-\epsilon)|A|$ vertices in A is less than $\epsilon|C_j|$.

Second Proof of Theorem 1 Let $\alpha>0$ be a fixed real number and suppose that G is a graph with n vertices and at least $\alpha\binom{n}{2}$ edges. Set $\epsilon=(\alpha/3)^l/25$, where $l=|L|$, and apply the regularity lemma.

Suppose without loss of generality that (C_1,C_2) is a ϵ-regular pair with density at least $\alpha/2$. From now on we shall only deal with the bipartite graph H spanned by this pair.

Set $u=|_1|=|C_2|$.

By Lemma 5 all but at most $2\epsilon u^2$ edges of H have both of their endpoints of degree at least $[\frac{\alpha}{2}-\epsilon]u>\frac{\alpha}{3}u$. Call these edges *good* and note that there are more than $\frac{\alpha}{3}u^2$ of them.

Set $k=\lceil 6/\alpha\rceil$ and let r denote the Ramsey number $r(k,k)$ (see [GRS]). Now let $\alpha'>0$ be a real number satisfying

$$a'M_0^2kr<\alpha/6.$$

Suppose that the edges of G, and consequently those of H, are colored by at most $\alpha'\binom{n}{2}$ colors. By the choice of a', some color, say red, will occur among the good edges in H with multiplicity at least kr. Therefore, either there are k good red edges forming a star or r good red edges forming a matching.

Lemma 6 Let X be a set of u elements, $0 < \alpha < 1, k = \lceil 6/\alpha \rceil$. Suppose that $B_1, \cdots, B_k \subset X$ satisfy $|B_i| \geq \alpha u/3$. Then there exist $1 \leq i < j \leq k$ with

$$|B_i \cap B_j| > \alpha^2 u/25.$$

Proof Suppose the contrary and let us bound the size of the union of the sets. Thus,

$$u \geq |B_1 \cup \cdots \cup B_k| \geq \sum |B_i| - \sum |B_i \cap B_j| \geq k\alpha u/3 - \binom{k}{2}\alpha^2 u/25 > 2u - \frac{49}{50}u,$$

which is a contradiction.

If there is a red star of size k then Lemma 6 implies that we can select two of its edges so that the endpoints have at least $\alpha^2 u/25$ common neighbors.

If we have a red matching, say $e_1, \cdots, e_r$, then let us form an auxiliary graph on $\{1, \cdots, r\}$ in the following way.

If the endpoints of e_i and e_j have fewer than $\alpha^2 u/25$ common neighbors in $C_t, t = 1, 2$, then join i and j by an edge of color t. By Lemma 6 no monochromatic complete graph of size k occurs this way. Thus, by the choice of r there are vertices $1 \leq i \leq j \leq r$ which are not connected by an edge. Thus, e_i and e_j have at least $\alpha^2 u/25$ common neighbors in both C_1 and C_2.

From now on the case of the star and the matching are very similar - we just find a "large" complete bipartite graph in the common neighborhood of the two red edges, thereby constructing a copy of L containing two red edges, i.e., which is not TMC.

This is done by repeated applications of Lemma 5. We only deal with the case of the matching, the other being nearly identical.

Let $e_i = (x_1, y_1), e_j = (x_2, y_2)$ with $x_1, x_2 \in C_1$ and $y_1, y_2 \in C_2$. Let $A \subset C_1$ be the set of common neighbors of y_i and y_j. The set $B \subset C_2$ is defined analogously. Recall that l denotes the number of vertices of L.

By Lemma 5 we can choose $x_3 \in A - \{x_1, x_2\}$ such that the neighborhood $B^{(3)}$ of x_3 in B satisfies

$$|B^{(3)}| > \frac{\alpha}{3}|B|.$$

Continuing in this way we find $x_1, \cdots, x_1 \in A$ such that they have at least $(\frac{\alpha}{3})^{l-2}|B|$ common neighbors in B. Let $y_1, y_2, \cdots, y_1$ be any l of them including y_1 and y_2.

Then these $2l$ vertices span a bipartite graph which is complete except possibly for the two edges (y_1, x_2) and (x_1, y_2). Therefore it contains a copy of L in which $(x_1, y_1), (x_2, y_2)$ are the two (red) strongly independent edges. This concludes the proof.

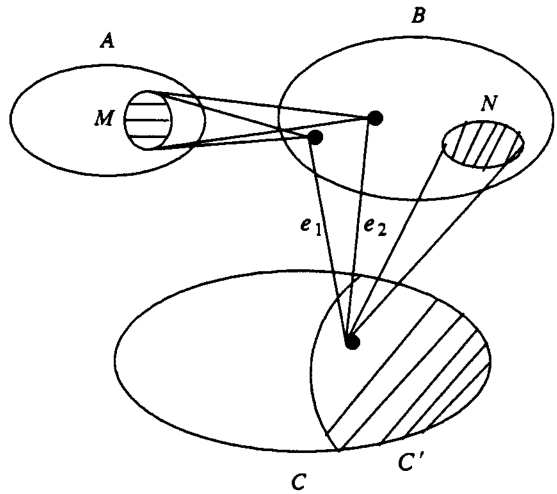

Figure 1

Using slightly more sophisticated arguments we will prove the following generalization in a later paper. Note that for $k = 2$ we get Theorem 1.

