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Intersection Theorems for *t*-Valued Functions

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This paper investigates the maximum possible size of families $\mathscr{F}$ of t-valued functions on an n-element set $S = \{1, 2, \ldots, n\}$, assuming any two functions of $\mathscr{F}$ agree in sufficiently many places. More precisely, given a family $\mathscr{B}$ of k-element subsets of S, it is assumed for each pair h, $g \in \mathscr{F}$ that there exists a B in $\mathscr{B}$ such that $h = g$ on B. If $\mathscr{B}$ is 'not too large' it is shown that the maximal families have t^{n-k} members.

Introduction

Recently, theories have been developed relating set systems which have some specific *intersection properties* with intersection properties of other structures.

Sets

A theorem of Erdös, Ko, and Rado [3] asserts if S is an n-element set and $\mathscr{A}$ is a family of k-element subsets of S any two of which have a non-empty intersection, then

$$|\mathscr{A}| \leqslant \binom{n-1}{k-1}, \qquad n \geqslant 2k. \tag{1}$$

This result is sharp as shown by the family of k-tuples containing a fixed element of S.

An analogous but much simpler assertion is the following observation.

If $\mathscr{A}$ is a family of subsets of an n-element set S such that the intersection of any two of them is non-empty, then

$$|\mathscr{A}| \leqslant 2^{n-1}. \tag{2}$$

This estimate is again sharp; simply take all subsets of S containing a fixed element x of S.

Problem 1. Assume S is an n-element set and $\mathscr{A}$ is a family of subsets of S such that the intersection of any two has at least k elements. What is the maximum cardinality of $\mathscr{A}$?

One family $\mathscr{A}$ satisfying the above condition is obtained by taking all supersets of a fixed k-element subset of S. For this family

$$|\mathscr{A}| = 2^{n-k}. \tag{3}$$

Unfortunately, this is not the largest family satisfying the condition. Indeed, if $n + k$ is even and $\mathscr{A}$ is the family of all subsets of S with at least $(n + k)/2$ elements, then any two of them intersect in at least k elements. The number of sets in this family is

$$N = \sum_{i=0}^{(n-k)/2} \binom{n}{i}. \tag{4}$$

This number is much greater than that given in (3) except when $k = 1$, when they are the same. Katona [7] proved that, indeed, (4) is the best possible result and also settled the case when $n + k$ is odd.

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DEFINITION 1. Let S be an n-element set and $\mathscr{B}$ a family of subsets of S. The *intersection problem* corresponding to $(S, \mathscr{B})$ is to find the maximum sized family $\mathscr{A}$ such that the intersection of any two members of $\mathscr{A}$ belongs to $\mathscr{B}$. The families attaining the maximum cardinality are called the *extremal families* corresponding to $(S, \mathscr{B})$.

Generally, one could distinguish between *strong* and *weak* intersection problems. If one requires that the intersection be an element of $\mathscr{B}$, then it is a *strong* intersection problem, while if one requires that the intersection only contains as a subset some element of $\mathscr{B}$, then it is a *weak* intersection problem.

REMARK. Here one should clarify that the distinction between strong and weak intersection problems is not a mathematical one, in the sense that $\mathscr{B}$ can be enlarged to contain all supersets of the original members of $\mathscr{B}$. The strong intersection problem corresponding to the enlarged $\mathscr{B}$ is identical with the weak intersection problem corresponding to the original $\mathscr{B}$.

Minimal Extremal Set Systems

Throughout, the *strong* version of the *intersection problem* is assumed, thus if $B \in \mathscr{B}$ and $B \subseteq B'$, then $B' \in \mathscr{B}$.

Surely the smaller $\mathscr{B}$ the smaller the extremal system corresponding to $(S, \mathscr{B})$. Whenever $\mathscr{B}$ contains some k-tuples, then by letting $\mathscr{A}$ be the family of all supersets of a fixed k-tuple in $\mathscr{B}$ the family $\mathscr{A}$ has 2^{n-k} elements each pair of which intersect in $\mathscr{B}$. This means that the minimal size of the extremal family corresponding to $(S, \mathscr{B})$ is 2^{n-k}. In the case when the extremal families contain at most 2^{n-k} members, the family or system is called a *minimal extremal system*. The aim of the paper is to investigate under which conditions *minimal extremal systems* are obtained.

