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Author(s): P. ERDÖS, M. SIMONOVITS, VERA T. SÓS and S. B. RAO

Source: *Sankhyā: The Indian Journal of Statistics, Series B (1960-2002)*, Vol. 54, SPECIAL VOLUME (1992), pp. 155-167

Published by: Indian Statistical Institute

Stable URL: <https://www.jstor.org/stable/44956227>

Accessed: 26-06-2020 12:58 UTC

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ON TOTALLY SUPERCOMPACT GRAPHS

By P. ERDÖS, M. SIMONOVITS, VERA T. SÓS
Mathematical Institute of The Hungarian Academy of Sciences

and

S. B. RAO
Indian Statistical Institute, Calcutta

SUMMARY. A graph is said to be *totally supercompact*, if for any two vertices x and y there is a vertex $z \neq x, y$ joined to one of them but not to the other. In this paper, (i) three problems of Chia and Lim [4], (problems 4,5,6, pp. 324–325) on totally supercompact selfcomplementary graphs are solved; (ii) we determine the maximum number of edges $f(n, p)$ in a totally supercompact graph of order n without K_p , solving a problem of Hoffman [12]. (We also solve a generalization of this problem for so-called k -asymmetrical graphs.)

1. INTRODUCTION

All graphs considered in this paper are finite and have neither loops nor multiple edges. For a graph G , we shall denote by $E(G)$ and $V(G)$ the set of edges and the set of vertices, respectively. We shall also admit the “null” graph (without vertices and edges), denoted by \emptyset . For graph theoretic definitions and notation not given here the reader is referred to Bondy and Murty [3].

Definition 1: A graph is said to be *totally supercompact*, if for any two vertices x and y there is a vertex $z \neq x, y$ joined to one of them but not to the other.

Since the basic concept of this paper is totally supercompactness, we give an alternative definition as well.

Two vertices, x and y , of the graph G are called symmetrical if they are joined to exactly the same vertices of $G - x - y$.

Definition 1': A graph G is totally supercompact if it has no pairs of symmetrical vertices.

Remark: Clearly, the definition of symmetry yields an equivalence relation. A graph G_n is totally supercompact iff each equivalence class is just a vertex. However, if we select one vertex from each equivalence class, we get a graph G' which is not necessarily totally supercompact. E. g., selecting one vertex from each equivalence class of $K_d(n_1, \dots, n_d)$ we get a K_d which is not totally supercompact.

Remark: Most graphs are totally supercompact. Moreover, if we take a random graph R_n with edge probability $p > (1 + \epsilon) \frac{\log n}{n}$, then — as proved by Erdős and Rényi [7], — the probability that R_n will have symmetries tends to 0, and if we delete any vertex of R_n , the probability that the remaining $G_n - x$ will have symmetrical vertices (or symmetries at all) also tends to 0.

Beside the random graphs let us see a few concrete examples. The path P_n ($n > 3$) or the cycle C_n ($n > 4$) are totally supercompact. By the way, C_n is a fairly symmetrical graph. The k -dimensional cube is also fairly symmetrical, still it is totally supercompact. $K_d(n_1, \dots, n_d)$, the complete d -partite graph is not totally supercompact.

In Lim [13] it has been shown that a graph is supercompact iff it is the intersection graph of some family \mathcal{S} of subsets of X such that \mathcal{S} satisfies the Helly property, and for any $x, y \in X, x \neq y$ there exists an $S \in \mathcal{S}$ with $x \in S$ and $y \notin S$. Lim's work was motivated by the work of de Groot [10], who introduced the graph theoretical representation of topological spaces. In [18,19] it was remarked that these concepts play an important role in certain works on empirical logic.

The complement \overline{G} of a graph G is the graph with $V(\overline{G}) = V(G)$ and

$$E(\overline{G}) = \{uv : u, v \in V(G), u \neq v \text{ and } uv \notin E(G)\}.$$

A graph G is said to be *selfcomplementary*, if G is isomorphic to $\overline{G}, G \simeq \overline{G}$.

