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TURÁN-RAMSEY THEOREMS AND SIMPLE ASYMPTOTICALLY EXTREMAL STRUCTURES

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This paper is a continuation of [10], where P. Erdős, A. Hajnal, V. T. Sós, and E. Szemerédi investigated the following problem:

Assume that a so called forbidden graph L and a function $f(n)=o(n)$ are fixed. What is the maximum number of edges a graph G_n on n vertices can have without containing L as a subgraph, and also without having more than $f(n)$ independent vertices?

This problem is motivated by the classical Turán and Ramsey theorems, and also by some applications of the Turán theorem to geometry, analysis (in particular, potential theory) [27–29], [11–13].

In this paper we are primarily interested in the following problem. Let (G_n) be a graph sequence where G_n has n vertices and the edges of G_n are coloured by the colours $\chi_1,\dots,\chi_r$, so that the subgraph of colour χ_ν contains no complete subgraph K_{p_ν}, $(\nu=1,\dots,r)$. Further, assume that the size of any independent set in G_n is $o(n)$ (as $n\to\infty$). What is the maximum number of edges in G_n under these conditions?

One of the main results of this paper is the description of a procedure yielding relatively simple sequences of asymptotically extremal graphs for the problem. In a continuation of this paper we shall investigate the problem where instead of $\alpha(G_n)=o(n)$ we assume the stronger condition that the maximum size of a K_p-free induced subgraph of G_n is $o(n)$.

Notation. In this paper we shall primarily consider graphs without loops and multiple edges. However, (as tools to prove our results) we shall also use coloured graphs with weighted edges and vertices. Given a graph G, $e(G)$ will denote the number of its edges, $v(G)$ the number of its vertices, $\chi(G)$ its chromatic number, $\alpha(G)$ the maximum size of an independent set in it. Given a graph, the (first) subscript will denote the number of vertices: G_n, S_n,... will always denote graphs on n vertices. $R(k_1,\dots,k_r)$ will denote the usual Ramsey number, i.e., the minimum t such that for every edge colouring of K_t in r colours K_t contains a monochromatic K_{k_ν} for some colour χ_ν. One more convention on the colouring of graphs: whenever we use two colours χ_1 and χ_2, we shall call the first colour RED, the second one BLUE.

Given two disjoint vertex sets, X and Y, in a graph G_n, $e_G(X,Y)$ denotes the number of edges joining X and Y, and $d_G(X,Y)$ denotes the edge-density between them:

$$d_G(X,Y)=\frac{e_G(X,Y)}{|X|\cdot|Y|}. \tag{1}$$

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The number of edges in a subgraph spanned by a set X of vertices of G will be denoted by $e_G(X)$. We shall say that X is "completely joined" to Y if every vertex of X is joined to every vertex of Y.

Given two points x, y in the Euclidean space $\mathbf{E}^h$, $\rho(x,y)$ will denote their distance.

Given two graphs G_n and H_n, their distance $\Delta(G_n, H_n)$ is defined as the minimum number of edges one has to delete from and add to G_n to get a graph isomorphic to H_n.

1. Introduction

This paper is a continuation of [10], where P. Erdős, A. Hajnal, V. T. Sós, and E. Szemerédi, investigated the following problem:

Assume that a so called forbidden graph L and a function $f(n)=o(n)$ are given. What is the maximum number of edges a graph G_n can have without containing L as a subgraph, and also without having more than $f(n)$ independent vertices?

This problem is motivated by the classical Turán and Ramsey theorems [25,26], [19], (see also [1,21]), and also by some applications of the Turán theorem to geometry, analysis (in particular, potential theory) [27–29], [11–13].

In 1930 Ramsey proved his famous theorem [19]:

Ramsey theorem for ordinary graphs. *Given r integers $k_1,\dots,k_r$, there exists a threshold integer $R = R(k_1,\dots,k_r)$ such that if a complete graph K_n is edge-coloured in r colours and $n \ge R$, then for some $\nu \le r$ it contains a K_{k_ν} in the νth colour.*

Motivated by this theorem, Turán posed the following question:

What is the maximum number of edges a graph G_n can have without containing a complete K_q?

Obviously, if we partition n vertices into $q-1$ classes as equally as possible and join two vertices iff they belong to different classes, then we obtain a graph not containing K_q. This graph will be denoted by $T_{n,q-1}$, and called the Turán graph on n vertices and $q-1$ classes.