Such questions were discussed in [2, 4, 6]. One result obtained independently in [2] and [4] is the following. Let S be an n-element set and let $X_1, X_2, \ldots, X_l$ be a partition of S into non-empty subsets. If $\mathscr{A}$ is a family of subsets of S in which the intersection of each pair of $\mathscr{A}$ contain k $(k \leqslant l)$ elements $Y_1, Y_2, \ldots, Y_k$ belonging respectively to k *cyclically consecutive* members of the partition $X_1, X_2, \ldots, X_l$, then $|\mathscr{A}| \leqslant 2^{n-k}$. Thus this extremal system is a minimal one and is already obtained by restricting oneself to a small intersection family.

Functions

In [4] and [6], in addition to intersecting families of sets the authors also consider *intersecting families of functions*. Given a family $\mathscr{F}$ of functions mapping the n-element set S to a t-element set, two functions $h, g \in \mathscr{F}$ are said to intersect or agree at $U \subseteq S$ if $U = \{i \in S\colon h(i) = g(i)\}$. Usually, when $h(i) = g(i)$ we simply say h and g agree at i.

Families of intersecting or agreeing functions are connected with families of intersecting sets. In particular, the family of characteristic functions defined on a family of intersecting sets gives an intersecting family of functions with $t = 2$. In the light of an earlier remark, it is not surprising that the following theorem holds.

THEOREM A [4]. *If $\mathscr{F}$ is a family of 2-valued functions on an n-element set S, and S is partitioned into l non-empty sets $X_1, X_2, \ldots, X_l$ such that each pair in $\mathscr{F}$ intersect or agree in at least k $(k \leqslant l)$ points $y_1, y_2, \ldots, y_k$ belonging respectively to k cyclically consecutive members of the partition $X_1, X_2, \ldots, X_l$, then $|\mathscr{F}| \leqslant 2^{n-k}$.*

Results

One of the questions left unanswered in [4] is whether Theorem A holds for t-valued functions. We establish this and more, showing that the agreement of pairs of functions at points of k consecutive members of the partition can be replaced by agreement at points of k members whose indices form either an *arithmetic* or *geometric* progression with a fixed increment or ratio. This is the content of the next three theorems.

Throughout the remainder of the paper it is always assumed that S is an n-element set, $\mathscr{F}$ is a family of t-valued functions defined on S, $X_1, X_2, \ldots, X_l$ is a partition X of S into non-empty sets, and k is a positive integer, $k \leqslant l$. In addition, the l members of the partition $X_1, X_2, \ldots, X_l$ will be assumed to be cyclically ordered.

THEOREM 1. *If each pair of functions in $\mathscr{F}$ agree at some point of each of k consecutive terms of the partition X, then $|\mathscr{F}| \leqslant t^{n-k}$.*

THEOREM 2. *Let d be a positive integer such that $id \not\equiv 0 (\text{mod } l)$, $1 \leqslant i \leqslant k - 1$. If each pair of functions in $\mathscr{F}$ agree at some point of each of k terms of an arithmetic progression of terms of X with increment d, then $|\mathscr{F}| \leqslant t^{n-k}$.*

THEOREM 3. *Let $l = p^m - 1$ for some prime p and let r be a positive integer such that $r^i \neq 1 (\text{mod } l + 1)$, $1 \leqslant i \leqslant k - 1$. If each pair of functions in $\mathscr{F}$ agree at some point of each of k terms of a geometric progression of terms of X with ratio r, then $|\mathscr{F}| \leqslant t^{n-k}$.*

Each of the above theorems result in a family $\mathscr{F}$ that is minimal extremal. It will be apparent from the proof given, that a slightly more general 'agreement condition' for the family $\mathscr{F}$ can be given such that $\mathscr{F}$ is again minimal extremal. Since this amounts to an appropriate permutation of the partition X, there is no need to include it.

These theorems have obvious set intersection theorem consequences.

COROLLARY 1 (set system version). *Let P be either the progression mentioned in Theorem 2 or the one in Theorem 3. If $\mathscr{A}$ is a family of subsets of S such that the intersection of each pair in $\mathscr{A}$ contains an element of each member of some progression P, then $|\mathscr{A}| \leqslant 2^{n-k}$.*

Clearly, when $d = 1$ and $t = 2$ the results of Theorem 2 and Corollary 1 reduce to ones given in [4].

Dropping the Consecutiveness

In an earlier paper [6], Frankl and Füredi consider the family $\mathscr{F}$ (of t-valued functions on n points) in which each pair of its members (functions) agree at k or more points of their domain S. They let $f(n, t, k)$ denote the maximum size of such a family. They prove the following theorem.