If G is a selfcomplementary graph, then $v(G) \equiv 0$ or $1 \pmod{4}$, since $\binom{n}{2}$ must be even; further any isomorphism σ of G onto \overline{G} is a permutation of a set $V(G)$ and is referred to as a *complementing permutation* of G . The set of all complementing permutations of G will be denoted by $\mathcal{C}(G)$.

Now we define three graphs U, W and Z being on the one hand illustrations to the definitions of this paper, on the other hand some building blocks in our constructions to come.

Let U be the graph in Figure 1, i.e., the graph on 5 vertices a, b, c, d, e , with

$$E(U) = \{(a, b), (b, c), (b, d), (a, d), (d, e)\}.$$

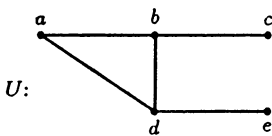


Figure 1

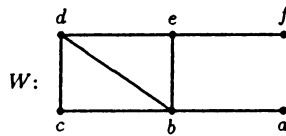


Figure 2

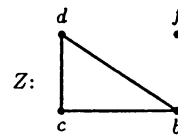


Figure 3

It is totally supercompact and selfcomplementary. Let W be the graph in Figure 2, i.e., the graph on 6 vertices a, b, c, d, e, f , where

$$E(W) = \{(a, b), (b, c), (c, d), (d, e), (e, f), (b, d), (b, e)\}.$$

It is totally supercompact but not selfcomplementary. Let Z be the graph in Figure 3, i.e. the graph on the 4 vertices b, c, d, f , where

$$E(Z) = \{(b, c), (c, d), (d, a)\}.$$

It is neither totally supercompact, nor selfcomplementary.

The first part of the paper investigates the “extension properties” of the selfcomplementary totally supercompact graphs. First we investigate the basic properties of three operations G^* , G^0 and G_0 to be defined below.

Given a graph G , let

$$G^* = \{v \in V(G) : G - v \text{ is totally supercompact}\} \quad \dots (1)$$

Also G^* will denote the subgraph of G induced by the corresponding vertices.

There are quite a few results on the structure of totally supercompact, selfcomplementary graphs, e.g. [18,19], [5], [9]. We mention just one result of Geoffroy [9], partly motivating some results of ours as well.

Theorem (Geoffroy): For any graph H there exists a connected, totally supercompact graph G for which $G^ = H$.*

We shall need the following

Proposition 1: If G is a selfcomplementary graph and $\sigma \in \mathcal{C}(G)$ is a complementing permutation, then G^ is also selfcomplementary and invariant under σ .*

Proof: Trivial and also will follow from the proof of Proposition 2 below.

The problems of Chia and Lim, formulated below ([4]), ask if one can deduce some information on the structure of G , knowing the structure of G^* .

Problem A: Let G be a totally supercompact graph. Suppose G^ is selfcomplementary and totally supercompact. Must G be selfcomplementary, too?*

The answer is NO:

Theorem A: There exist infinitely many totally supercompact nonselfcomplementary graphs G for which G^ is selfcomplementary. Moreover, for every totally supercompact selfcomplementary graph H there exist infinitely many totally supercompact but not selfcomplementary graphs G with $G^* = H$.*

Problem B: Let H be a selfcomplementary graph. Does there exist a selfcomplementary totally supercompact G such that $G^ = H$?*

Here the answer is YES:

Theorem B: If H is a selfcomplementary graph, then there exist infinitely many totally supercompact selfcomplementary graphs G for which $G^ \simeq H$.*

One can unify Theorems A and B by saying

Theorem AB: For every selfcomplementary H there exist infinitely many selfcomplementary totally supercompact graphs G with $G^ = H$, and at the same time infinitely many nonselfcomplementary totally supercompact graphs G with $G^* = H$, and infinitely many nonselfcomplementary and not totally supercompact graphs G with $G^* = H$.*

For the next problem we need some more definitions. A non-null graph G is said to be *vertex-distinguishing* or *supercompact* if distinct adjacent vertices cannot

be symmetrical, (though nonadjacent pairs are allowed to be symmetrical). On the other hand, G is said to be *vertex-determining* if distinct *nonadjacent* vertices cannot be symmetrical. Clearly, G is vertex-distinguishing iff \overline{G} is vertex-determining. Let

$$G_0 = \{v \in V(G) : G - v \text{ is vertex-distinguishing}\}; \quad \dots(2)$$

$$G_0 = \{v \in V(G) : G - v \text{ is vertex-determining}\}. \quad \dots(3)$$

Again, G^0 and G_0 will also denote the subgraphs of G induced by the corresponding vertices.