P. Turán proved (1940) [25,26], that

Turán theorem. *Given n and q, $(1<q\le n)$, all the graphs G_n on n vertices not containing a K_q have at most $t(n,q-1)=e(T_{n,q-1})$ edges, and this maximum is attained only by $T_{n,q-1}$.*

Note that

$$t(n,q-1) = \left(1-\frac{1}{q-1}\right)\binom{n}{2} + O(1).$$

As Turán observed, both Ramsey's and his theorems are, in some sense, generalizations of the Pigeon Hole Principle, and therefore they are applicable in many different situation [21]. He himself started a new line of applications [27–29], [11–13] in geometry and analysis (primarily potential theory). Another line was started by G. Katona, where some Turán type theorems were used to obtain inequalities in Probability Theory. One of the limitations of these applications seemed to be that in Turán's theorem the graphs attaining the maximum (called extremal graphs) are too special e.g. they have "large" independent sets. The question is, how stable the extremal graph is? In [22] the following was asked:

How many edges can a graph G_n have if it contains neither a K_q nor such an "enormous" independent set?

Since that a whole theory has emerged around this and similar questions. We first formulate a general question in this field.

Question. Let $L_1,\dots,L_r$ be given graphs and a graph G_n on n vertices be coloured by r colours $\chi_1,\dots,\chi_r$. Assume that the subgraph of colour χ_ν contains no L_ν, (for $\nu=1,\dots,r$), and $\alpha(G_n)\le m$. What is the maximum of $e(G_n)$ under these conditions? This maximum will be denoted by $RT(n,L_1,\dots,L_r,m)$, and the L_ν's will be called forbidden graphs.

In the case when the forbidden graphs are complete graphs, $L_\nu = K_{k_\nu}$, we shall use the simpler notation $RT(n,k_1,\dots,k_r,m)$. Mainly we will be interested in the case $m=f(n)=o(n)$, and use the (simplified) notation $RT(n,k_1,\dots,k_r,o(n))$.

In [10] primarily the problem of $RT(n,L,o(n))$, i.e. the case of $r=1$ and arbitrary L was investigated. Here we consider the problem for $r>1$.

It is probably hopeless to give an exact description of the optimum. Quite often, instead of looking for the optimal (or so-called extremal) graphs, we try to find an asymptotically extremal sequence of graphs of relatively simple structure. "Relatively simple" means that its structure depends very loosely on n.

Definition 1. (Asymptotically extremal graphs) Given the forbidden graphs $L_1,\dots,L_r$, and the function f, a sequence of graphs (S_n) will be called an asymptotically extremal sequence for $RT(n,L_1,\dots,L_r,f(n))$ if the edges of S_n can be r-coloured so that the νth colour contains no L_ν, for $\nu=1,\dots,r$, $\alpha(S_n)\le f(n)$ and $e(S_n)=RT(n,L_1,\dots,L_r,f(n))+o(n^2)$.

In this paper we give upper bounds on $RT(n,L_1,\dots,L_r,o(n))$ and show that a generalization of the Bollobás-Erdős graph [2] forms an asymptotically extremal sequence for the problem of $RT(n,k_1,\dots,k_r,o(n))$. In Sections 2,3 we shall define the generalized Bollobás-Erdős graphs, and formulate our corresponding results. We shall also establish some results concerning particular cases of this problem. In Section 4 we give the proofs.

In a continuation of this paper we shall investigate the problem where instead of $\alpha(G_n)=o(n)$ we assume the stronger condition that the maximum size of a K_p-free induced subgraph of G_n is $o(n)$.

Since we will be able to state our main result only after having introduced some involved definitions, here we give a simplified version of it.

It is easy to see, that for every $L_1,\dots,L_r$ there exists a minimum $\vartheta(L_1,\dots,L_r)$ such that whenever $f(n)=o(n)$, then

$$RT(n,L_1,\dots,L_r,f(n))\le \vartheta(L_1,\dots,L_r)n^2+o(n^2). \tag{2}$$

(The equality is attained for some functions $f(n)$ for which $f(n)/n\to 0$ sufficiently slowly. Of course, if $f(n)/n\to 0$ too fast, then this ϑ can be replaced by a smaller constant. Still, when we speak (sometimes a little loosely) about $RT(n,L_1,\dots,L_r,o(n))$, we will always mean the determination of this minimum ϑ.