Theorem 2 Let L be a graph which is not the disjoint union of complete graphs, and which has two strongly independent edges e and e'. Suppose the vertices of L can be partitioned into k independent sets so that e and e' both have endpoints on the same two sets. Let α be a real number satisfying $\frac{k-2}{k-1} < \alpha < 1$. Then there is a $\beta > 0$ depending only on L and α, such that if $e > \alpha\binom{n}{2}$,

$$\chi_S(n, e, L) \geq \beta\binom{n}{2}.$$

The next result applies to graphs not having two strongly independent edges. We first state an auxiliary result.

Lemma 7 There is a function f such that if $y \leq f(x)$ then

$$\binom{a+b}{a} < n$$

for all a, b which satisfy $1 \le a \le x \log n$ and $1 \le b \le y\log n$.

Proof First we note that

$$\binom{a+b}{a} < \frac{(a+b)^{a+b}}{a^a b^b}$$

for all $a, b \ge 1$. This follows immediately from Robbins' form of Stirling's formula:

$$n! = \sqrt{2\pi n}[\frac{n}{3}]^n e^{\alpha_n} \text{ where } \frac{1}{12n+1} < \alpha_n < \frac{1}{12n}.$$

Since $\binom{a+b}{a}$ increases with both a and b, we have by a simple calculation

$$\log \binom{a+b}{a} < \log [\frac{(x+y)^{x+y}}{x^x y^y}] \log n$$

for all a, b satisfying $1 \le a \le x \log n$ and $1 \le b \le y \log n$. Thus to obtain

$$\binom{a+b}{a} < n,$$

it suffices to have

$$\frac{(x+y)^{x+y}}{x^x y^y} < e. \tag{1}$$

For fixed $x > 0$, the left-hand side of (1) approaches 1 as $y \downarrow 0$ and thus the existence of the desired function f is assured. In fact, writing (1) as

$$x \log(1 + y/x) + y \log(1 + x/y) < 1$$

and using the fact that $\log(1+t) < \min(t, \sqrt{t})$ for all $t > 0$, it follows that (1) is satisfied if $y + \sqrt{xy} \le 1$ and we may take $f(x) = 1/(x+2)$.

Theorem 3 Let $0 < \epsilon < \frac{1}{2}$ be fixed and set $e(n) = \lfloor \binom{n}{2} - \epsilon n^2 \rfloor$. Assume that n is appropriately large. There exists a graph G with n vertices and $e(n)$ edges which can be colored using $C(e)n^2/logn$ or fewer colors so that for every graph L with no two strongly independent edges, every copy of L in

G is TMC. In particular, for every $q \leq e(n)$ and every graph L with no two strongly independent edges,

$$\chi s(n,q,L) \leq \frac{C(\epsilon)n^2}{\log n}$$

for all sufficiently large n.

Proof Let L be an arbitrary graph having no two strongly independent edges. We shall show that there is a graph G with n vertices and at least $e(n)$ edges which can be colored using $C(\epsilon)n^2/\log n$ or fewer colors so that every copy of L in G is TMC. To prove this existence of such a graph, we use the probabilistic method. Consider the random graph $G_{n,p}$ on n vertices in which each edge is chosen with independent probability $p = 1 - \epsilon$. The expected number of edges of $G_{n,p}$ is $(1-\epsilon)\binom{n}{2}$ and it has at least $\binom{n}{2} - \epsilon n^2$ edges almost surely. Extend the notion of strong independence by defining disjoint pairs of vertices $\{u_1, v_1\}$ and $\{u_2, v_2\}$ in G to be strongly independent if none of four pairs $u_1u_2, u_1v_2, v_1u_2, v_1v_2$ is an edge in G (whether or not u_1v_1 and/or u_2v_2 are edges). Let X be the number of m-tuples of pairs of vertices in $G_{n,p}$ no two of which are strongly independent. The probability that two disjoint pairs are not strongly independent is $1-\epsilon^4$ and the expected value of X is

$$\begin{aligned} E(X)&\frac{n!}{2^m m!(n-2m)!}(1-\epsilon^4)^{\binom{m}{2}} \\ &< (\frac{en^2(1-\epsilon^4)^{(m-1)}/2}{2m})^m \\ &(\frac{en^2 \mathrm{enp}(-(m-1)\epsilon^4/2)}{2m})^m. \end{aligned}$$

Now if $m = \lceil 5\log n/\epsilon^4 \rceil$ then

$$E(X) = o(1) \quad (n \to \infty).$$

Thus, for all sufficiently large n, there exist graphs with n vertices and at least $\binom{n}{2} - \epsilon n^2$ edges such that every list of $m = \lceil 5\log n/\epsilon^4 \rceil$ disjoint pairs of vertices has two strongly independent pairs. Pick such a graph and simply refer to it as G.