Examples: If $G_1 = P_4$ is the path of 4 vertices, then G_1 is a totally supercompact selfcomplementary graph and $G_1^* = \emptyset$. For $G_2 = U$ of Figure 1, $U^* = K_1$ is just a vertex. For $G_3 = C_5$, i.e. for a pentagon graph, $G_3^* = G_3$. These graphs are selfcomplementary, whereas the graph W of Figure 2 is not selfcomplementary but is a totally supercompact graph with $W^* = K_2$, and $W^0 \simeq \overline{W}_0$.

Proposition 2: *If G is totally supercompact selfcomplementary graph, then $\overline{G}_0 \simeq G^0$.*

Proof of Proposition 2: Assume that G is selfcomplementary, with a complementing permutation σ . Then this σ (extended to the edges as well) will send \overline{G}_0 onto G^0 and conversely. Hence for any selfcomplementary graph $\overline{G}_0 \simeq G^0$. Further, since $G^* = \overline{G}_0 \cap G^0$, for a selfcomplementary graph G in such a case G^* is invariant under σ and is also a selfcomplementary graph (possibly null). \square

Problem C: *Let G be a totally supercompact graph. If $\overline{G}_0 \simeq G^0$, or, equivalently, $\overline{G}^0 \simeq G_0$, then would G be selfcomplementary?*

The answer is again NO:

Theorem C: *There exist infinitely many totally supercompact nonselfcomplementary graphs G such that $\overline{G}_0 \simeq G^0$.*

Remark: Roughly, all these results show that given some reasonable properties of a small graph G^* in terms of selfcomplementedness and supercompactness, etc., this small graph can be extended in many ways, fairly easily, so that the large graph G satisfies some conditions, but not the others.

The following problem posed by Hoffman [12] differs from the previous ones in style: it is an extremal problem and selfcomplementedness is not involved here.

Problem D: *Let n and p be fixed. Find the maximum number $f(n, p)$ of edges in a supercompact graph of n vertices, not containing the complete graph K_p .*

In Section 3 we shall solve the problem of Hoffman on $f(n, p)$ and a generalization of it as well. These are, of course, perturbation problems related to Turan's Theorem.

2. PROOFS OF THEOREMS A–C

To prove Theorems A–C, we shall need the following

Definition 2: The lexicographic product $G \oplus \mathcal{F}$ of a non-null graph G with a family

$$\mathcal{F} = \{G_x : x \in V(G)\}$$

(indexed by the vertices of G) is the graph obtained from G by substituting each vertex $x \in V(G)$ by the graph G_x , and joining every vertex of G_x to every vertex of G_y (respectively no pairs $u, v : u \in V(G_x), v \in V(G_y)$) depending on whether xy is an edge or not.

More formally, $G \oplus \mathcal{F}$ is the graph defined by

$$V(G \oplus \mathcal{F}) = \bigcup \{\{x\} \times V(G_x) : x \in V(G)\},$$

$$E(G \oplus \mathcal{F}) = \{((x_1, y_1)(x_2, y_2)) : (x_1, x_2) \in E(G) \text{ or } x_1 = x_2 \text{ and } (y_1, y_2) \in E(G_{x_1})\}.$$

If $G_x = \emptyset$ for every $x \in V_0 \subseteq V$ and $\mathcal{F} = \{G_x : x \in V(G) - V_0\}$, then for $V_1 = V(G) - V_0$ clearly, $G \oplus \mathcal{F} \simeq G(V_1) \oplus \mathcal{F}_1$, - where $G(A)$ is the subgraph of G induced by A . Below we shall always assume that none of the graphs $G_x = \emptyset$.

Definition 3 was introduced by G. Sabidussi [15], called *X-join of a graph family* and used — e.g. in [9] — to study the structure of totally supercompact graphs.