Theorem 1. *Given the integers $k_1,\dots,k_r \geq 3$, for $RT(n,k_1,\dots,k_r,o(n))$ there exists a fixed t and a sequence of asymptotically extremal graphs (S_n) such that the vertices of S_n can be partitioned into t classes $X_1,\dots,X_t$ where*

— $e_{S_n}(X_i)=o(n^2)$ *for* $i=1,2,\dots,t$,

and for $1\leq i<j\leq t$

— *either* $d_{S_n}(X_i,X_j)=\frac{1}{2}+o(1)$ *or* $d_{S_n}(X_i,X_j)=1+o(1)$.

To get some insight first we construct some graphs which later will turn out to be extremal in most of our problems.

The Erdős graph. The simplest statement in our field is as follows. If $\alpha(G_n)=o(n)$ and $K_3 \not\subseteq G_n$, then $e(G_n)=o(n^2)$. One immediately wants to know if such graphs (with $\alpha(G_n)=o(n)$ and $K_3 \not\subseteq G_n$) do exist at all. The existence of such graphs can be proved by using probabilistic arguments but there is a more useful way to get such graphs. In [14] Erdős and Rogers constructed a graph sequence (G_m) for which $\alpha(G_m)=o(m)$ but $K_3 \not\subseteq G_m$.

Combining the Erdős (or Erdős-Rogers) graph with Turán's graph we get some very useful graph sequences.

Definition 2. (*Canonical colouring* with respect to a vertex-partition) Given a graph G_n the vertex set of which is partitioned into the classes $X_1,\dots,X_q$, an r-colouring will be called **canonical**, if the colour of an edge depends only on the classes its endpoints belong to: all the edges $(x,y): x,y\in X_i$ have the same colour χ_{ν_i}, and for $1\leq i<j\leq q$ all the edges $(x,y): x\in X_i, y\in X_j$ have the same colour $\chi_{\nu_{i,j}}$.

Construction 1. Given the integer d, consider $T_{n,d}$, with the classes $C_1,\dots,C_d$, and put into each C_i an Erdős graph H of $\left[\frac{n}{d}\right]$ vertices, with $\alpha(H)=o(n)$. Thus we get a graph $U_n=U(n,d)$ with $\alpha(U_n)=o(n)$.

(a) Since $K_{2d+1}\not\subseteq U_n$, therefore

$$RT(n,2d+1,o(n)) \geq e(T_{n,d}). \tag{3}$$

(b) Colouring the edges of $T_{n,d}$ by RED and the edges in the classes C_i by BLUE we obtain a graph not containing RED K_{d+1}, nor BLUE K_3:

$$RT(n,d+1,3,o(n)) \geq e(T_{n,d}). \tag{4}$$

We shall prove (see Theorem 4) that this is asymptotically sharp for $d=2$ and $d=3$. However for $d=4$ we get the extremal sequence by the following:

Construction 2. Let $t=R(q,s)-1$ (where $R(q,s)$ is the Ramsey function). Colour $T_{n,t}$ by RED and BLUE canonically (with respect to the classes of $T_{n,t}$) so that the coloured graph should contain neither RED K_q, nor BLUE K_s. Put into each class of this graph a RED Erdős graph. Then the resulting graph $U_n=U(n,q,s)$ will contain neither a RED K_{2q-1}, nor BLUE K_s. Clearly,

$$e(U_n) > \left(1-\frac{1}{t}\right)\binom{n}{2},$$

and

$$\alpha(U_n) = o(n).$$

Hence

(5) $$RT(n, 2q-1, s, o(n)) \geq e(T_{n,t}).$$

Let us list the already known upper bounds on $RT(n, k_1, \ldots, k_r, m)$. If $m = n$, then there is no real restriction on $\alpha(G_n)$. Put $T(n, k_1, \ldots, k_r) := RT(n, k_1, \ldots, k_r, n)$. This case is described by

Theorem [22]. *Let $k_1, \ldots, k_r \geq 3$ be given integers, and let $T(n, k_1, \ldots, k_r)$ denote the maximum number of edges an r-coloured G_n can have under the condition that it does not contain a K_{k_ν} in its νth colour. Then*

$$T(n, k_1, \ldots, k_r) \leq \left(1 - \frac{1}{R(k_1, \ldots, k_r) - 1}\right)\binom{n}{2}.$$

This inequality immediately follows from Turán's and Ramsey's theorems. Indeed, if

$$e(G_n) > \left(1 - \frac{1}{R(k_1, \ldots, k_r) - 1}\right)\binom{n}{2},$$

then it contains a complete K_R for $R = R(k_1, \ldots, k_r)$. This is coloured by r colours, hence for some ν it contains a K_{k_ν} in the νth colour.