THEOREM B [6]. *For $t \geqslant 3$, $t^n/t^k \leqslant f(n, t, k) \leqslant t^n/(t-1)^k$ and for $k \geqslant 15$, $f(n, t, k) = t^{n-k}$ if and only if $t \geqslant k + 1$ or $n \leqslant k + 1$.*

Since then, Richard Wilson has shown that the condition $k \geqslant 15$ can be dropped in this theorem. We consider a generalization of the Frankl–Füredi bound.

THEOREM 4. *If each pair of functions in $\mathscr{F}$ agree at some point of each of k members of the partition X, then $|\mathscr{F}| \leqslant f(l, t, k)t^{n-l}$.*

In particular, the Frankl–Füredi result shows that the family $\mathscr{F}$ of Theorem 4 satisfies $|\mathscr{F}| \leqslant f(l, t, k)t^{n-l} = t^{l-k} \cdot t^{n-l} = t^{n-k}$ and is minimal extremal when $t \geqslant k + 1$ or $n \leqslant k + 1$. Also, the inequality of Theorem B shows $t^n/t^k \leqslant f(l, t, k)t^{n-l} \leqslant t^n/(t-1)^k$.

Erdös posed and Kleitman [8] showed that

$$f(l, 2, k) = \begin{cases} \displaystyle\sum_{i=0}^{(l-k)/2} \binom{l}{i} & \text{if } l - k \text{ is even;} \\ \displaystyle 2\sum_{i=0}^{[(l-k)/2]} \binom{l-1}{i} & \text{if } l - k \text{ is odd.} \end{cases} \tag{5}$$

This gives an exact upper bound on $|\mathscr{F}|$ in Theorem 4 for $t = 2$.

When t is a power of some fixed positive integer one can prove the following theorem, which in some cases gives a more useful upper bound than the one in Theorem 4.

THEOREM 5. *If $t = d^m$ and $\mathscr{F}$ satisfies the condition of Theorem 4, then $|\mathscr{F}| \leqslant [f(l, d, k)]^m \cdot t^{n-l}$.*

To demonstrate the usefulness of the bound of Theorem 5 consider the case when $d = 2$ and, consequently, $f(l, 2, k)$ is known exactly. In particular, consider a comparison of the bounds of Theorems 4 and 5 in the case when $l - k = d$ and m are both fixed with l large. To do this, observe by (5) that $(f(l, 2, k))^m \leqslant l^{dm/2}$, a polynomial upper bound in l, while $f(l, 2^m, k) \leqslant t^l/(t-1)^k = (t-1)^d(t/(t-1))^l$ by Theorem B, an exponential upper bound in l. Hence this is an instance where the bound of Theorem 5 is considerably more effective to use than the one of Theorem 4. Similarly, Theorem 5 is better in cases when m and $l - k$ are not fixed but tend to infinity slowly (as functions of l).

One of the most interesting open questions left unanswered is a slight generalization of one initially posed in [2]. Select any k element set T of indices from the index set $L = \{1, 2, \ldots, l\}$ of the partition $X = \{X_1, X_2, \ldots, X_l\}$. Let $\mathscr{B}$ have as elements the set T together will all its cyclic translates in L. If each pair of functions in $\mathscr{F}$ agree at some point of each element of the partition indexed by an element B in $\mathscr{B}$, then is $|\mathscr{F}| \leqslant t^{n-k}$? Some evidence is given in [2] and [4] that the answer to this question is yes.

PROOFS

In order to prove Theorems 1, 2 and 3 a special case of the theorem is needed.

LEMMA 1. *Let $l = n \leqslant 2k$ so that the partition X consists of singleton sets. If each pair of functions in $\mathscr{F}$ agree at k consecutive terms of the partition X, then $|\mathscr{F}| \leqslant t^{n-k}$.*

This lemma was proved in [4] for $t = 2$, and the proof for arbitrary t is similar. To make the paper self-contained an outline of the proof is provided.

PROOF (outline). Let $X_i = \{i\}$ for each member of the partition and let $Y \subseteq S = \{1, 2, \ldots, n\}$ be the set on which all elements of $\mathscr{F}$ agree (have the same values). Surely if $|Y| \geqslant k$ then the result follows. Using the 'agreement condition' for pairs of functions in $\mathscr{F}$ it follows when i and j are at a distance at most k in either direction along the n-cycle (i.e. when $2n - k \leqslant |i - j| \leqslant k$), that either i or j belong to Y. Thus for each $i \notin Y$ there are $2k - n + 1$ consecutive elements of S in Y, and each additional element not in Y accounts for an additional element in Y. Hence $|Y| \geqslant 2k - n + |S - Y| \geqslant k$.