Below we shall formulate and prove a few easy lemmas, all of which have the same idea:

Assume that G and H are given graphs and we replace each vertex of G either by H or by K_1 (where replacing by a K_1 means that we leave those vertices untouched):

$$G_x = \begin{cases} H & \text{if } x \in X_0 \\ K_1 & \text{if } x \notin X_0 \end{cases}$$

Then the symmetry and selfcomplementedness properties of the $G \oplus \mathcal{F}$ can immediately be seen from the symmetry and selfcomplementedness properties of G and H .

Below, in our lemmas and proofs the one-vertex graph K_1 will be considered as a totally supercompact, selfcomplemented graph *without* isolated vertices and vertices of full degree (!!!).

Lemma 1: (a) *If G is a selfcomplementary non-null graph and $\mathcal{F} = \{G_x : x \in V(G)\}$ is a family of non-null graphs where none of the graphs G_x and \overline{G}_x contains isolated vertices, and $\sigma \in C(G)$ and $\overline{G}_x \simeq G_{\sigma(x)}$ for every $x \in V(G)$, then $G \oplus \mathcal{F}$ is a selfcomplementary graph.* (b) *In particular, if $G_x \simeq H$ for every $x \in V(G)$ where H is a selfcomplementary graph, then $G \oplus \mathcal{F}$ is a selfcomplementary graph.*

Remark: Obviously, selfcomplementary graphs cannot have isolated vertices, neither vertices of full degree.

Proof of Lemma 1: Clearly, $\overline{G \oplus \mathcal{F}} = \overline{G} \oplus \overline{\mathcal{F}}$, where $\overline{\mathcal{F}} = \{\overline{G}_x : x \in V(G) = V(\overline{G})\}$. Let $\sigma \in C(G)$ and $\tau_x : G_x \rightarrow \overline{G}_{\sigma(x)}$ be an isomorphism, $x \in V(G)$. Define

$$\begin{aligned} \Sigma : V(G \oplus \mathcal{F}) &\rightarrow V(\overline{G \oplus \mathcal{F}}) \\ &= V(G \oplus \mathcal{F}), \end{aligned}$$

by $\Sigma(x, y) = (\sigma(x), \tau_x(y))$.

It is easy to check that $\Sigma \in C(G \oplus \mathcal{F})$ and consequently $G \oplus \mathcal{F}$ is a selfcomplementary graph, proving Lemma 1. \square

Lemma 2: If G is a non-null graph and $\mathcal{F} = \{G_x : x \in V(G)\}$ is a family of non-null graphs where none of the graphs G_x and \overline{G}_x contains isolated vertices, then $G \oplus \mathcal{F}$ is totally supercompact iff each $G_x (x \in V(G))$ is totally supercompact and for any pair x, y of vertices satisfying $v(G_x) = 1, v(G_y) = 1, x \neq y, x$ and y are asymmetric.

Corollary 1: There exist infinitely many totally supercompact selfcomplementary graphs with $n \equiv 1 \pmod{4}$ vertices.

Proof of Corollary 1: Define recursively the following infinite sequence: $H_5 = C_5$, the cycle of order 5, and for $n = 5^k, k > 1$, let H_n be the lexicographic product of $H_{n/5}$ and C_5 , more precisely,

$$H_n = H_{n/5} \oplus \{G_x\},$$

where $G_x \simeq C_5$ for every x . By Lemma 1 H_k is selfcomplementary and by Lemma 2 it is totally supercompact. \square

Remark: We can prove Corollary 1 also by using random graph methods.

Corollary 2: There exist infinitely many totally supercompact selfcomplementary graphs G with $G^* = K_1$, on $n \equiv 1 \pmod{4}$ vertices.

Proof of Corollary 2: Let H be an arbitrary totally supercompact selfcomplementary graph. Using the lexicographical product method replace all but one of its vertices by P_4 , the path on 4 vertices; replace one of its vertices by U of Figure 1. Clearly, for the obtained G we have $G^* = K_1$. \square

Lemma 3: If G is a non-null graph and $\mathcal{F} = \{G_x : x \in V(G)\}$ is a family of non-null graphs where none of the graphs G_x and \overline{G}_x contains isolated vertices, then

(i) $G \oplus \mathcal{F}$ is vertex-distinguishing (resp. vertex determining) iff (a) each G_x is vertex-distinguishing (resp. vertex-determining), and (b) no pair x, y of vertices of full or null-degree are symmetric.