The theorem is sharp up to an additive constant $O(1)$.

If $r = 1$, then this theorem reduces to Turán's theorem.

Remark 1. Clearly, if $m > \frac{n}{R(k_1, \ldots, k_r) - 1}$, then

$$RT(n, k_1, \ldots, k_r, m) = T(n, k_1, \ldots, k_r).$$

On the other hand we shall always assume that m is so large that $n < R(k_1, \ldots, k_r, m)$.

The problem of $T(n, L_1, \ldots, L_r) := RT(n, L_1, \ldots, L_r, n)$ is still "easy":

Theorem. [7] *Let $t = t(L_1, \ldots, L_r)$ be the smallest integer for which there exists a $v > 0$ such that*

$$K_t(v, \ldots, v) \to (L_1, \ldots, L_r),$$

(or, in less formal language), for any r-colouring of $K_t(v, \ldots, v)$ there exists a colour ν and a monochromatic L_ν in this colour. Then for some constant $c > 0$,

$$T(n, L_1, \ldots, L_r) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + O(n^{2-c}).$$

This is related to some results of Burr, Erdős and Lovász [7] and Chvátal. Applying their results and the Erdős–Stone Theorem [16] a slightly weaker form of the above theorem immediately follows. The proof of this version uses a strengthening of Erdős–Stone Theorem [16], the Erdős–Simonovits Theorem [8], [20], and is easy.

These are results where we used colourings but actually there were no restrictions on the independent sets. The first theorem where $\alpha(G_n) = o(n)$ was assumed is

Theorem. [15] *For* $q=2k+1$

$$RT(n,q,o(n)) = \left(1-\frac{1}{k}\right)\binom{n}{2} + o(n^2). \tag{6}$$

(See Construction 1.)

Intuitively the theorem asserts that for large q, if we add (in Turán's theorem on K_q) the extra condition that $\alpha(G_n)=o(n)$, this will have roughly the same effect as excluding a complete graph of half the original size.

For $q=3$ the theorem is trivial, since in that case K_3 is excluded, which implies that each degree is $o(n)$. The previous theorem leaves open the question of the **even** values, which is much more difficult. For quite a long while the question whether

$$RT(n,K_4,o(n)) = o(n^2)$$

seemed to be untractable. Then Szemerédi found a nontrivial upper bound:

Theorem. [23]

$$RT(n,K_4,o(n)) \le \frac{n^2}{8} + o(n^2). \tag{7}$$

It came as a surprise — when Bollobás and Erdős proved — that (7) is sharp:

Theorem. [2] (Construction)

$$RT(n,K_4,o(n)) \ge \frac{n^2}{8} - o(n^2). \tag{8}$$

Finally, the problem of K_{2k} and many related problems were settled in [10].

Theorem. [10] *For* $q=2k$,

$$RT(n,K_q,o(n)) = \frac{1}{2}\frac{3q-10}{3q-4}n^2 + o(n^2). \tag{9}$$

2. The Bollobás-Erdős graph and its generalization

First we describe the original construction of Bollobás and Erdős used to prove (8). We shall fix n points on the surface of a high dimensional unit sphere $\mathbf{S}$ "uniformly". Speaking of the relative measure of a set X on $\mathbf{S}$ we will mean the measure of X divided by the total measure of $\mathbf{S}$.

In the definition of the BE-graph [2] the following fact is used:

Given $\delta,\eta>0$, for any $\varepsilon>0$ small enough and integer $h>h_0(\varepsilon)$, if $\mathbf{S}$ is a unit sphere in the h-dimensional euclidean space $\mathbf{E}^h$, then for $\mu=\varepsilon/\sqrt{h}$

(∗) for every sufficiently large n $\mathbf{S}$ can be partitioned into n sets of equal measure and diameter $<\mu/10$;

(**) the relative measure of the spherical cap of diameter $2-\frac{1}{2}\mu$ is at most δ;

(***) given a point x, the relative measure of the set of points y with $\rho(x,y)<\sqrt{2}-\mu$ is almost $\frac{1}{2}$, more precisely, at least $\frac{1}{2}-\eta$.