PROOF (Theorem 1). For $l = uk + \varrho$, $0 \leqslant \varrho \leqslant k$, partition the index set of the partition $X = \{X_1, X_2, \ldots, X_l\}$ into $k + \varrho$ subsets $\{Y_i\}_{i=1}^{k+\varrho}$ by letting $Y_i = \{i, k + i, \ldots, (\mu - 1)k + i\}$ for $1 \leqslant i \leqslant k$ and $Y_{k+i} = \{\mu k + i\}$ for $1 \leqslant i \leqslant \varrho$. Note that any two distinct integers in the same term of this partition differ by at least k, so any k consecutive integers (1 and l are assumed consecutive) will be in k cyclically consecutive terms of the partition $Y_1, Y_2, \ldots, Y_{k+\varrho}$ of the index set of X. Let $W_1, W_2, \ldots, W_{k+\varrho}$ be the partition of S defined by $W_i = \bigcup_{j \in Y_i} X_j$ for $1 \leqslant i \leqslant k + \varrho$. Due to the choice of the Y_i's each pair of functions in $\mathscr{F}$ agree at some point of each of k cyclically consecutive terms of the partition $W_1, W_2, \ldots, W_{k+\varrho}$.

Let $\mathscr{F}^*$ be the set of all t-valued functions defined on S. Clearly, $\mathscr{F}^*$ has t^n functions which will be partitioned into $t^{n-\varrho-k}$ classes as follows. For each $g, h \in \mathscr{F}^*$ define $g \sim h$ (equivalent to) if $g(x) - h(x)$ has a constant value on each W_j. Clearly '$\sim$' is an equivalence relation. Let $[g]$ denote the equivalence class containing g. Observe that each class $[g]$ contains $t^{k+\varrho}$ functions.

Let $w_j \in W_j$, $1 \leqslant j \leqslant k + \varrho$, be *fixed* elements of the partition $W_1, W_2, \ldots, W_{k+\varrho}$. Let $\mathscr{F}^{**}$ be the set of all t-valued functions with domain $\{1, 2, \ldots, k + \varrho\}$. For each class $[g]$ define a function γ: $[g] \to \mathscr{F}^{**}$ by $\gamma(h) = \tilde{h}$, $h \in [g]$, where $\tilde{h}(j) = h(w_j)$ for all j. Observe that $g(x) - h(x) = g(w_j) - h(w_j)$ for all j. Clearly γ is a one-to-one function. Also if $h_1, h_2 \in [g] \cap \mathscr{F}$, then h_1 and h_2 agree at points of at least k cyclically consecutive terms of $W_1, W_2, \ldots, W_{k+\varrho}$, so that $\tilde{h}_1$ and $\tilde{h}_2$ agree at k cyclically consecutive points of $\{1, 2, \ldots, k + \varrho\}$. Hence from the one-to-one correspondence of γ it follows from Lemma 1 that $|[g] \cap \mathscr{F}| \leqslant t^{(\varrho+k)-k} = t^{\varrho}$. Since this is true for each equivalence class $[g]$, $|\mathscr{F}| \leqslant t^{n-\varrho-k}t^{\varrho} = t^{n-k}$.

Since the proofs of Theorems 2 and 3 are similar adaptions of the strategy used in the proof of Theorem 1, their proofs will be given as a single proof.

Proof (Theorem 2 and Theorem 3). Consider a maximal length progression $X^{(1)} = \{X_{m_1}, X_{m_2}, \ldots, X_{m_s}\}$ of distinct terms of the partition $X = \{X_1, X_2, \ldots, X_l\}$ which is arithmetic with increment d in the case of Theorem 2 and geometric with ratio r in the case of Theorem 3. The conditions in each of the theorems make $s \geqslant k$. Consider this subpartition $X^{(1)} = \{X_{m_1}, X_{m_2}, \ldots, X_{m_s}\}$ of X ordered cyclically as listed. For $s = \mu k + \varrho$, $0 \leqslant \varrho < k$, partition the set of indices of $X^{(1)}$ into $k + \varrho$ subsets $\{Y_i^{(1)}\}_{i=1}^{k+\varrho}$ by letting $Y_i^{(1)} = \{m_i, m_{k+i}, \ldots, m_{(\mu-1)k+i}\}$ for $1 \leqslant i \leqslant k$ and $Y_{k+i}^{(1)} = \{m_{\mu k+i}\}$ for $1 \leqslant i \leqslant \varrho$.