(ii) Let

$$\mathcal{F}_0 = \{(G_x)_0 : x \in V(G)\}$$

and

$$\mathcal{F}^0 = \{(G_x)^0 : x \in V(G)\}$$

and

$$\mathcal{F}^* = \{(G_x)^* : x \in V(G)\}.$$

If $G \oplus \mathcal{F}$ is totally supercompact, then

(a) $(G \oplus \mathcal{F})_0 = G \oplus \mathcal{F}_0$, whenever for no $x \in V(G)$ have both $G_x - y$ with $y \in (G_x)_0$ full degree vertices;

(b) $(G \oplus \mathcal{F})^0 = G \oplus \mathcal{F}^0$, whenever for no $x \in V(G)$ have both G_x and $G_x - y$ with $y \in (G_x)^0$ isolated vertices;

(c) $(G \oplus \mathcal{F})^* = G \oplus \mathcal{F}^*$, if both (a) and (b) hold.

The proofs of Lemma 2 and Lemma 3 are straightforward applications of the definitions of H_0, H^0 , and H^* for a graph H and are omitted. \square

Proof of Theorem A: Let H be any totally supercompact non-null selfcomplementary graph, U be the graph of Figure 1, or more generally, let U be an arbitrary totally supercompact, selfcomplementary graph with $U^* = K_1, v = v(U) = 4k + 1$ for some $k \geq 1$. (Among others, the graphs of Corollary 2 will do.) Such graphs have $\frac{1}{2} \binom{v}{2} = 4k^2 + k$ edges. Let x, y be fixed nonadjacent distinct vertices of H such that $d_H(x) + d_H(y) \neq v(H) + 2k - 1$. For example either $z, \sigma(z)$ or $\sigma(z), \sigma^2(z)$ will do for any $\sigma \in \mathcal{C}(H)$ provided $z \neq \sigma(z)$. Define $\mathcal{F} = \mathcal{F} = \{G_w : w \in V(H)\}$ where $G_x \simeq G_y \simeq U$ and $G_w = K_1$, for every $w \in v(H) - \{x, y\}$. Then by Lemma 2, $G = H \oplus \mathcal{F}$ is a totally supercompact graph and by Lemma 3(i), $G^* = (H \oplus \mathcal{F})^* \simeq H$ is a selfcomplementary graph. We have to prove that G is not selfcomplementary. Clearly $v(G) = v + 8k$. Since x and y are nonadjacent, therefore $e(G) - e(H) = 4k(d_H(x) + d_H(y)) + 8k^2 + 2k$. If now G is a selfcomplementary graph, then

$$e(G) - e(H) = \frac{1}{2} \left(\binom{v(G)}{2} - \binom{v(H)}{2} \right) = \frac{1}{2} \left(\binom{v + 8k}{2} - \binom{v}{2} \right) = 4kv + 16k^2 - 2k.$$

Therefore

$$e(G) - e(H) = 4k(d_H(x) + d_H(y)) + 8k^2 + 2k = 4kv + 16k^2 - 2k,$$

i.e. $d_H(x) + d_H(y) = v + 2k - 1$, a contradiction. Hence G is not a selfcomplementary graph. \square

Remark: We could prove Theorem A in various ways. One question here would be if we need to replace 2 or more vertices of H or replacing just one would do. If H is not regular, then it has a vertex x of degree $d(x) \neq \frac{1}{2}(v + 1)$ and replacing this vertex by U would also do.

Proof of Theorem B: Let U be as above : either U be the graph of Figure 1 or any graph described in Corollary 2 : let U be selfcomplementary, totally supercompact, $v(U) \geq 5$ and $v(U^*) = 1$. Define $\mathcal{F}_H = \{G_x : x \in V(H)\}$, where for every $x, (G_x)^* \simeq U$ and $G = H \oplus \mathcal{F}_H$. Then G_x is a totally supercompact selfcomplementary graph with $(G_x)^* = K_1$. By Lemma 2, G is totally supercompact. It is trivially supercomplementary. Further, by Lemma 3(c), $G^* \simeq H$.