Based on this, let us partition **S** into $\frac{n}{2}$ subsets $A_1,\dots,A_{n/2}$ as described in (*), select two sets of vertices, X_1,X_2 of size $\frac{n}{2}$, each containing exactly one point from each A_i, and then join two vertices x and y by an edge iff

(i) either $x\in X_i$ and $y\in X_{3-i}$, and $\rho(x,y)<\sqrt{2}-\mu$

(ii) or $x,y\in X_i$ $i=1$ or 2 and $\rho(x,y)\geq 2-\mu$.

Denote the obtained graph by $BE(n,h,\varepsilon)$. It is not uniquely determined by the above construction, but any realization of it will have the properties we need:

(a) $\alpha(BE(n,h,\varepsilon))<2\delta n$.

(b) the degrees of the vertices are $>\left(\frac{1}{2}-\eta\right)n$.

(c) $K_4\not\subseteq BE(n,h,\varepsilon)$.

(d) the subgraphs spanned by X_1 and X_2 are K_3-free.

In our main theorems the asymptotically extremal graph sequences will be given by some generalizations of the BE-graph. In this generalization we use many sets $X_1,\dots,X_t$ on the sphere and of different sizes.

Definition 3. (Weighted t-partite BE graphs, construction). The graph to be defined below will depend on the integers h, t, $n_1,\dots,n_t$, and on a small positive number μ, and will be denoted by

$$B(h,t|n_1,\dots,n_t|\mu).$$

We shall subdivide the points of the h-dimensional sphere **S** into n_i sets according to (*), for each i. For $i=1,\dots,t$, $k=1,\dots,n_i$ choose a vertex x_{ik} from the kth set of the ith partition. Put $X_i=\{x_{ik},\ 1\leq k\leq n_i\}$ for $1\leq i\leq t$ and let $V(B)=\bigcup_{i=1}^{t} X_i$. For each pair $x,y\in X_i$ we join them iff $\rho(x,y)>2-\mu$. For some pairs (i,j) $(i\neq j)$ we join every vertex of X_i to every vertex of X_j, for all the other pairs we build a Bollobás-Erdős graph between the two classes: join $x\in X_i$ to $y\in X_j$ iff $\rho(x,y)<\sqrt{2}-\mu$. The resulting graph is the monochromatic generalized Bollobás-Erdős or shortly GBE graph. If two classes are joined completely, we shall call this a "full" connection; if they are joined by a Bollobás-Erdős graph, we shall call this a "half" connection.

Remark 2. The graph $B(h,t|n_1,\dots,n_t|\mu)$ is not completely determined by the parameters listed, even when the "full" connections and "half" connections are determined, since the embedding of the vertices into the sphere also can slightly influence its structure. However, in all our statements these minor variances can be neglected. We shall choose the parameters so that with $\varepsilon\to 0$ slowly enough and $h\to\infty$ (at a speed, depending on ε), $\mu=:\varepsilon/\sqrt{h}$ and $n>n_0(h,\varepsilon)$. Under these assumptions our assertions will not depend on the choice of these "hidden parameters".

3. Main results

Our main results solve asymptotically the problem of $RT(n,k_1,\dots,k_r,o(n))$, (Theorem 2) and yield an upper bound on $RT(n,L_1,\dots,L_r,o(n))$, (Arboricity Theorem).

The structure of the asymptotically extremal graphs is given by

Theorem 2. *Let* $k_1,\dots,k_r\geq 3$ *be given integers, then for some fixed* t *there exists a sequence of graphs* $B(h,t|n_1,\dots,n_t|\mu)$ *asymptotically extremal in the problem of* $RT(n,k_1,\dots,k_r,o(n))$. *(Meanwhile* $\mu\to 0$ *and* $h\to\infty$.*)*

Obviously Theorem 2 immediately implies Theorem 1.

Let A be a $t\times t$ matrix, whose diagonal entries are all 0's and $a_{ij}=\frac{1}{2}$ or 1 depending on whether the ith and jth classes of $B(h,t|n_1,\dots,n_t|\mu)$ are joined by a BE-graph or completely. Let us normalize the integers $n_1,\dots,n_t$ by putting

$$u_i=\frac{n_i}{\sum n_j}.$$

Then

$$e(B(h,t|n_1,\dots,n_t|\mu))=uAu^*n^2+o(n^2) \tag{10}$$

(as $h\to\infty$, $\varepsilon\to 0$, $n\to\infty$). If uAu^* attains its maximum on the standard simplex

$$\left\{\mathbf{u}:\quad \sum_{i=1}^{t}u_i=1,\quad u_i\geq 0\ (i=1,\dots,t)\right\}$$

on the boundary, i.e. for some $u_k=0$, then the GBE-graph will be called **degenerate**. If there are no maxima on the boundary, then one can easily see that there is exactly one maximum and the structure of this GBE-graph will be called **dense**.