Partition the edges of G of into n or fewer matching. We claim that the edges of an arbitrary matching in G can be colored using $C(\epsilon)n/\log n$ or fewer colors so that every pair of edges which receive the same color are strongly independent. The truth of this claim means that the edges of G can be colored with $C(\epsilon)n^2/\log n$ or fewer colors so that each copy of L is TMC. If a matching has $n/\log n$ or fewer edges then the claim is trivial; just give each edge a different color. Thus consider a matching M which contains between $n/\log n$ and $n/2$ edges.

There is a positive number $\kappa(\epsilon)$ so that M contains at least $\kappa \log n$ strongly independent edges. To see this, consider the graph with vertex set M in which two vertices are independent whenever the corresponding edges in M are strongly independent. By the previous conclusion, this graph has no clique of size $m = \lceil 5\log n/\epsilon^4 \rceil$. It must then have an independent set of k vertices (so M must contain a set of k strongly independent edges) if $|M| \geq r(k,m)$, where $r(k,m)$ denotes the classical Ramsey number. Since (see [GRS])

$$r(k,m) \leq \binom{k+m-2}{k-1},$$

we certainly get the desired result if $\kappa(\epsilon)$ can be chosen so that $k = \lceil \kappa(\epsilon)\log n \rceil$ satisfies

$$\binom{k+m}{k} < \frac{n}{\log n}.$$

This follows easily from Lemma 7 (by setting $x > 5/\epsilon^r$ and replacing n by $n/\log n$).

Set

$$C(\epsilon) = \frac{1}{2\kappa(\epsilon)} + 1.$$

Since we may remove sets of $\lceil \kappa(\epsilon)\log n \rceil$ or more strongly independent edges, giving each such set a distinct color, until there are at most $n/\log n$ edges of M left, it follows that the edges of M can be colored with $C(\epsilon)n/\log n$ or fewer colors so that every pair of edges receiving the same color are strongly independent. This completes the proof of the earlier claim and thus of the theorem.

Note that we have really done more than we needed to. The coloring of G that we have given actually makes every copy of every graph in the entire class of graphs without strongly independent edges TMC, not just copies of L.

4. Concluding Remarks

As mentioned in [BEGS], our knowledge of the behavior of $\chi_S(n, e, L)$ is strikingly incomplete even for some very simple graphs L. For example, how does $\chi_S(n, e, C_5)$ behave? We only know:

$$\chi_S(n, e, C_5) > c_n N \text{ for } e = (\frac{1}{4} + \epsilon)n^2, \epsilon > 0,$$

where $c_n \to \infty$ (very slowly) as $n \to \infty$; and

$$\chi_S(n, e, C_5) = 0(n^2/\log n) \text{ for } e = (\frac{1}{2} - \epsilon)n^2, \epsilon > 0.$$

The gap between these two bounds is embarrassing.

For P_4, the path of length 3, the situation is only slightly better.

$$\chi_S(n, e, P_4) > c'_n n \text{ for } e = \epsilon n^2, \ \epsilon > 0$$

where $c'_n \to \infty$ (very slowly) as $n \to \infty$; and

$$\chi_S(n, e, P_4) \le n \text{ for } e = n^2/\exp(c\sqrt{\log n})$$

for a suitable $c > 0$.

Clearly a great deal more remains to be done in this area.

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REFERENCES

[1] S. A. Burr, P. Erdös, R. L. Graham and V. T. Sós, [BEGS] "Maximal anti-Ramsey graphs and the strong chromatic number" (to appear in the *Journal of Graph Theory*, 1989).

[2] S. A. Burr, R. Faudree, C. C. Rousseau and R. Schelp, [BFRS] "Graph - ical Ramsey Theory (to appear; title tentative).

[3] R. L. Graham, B. L. Rothschild and J. H. Spencer, [GRS] "Ramsey Theory", John Wiley and Sons, New York, 1980.

[4] R. L.Graham, D. E. Knuth and O. Patashnik [GKP], "Concrete Mathematics" Addison- Wesley, Reading, MA, 1989.

[5] F. Harary [H], *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

[6] E. Szemerédi, "Regular partitions of graphs", in *Problèmes Combinatories et Théorie des Graphes*, (J. C. Bermond et. al., eds.) Editions du CNRS, Paris, 1978, pp. 399-401.