Proof of Theorem C: Let H be a totally supercompact selfcomplementary graph with $v(H) \equiv 1 \pmod{4}$ (for example H may be taken as $C_5^{(n)}$ of Proposition 1) and σ be a complementing permutation of H and u be a fixed point of σ (see Rao [14] for several properties of σ and the existence of a fixed point in case $v(H) \equiv 1 \pmod{4}$). Let

$$G_x = \begin{cases} W \text{ of Figure 2} & \text{if } x = u \\ K_1 & \text{if } x \neq u; \end{cases}$$

and

$$\mathcal{F} = \{G_x : x \in V(H)\}.$$

Then $G = H \oplus \mathcal{F}$ is a nonselfcomplementary graph since $v(G) \equiv 2 \pmod{4}$. Further, by Lemma 2, G is totally supercompact. Also as

$$(G_x)'_0 \simeq \begin{cases} Z \text{ of Figure 3} & \text{if } x = u, \\ K_1 & \text{if } x \neq u, \end{cases}$$

and

$$(G_x)_0 \simeq \begin{cases} \bar{Z} \text{ of Figure 3} & \text{if } x = u, \\ K_1 & \text{if } x \neq u, \end{cases}$$

it follows by Lemma 3(ii), that $G_0 \simeq \bar{G}^0$ under the complementing permutation σ . \square

3. THE EXTREMAL PROBLEM : DETERMINATION OF $f(n, p)$

Given a graph L , the following problem is called the Turán type extremal problem corresponding to L .

What is the maximum number of edges a graph G_n can have without containing L as a (not necessarily induced) subgraph?

Turán proved that if $T_{n,p}$ is the graph obtained by partitioning n vertices into p classes as equally as possible, and joining two vertices iff they belong to different classes, then $T_{n,p-1}$ contains no K_p and has more edges than any other graph G_n on n vertices and not containing any K_p [21,22].

(Here $K_p \not\subseteq T_{n,p-1}$ is trivial.)

Replacing K_p with an arbitrary "forbidden graph" L we get a general theory (see e.g. [3], [16]). The graphs having the maximum number of edges among the ones not containing L will be called *extremal graphs* for L , and the number of edges of $T_{n,p-1}$ will be denoted by $t(n, p-1)$.

Without trying to describe the general theory, we mention just one result needed below.

Definition 3: An edge e of a graph G will be called *critical* if $\chi(L-e) = \chi(L)-1$.

Theorem ([17]): *Let L be an arbitrary p -chromatic graph with a critical edge e . Then for sufficiently large n , $T_{n,p-1}$ is extremal for L and there are no other extremal graphs for the Turán type extremal problem of L .*

Given a property \mathcal{P} such that $T_{n,p-1}$ does not have this property, we shall call *perturbated Turán problem corresponding to \mathcal{P}* the following problem.

Problem E: *Let L be an arbitrary p -chromatic graph with a critical edge e . What is the maximum number of edges of a graph G_n having the property \mathcal{P} and not containing L ?*

Given a graph L , we shall call *Hoffman type extremal problem* the following “perturbated” Turán type extremal problem.

Problem F: *What is the maximum number of edges in a totally supercompact graph G_n not containing L ?*

We can strengthen the notion of totally supercompact graphs and then generalize the above question.

Definition 4: A graph G_n is *k -asymmetrical* if for any $x, y \in V(G_n)$, the symmetric difference of $N(x)$ and $N(y)$ has at least k vertices.

Remark: For any fixed k and n most graphs on n vertices are not only totally supercompact, but k -asymmetric. Moreover, if we fix an $\epsilon > 0$ and an $\eta > 0$ and take a random graph R_n with edge probability $p > (1 + \epsilon)\frac{\log n}{n}$, then, — as proved by Erdős and Rényi [7] — almost surely, for any pair x, y of vertices of R_n we have $|N(x) \cap N(y)| > p^2 n - \eta n$. This — applied also to the complementary graph — means that the random graphs are almost surely k -asymmetric for $k = (p^2 + (1 - p)^2 - \eta)n$.