An equivalent form of Theorem 2 is

Theorem 2'. *Given the integers* $k_1,\dots,k_r\geq 3$, *then for some fixed* t *there exists a sequence of dense graphs* $B(h,t|n_1,\dots,n_t|\mu)$ *asymptotically extremal for* $RT(n,k_1,\dots,k_r,o(n))$.

Indeed, assume that Theorem 2 is already proved. If there exists a degenerate asymptotically extremal structure $B(h,t|n_1,\dots,n_t|\mu)$ for the problem of $k_1,\dots,k_r$, then we may assume that $n_t=0$, i.e. we got an asymptotically extremal structure with fewer classes. Consider an asymptotically extremal structure $B(h,t|n_1,\dots,n_t|\mu)$ with minimum t. It must be dense.

Remark 3. Our problem is strongly related to multigraph and digraph extremal problems. The notion of dense structures was introduced by Brown, Erdős, and Simonovits in [3], (see also [4–6]).

To get some more information on $RT(n;k_1,\dots,k_r,o(n))$ first we define the generalized complete graphs, then define a Ramsey number $\beta=\beta(k_1,\dots,k_r)$ for generalized complete graphs, and finally prove that $\vartheta(k_1,\dots,k_r)=\beta(k_1,\dots,k_r)$.

Definition 4. (*Generalized complete subgraphs*) Let R be a graph some vertices of which are "marked", the edges of which are weighted by the weights 0, $\frac{1}{2}$ and 1.

$X \subseteq V(R)$ and $Y \subseteq X$ span (by definition) a generalized complete subgraph of the size $|X|+|Y|$ if

(a) all the vertices of Y are "marked";

(b) all the edges in Y have weight 1,

(c) all the edges in X have weight $\geq \frac{1}{2}$.

Definition 5. (*Weighted Ramsey numbers*). Assume that the *vertices* and the *edges* of a K_t are coloured by the colours $\chi_1, \dots, \chi_r$, and the edges are weighted by $\frac{1}{2}$ and 1. For the colour χ_ν we define R^ν as a weighted graph, where a vertex is "marked" iff its colour is χ_ν, and the edge e gets weight $\frac{1}{2}$ or 1 if its colour is χ_ν and its weight is $\frac{1}{2}$ or 1 respectively. (If its colour differs from χ_ν, then its weight is 0 in R^ν.) Finally, assume that there is a distribution $\mathbf{u} = (u_1, \dots, u_t)$ $(u_i > 0, \sum_{i=1}^{t} u_i = 1)$ on the vertices of K_t. If we wish to emphasize that K_t is weighted, we shall write $K_t(w)$. Let

$$(11) \qquad g(K_t(w), \mathbf{u}) = \sum_{1 \leq i < j \leq t} w_{i,j} u_i u_j.$$

where $w_{i,j}$ is the weight of (i,j). We define the *edge density* of such a weighted graph as

$$(12) \qquad g(K_t(w)) = g(K_t(w), \mathbf{u}) = \max_{\mathbf{u}} \sum w_{i,j} u_i u_j.$$

The weighted Ramsey number $\beta(k_1, \dots, k_r)$ is the maximum B such that

(∗) *there exist a t and a weighted colouring of K_t with edge density B for which none of the subgraphs R^ν of colours χ_ν (in K_t) contains a generalized K_{k_ν}.*

Clearly,

$$(12^*) \qquad t < R(k_1, \dots, k_r).$$

Theorem 2". *Let $k_1, \dots, k_r \geq 3$ be given integers, then*

$$(13) \qquad \vartheta(k_1, \dots, k_r) = \beta(k_1, \dots, k_r).$$

In other words,

$$(13^*) \qquad RT(n, k_1, \dots, k_r, o(n)) = \beta(k_1, \dots, k_r) n^2 + o(n^2).$$

Corollary 1. *There exists a finite algorithm to find an asymptotically extremal sequence for any $RT(n, k_1, \dots, k_r, o(n))$.*

Motivation of the previous definitions. We shall consider a graph G_n the edges of which are coloured with r colours, and will approximate it by some (canonically coloured) $H_n = B(h, t | n_1, \dots, n_t | \mu)$. This approximation will be "encoded" using a graph R_t defined on the index set $\{1, 2, \dots, t\}$ of the classes X_i of H_n, (see Definition 3).