If $s < l$ then find another maximal length progression $X^{(2)}$ of distinct terms of X disjoint from $X^{(1)}$. Clearly, its length is also s. Form the analogous sequence of indices $\{Y_i^{(2)}\}_{i=1}^{k+\varrho}$. Repeat this process sequentially until the maximal progressions exhaust all terms of X, giving subpartitions $X^{(1)}, X^{(2)}, \ldots, X^{(v)}$ (each cyclically ordered) with corresponding sequences of vertices $\{Y_i^{(j)}\}_{i=1}^{k+\varrho}$, $1 \leqslant j \leqslant v$. Let $Y_i = \bigcup_{j=1}^{v} Y_i^{(j)}$ for $1 \leqslant i \leqslant k + \varrho$.

At this point the proof becomes identical with the proof of Theorem 1. Set $W_i = \bigcup_{j \in Y_i} X_j$ for $1 \leqslant i \leqslant k + \varrho$. Note that if a pair of functions in $\mathscr{F}$ agree at some point of each of k terms of a progression of terms of X, then they agree at some point of each of k cyclically consecutive terms of the partition $W_1, W_2, \ldots, W_{k+\varrho}$. Hence $|\mathscr{F}| \leqslant t^{n-k}$ as required.

Proof (Theorem 4). This proof is similar to part of the proof of Theorem 1. Let $\mathscr{F}^*$ be the set of all t-valued functions defined on S. Surely $\mathscr{F}^*$ has t^n functions which we partition into t^{n-l} classes as follows. For each $g, h \in \mathscr{F}^*$ define $g \sim h$ if $g(x) - h(x)$ is constant on each X_i, $1 \leqslant i \leqslant l$. Thus the equivalence class $[g]$ containing g has t^l elements. Select fixed elements $x_i \in X_i$, $1 \leqslant i \leqslant l$, and let $\mathscr{F}^{**}$ be the set of all t-valued functions with domain $\{1, 2, \ldots, l\}$. For each class $[g]$ define a function γ: $[g] \to \mathscr{F}^{**}$ by $\gamma(h) = \tilde{h}$, $h \in [g]$, where $\tilde{h}(j) = h(x_j)$ for all j. Surely γ is one to one and if $h_1, h_2 \in [g] \cap \mathscr{F}$ then $\tilde{h}_1$ and $\tilde{h}_2$ have values which agree at k points of their domain. Hence $|[g]| \cap \mathscr{F}| \leqslant f(l, t, k)$, so that $|\mathscr{F}| \leqslant f(l, t, k)t^{n-l}$.

Before Theorem 5 is proved some observations are needed. A family $\mathscr{F}^*$ of t-valued functions defined on the n-element set S can be replaced by $t = ab$-valued functions where the set of values is $\{(z, w) | 1 \leqslant z \leqslant a, 1 \leqslant w \leqslant b\}$. For $\mathscr{C} \subseteq \mathscr{F}^*$ let $P_1(\mathscr{C})$ $(P_2(\mathscr{C}))$ be the projection of members of $\mathscr{C}$ onto the first (second) coordinate. Surely $|\mathscr{F}^*| = |P_1(\mathscr{F}^*)| \cdot |P_2(\mathscr{F}^*)|$ with $|P_1(\mathscr{F}^*)| = a^n$, $|P_2(\mathscr{F}^*)| = b^n$, and $|\mathscr{C}| \leqslant |P_1(\mathscr{C})| \cdot |P_2(\mathscr{C})|$. Also, given the equivalence defined in the proof of Theorem 4, for $g \in F^*$, $|[g]| = |P_1[g]| \cdot |P_2[g]| = a^l \cdot b^l$.

Proof (Theorem 5). We show by induction on m that $|[g] \cap F^*| \leqslant [f(l, d, k)]^m$ where F^*, is as given above, $ab = d^m = t$, $g \in \mathscr{F}^*$, and $[g]$ is the equivalence relation defined in the proof of Theorem 4. It is clear that one may assume $a = d$ and $b = d^{m-1}$. Further, since $[g] \cap \mathscr{F}$ satisfies the conditions of Theorem 4 so do $P_1([g] \cap \mathscr{F})$ and $P_2([g] \cap \mathscr{F})$. Thus as in the proof of Theorem 4 $|P_1([g] \cap \mathscr{F})| \leqslant f(l, d, k)$ and by induction on m, when $m > 1$, $|P_2([g] \cap \mathscr{F})| \leqslant f(l, d, k)]^{m-1}$.

Thus $|[g] \cap \mathscr{F}| \leqslant |P_1([g] \cap \mathscr{F})| |P_2([g] \cap F)| \leqslant [f(l, d, k)]^m$. Since this holds for each of the t^{n-l} equivalence classes $|\mathscr{F}| \leqslant [f(l, d, k)]^m \cdot t^{n-l}$.

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