Problem G: *Given k and L , what is the maximum number of edges in a k -asymmetrical graph G_n not containing L ?*

For many L the extremal graphs of the Turán problem are already k -asymmetrical and in these cases they are also extremal for Problem G. Then Problem G is uninteresting. Without going into technical explanation we state that in some sense the most interesting case is, when the extremal graphs for the Turán type extremal problem (for L) are very symmetrical, and below we shall restrict our investigations to the case when the extremal graph is just $T_{n,p-1}$. A necessary and sufficient condition for this is that L has a critical edge, [17], see above.

Below we shall determine — with a small error term — the extremal number for Problem G in the case when L has a critical edge. We start with some constructions, yielding the lower bounds. Actually, in some cases we could determine the exact optimum: we shall prove that the optimal graph is obtained from a complete p -partite graph by deleting some edges, forming a subgraph Q .

Before really giving these constructions, we give an informal description of theirs. Clearly, if we have a $K_d(n_1, \dots, n_d)$, with almost equal classes, and x and y belong to two different classes, then their neighbourhoods differ in at least cn vertices.

Therefore we may forget this case. Let now x and y belong to the same class. If we delete now the edges $(x, u_1), \dots, (x, u_d)$ and $(y, v_1), \dots, (y, v_d)$ and all the u_i 's and v_j 's are different, then these vertices will be 2-asymmetrical. However, if $n_d > n$, then we cannot delete those edges so that for any pair x, y the second endvertices of the deleted edges are different. But then we can delete those edges so that for any pair x, y the "missing" neighbourhoods intersect in at most one vertex (depending on the pair). For this we have to use finite geometrical constructions.

Construction 1: Let us delete a 1-factor Q_n from $T_{n,p-1}$. (If n is odd, we allow one vertex to be isolated in Q_n .) The resulting graph Z_n will contain no L and there will be no symmetric pairs of vertices in it. This shows that if S_n is extremal for the Hoffman problem, then

$$e(S_n) \geq t(n, p-1) - \frac{1}{2}n - O(1).$$

Construction 2: Assume that $d < p$ is fixed and n is large. We can delete a d -factor Q_n^* from $T_{n,p-1}$ so that the resulting graph be $2d$ -asymmetric. (If n is odd, we allow one vertex to be of degree $d+1$ in Q_n^* .) This shows that if S_n is extremal for Problem G with $k = 2d$, then

$$e(S_n) \geq t(n, p-1) - \frac{1}{2}dn - O(1).$$

Construction 3: Assume that $d \geq p$. One can delete a d -regular $Q_n^{**} - T_{n,p-1}$ with $C_4 \not\subseteq Q_n^{**}$, so that the resulting graph is $(2d-1)$ -asymmetric. (If n is odd, we allow $o(n)$ vertices to be of degree $d-1$ in Q_n^{**} .) This shows that if S_n is extremal for Problem G with $k = 2d-1$, then

$$e(S_n) \geq t(n, p-1) - \frac{1}{2}dn - O(1).$$

Theorem D: If S_n is extremal for the Hoffman problem, i.e. S_n is totally supercompact, then

$$e(S_n) = t(n, p-1) - \frac{1}{2}n + O(1).$$

Theorem E: If $k = \frac{1}{2} \log n$ and S_n is extremal for the k -asymmetrical problem, then $\chi(S_n) \leq p-1$. Further, the graphs described in Constructions 1-3 are extremal upto an error term $O(1)$ or $o(n)$ respectively. Thus, e.g., for $k < 2p-1$,

$$e(S_n) \geq t(n, p-1) - \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil n + O(1).$$

Proof of Theorem E: (a) Let $d(G_n)$ denote the minimum degree in G_n . Andrásfai, Erdős and Sós [1] proved for K_p that if $K_p \not\subseteq G_n$ and $\chi(G_n) \geq p$, then $d(G_n) < (1 - \frac{1}{p-4/3})n$.

Soon Erdős and Simonovits [8] generalized the assertion to every L having a critical edge. In fact, they have shown that if $L \neq K_p$ has chromatic number p and

e is a critical edge in L , then for every G_n not containing L and having chromatic number $\geq p$ we have $d(G_n) < (1 - \frac{1}{p-3/2})n$.

We shall need below only the somewhat weaker assertion that there exists a positive constant C_L such that

$$\chi(G_n) \geq p, \quad L \not\subseteq G_n$$

imply $d(G_n) < (1 - \frac{1}{p-1})n - C_{L_n}$.

So take an arbitrary graph G_n not containing L and define

$$V_p := \left\{ v \in V(G_n) : d(v) < \left(1 - \frac{1}{p-1}\right)n - \frac{3}{4}C_{L_n} \right\}. \quad \dots (4)$$

If $t = |V_p| = o(n)$, then $V(G_n) - V_p$ contains no vertices of degree $< \left(1 - \frac{1}{p-1}\right)n - C_{L_n} + \frac{1}{2} \log n$. Hence, by the Andrásfai-Erdős-Simonovits-T. Sós Theorem, $\chi(G_n - V_p) \leq p - 1$ therefore

$$e(G_n - V_p) < \left(1 - \frac{1}{p-1}\right) \binom{n-t}{2} + O(t).$$

Hence

$$e(G_n) < \left(1 - \frac{1}{p-1}\right) \binom{n-t}{2} + O(1) + t \left(\left(1 - \frac{1}{p-1}\right)n - C_{L_n} \right) + \binom{t}{2}, \quad \dots (5)$$

and therefore — as an easy calculation shows —

$$e(G_n) < \left(1 - \frac{1}{p-1}\right) \binom{n}{2} - \frac{1}{2}C_{L_n}n. \quad \dots (6)$$

(b) By constructions 1-3, if S_n is extremal for the k -asymmetrical problem, then

$$e(S_n) \geq t(n, p-1) - \left\lceil \frac{k}{2} \right\rceil \frac{n}{2} + O(1).$$

This and (6) immediately yield that $t = |V_p| \leq \frac{1}{2} \log n$. Since $V(G_n - V_p) \leq p - 1$, we can partition $V(G_n)$ into p classes V_1, \dots, V_p so that V_1, \dots, V_{p-1} are independent sets of vertices and $|V_p| = O(k)$.

(c) Later we shall see that $V_p = \emptyset$ for the extremal graph S_n . Now we give only an upper bound on $e(G_n)$. This is the point where we shall really use that S_n is k -asymmetrical.

We can partition the sets V_i ($i = 1, \dots, p - 1$) into $\leq T = 2^t < \sqrt{n}$ subsets V_{ij} according to the connections to V_p : two vertices $x, y \in V_i$ belong to the same class V_{ij} if $N(x) \cap V_p = N(y) \cap V_p$. Let us call a pair (x, u) a ‘missing edge’ if x, u belong to different classes V_i, V_h ($i, h < p$) and they are not joined in S_n . Then in each V_{ij} there is at most one x incident to less than $\lceil \frac{k}{2} \rceil$ missing edges. Indeed, if say, $x, y \in V_{ij}$ are both adjacent to $< \lceil \frac{k}{2} \rceil$ “missing edges” then since they are

joined to exactly the same vertices in V_p -the size of the symmetrical difference of their neighbourhoods is $< k$, a contradiction. This implies that if M denotes the number of "missing edges",

$$e(G_n) < \left(1 - \frac{1}{p-1}\right) \binom{n}{2} = \frac{1}{2}c_L^t - M < \left(1 - \frac{1}{p-1}\right) \binom{n}{2} - \frac{1}{2}c_L^t - \left\lfloor \frac{k}{2} \right\rfloor \frac{n}{2} + o(n).$$

This shows that, if Z_n is the graph of Constructions 2-3, then $e(S_n) < e(Z_n) - c'|V_p|n < e(S_n)$, unless $V_p = \emptyset$. Thus we proved that $V_p = \emptyset$, or in other words, S_n is $(p-1)$ -chromatic. This easily implies Theorems D and E. \square

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P. ERDŐS, M. SIMONOVITS
AND VERA T. SÓS
MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY

S. B. RAO
STAT-MATH. DIVISION
INDIAN STATISTICAL INSTITUTE
203 B. T. ROAD
CALCUTTA 700 035
INDIA