(a) The encoding goes as follows. If two sets X_i and X_j are joined completely, (in Definition 3) in RED, then i and j will be joined in R_t by a RED edge of weight 1. If they are joined by a RED BE-graph, i and j will be joined in R_t by a RED edge of weight $\frac{1}{2}$. If the edges in X_i are RED, "mark" i in RED.

(b) This encoding will be useful, since the maximum size of a RED ordinary complete subgraph of G_n will be at least as large as that of $B(h,t|n_1,\dots,n_t|\mu)$ and this will be exactly as large as the maximum size of a RED generalized complete subgraph of R_t.

We shall characterize below the relative sizes of the classes X_i, $i=1,\dots,t$ by the distribution vector $\mathbf{u}$.

(c) The size $|X|+|Y|$ of a generalized complete graph is used because if R_t contains a generalized K_q with the vertex-sets X and $Y\subseteq X$ (as described above), then we shall be able to find a $K_q \subseteq G_n$ with $q=|X|+|Y|$: each vertex of Y will yield 2 vertices of this ordinary K_q and each vertex of $X-Y$ will yield one vertex of K_q.

Until now we restricted our considerations to the case of complete graphs. As described in [10], $RT(n,L,o(n))$ depends in some sense on the arboricity of L defined below — differently from the usual one — as follows. To consider arbitrary $L_1,\dots,L_r$ in our problem, we need the

Definition of Arboricity. (a) $L\in Arb(2k)$ if the vertices of L can be k-coloured so that the subgraph spanned by the νth colour is a forest, for $\nu=1,2,\dots,k$.

(b) $L\in Arb(2k+1)$ if the vertices of L can be $k+1$-coloured so that the subgraph spanned by the νth colour is a forest, for $\nu=1,2,\dots,k$ and the vertices of colour $k+1$ are independent.

Remark 4. A slightly different definition of arboricity used to be given as the minimum k for which L can be coloured in k colours so that each colour-class spans a forest.

Arboricity Theorem. *Given r graphs $L_1,\dots,L_r$ with $L_\nu\in Arb(k_\nu)$, then*

$$RT(n,L_1,\dots,L_r,o(n)) \le RT(n,k_1,\dots,k_r,o(n)) + o(n^2).$$

Unfortunately, we have only upper bound for the general case. We do not even know the truth for $r=1$, not even for one of the simplest cases: we do not know if $RT(n,K(2,2,2),o(n))=o(n^2)$ or not. The Arboricity Theorem will not explicitly be proven here: we shall prove Theorem 2 and the reader can easily generalize the upper bound of Theorem 2 to the Arboricity Theorem, applying the Tree-building lemma, and the ideas given (in details) in [10].

We determine $RT(n;L_1,L_2,o(n))$ for some special L_1,L_2.

Definition 5. We shall say that for a problem $RT(n,L_1,\dots,L_k,o(n))$ the **weak stability** property holds if for any two sequences of asymptotically extremal graphs, say $(S_n)_\infty$ and $(Z_n)_\infty$ their distance is $o(n^2)$: $\Delta(S_n,Z_n)=o(n^2)$.

Theorem 3.

(a) $\vartheta(K_3,K_3)=\frac{1}{4}$. *$U(n,2)$ is an asymptotically extremal sequence.*

(b) $\vartheta(K_3,K_4)=\frac{1}{2}\left(1-\frac{1}{3}\right)$. *$U(n,3)$ is an asymptotically extremal sequence.*

(c) $\vartheta(K_3,K_5)=\frac{1}{2}\left(1-\frac{1}{5}\right)$. *$U(n,3,3)$ is an asymptotically extremal sequence.*

(d) $\vartheta(K_4,K_4) = \frac{1}{2}\left(1-\frac{3}{14}\right)$. *The sequence in Construction 4 below is an asymptotically extremal structure.*

(e) *In all these cases the weak stability holds.*

Theorem 4. *If p and q are odd integers, then*

$$RT(n,C_p,C_q,o(n)) = \frac{n^2}{4} + o(n^2). \tag{14}$$

4. Proof of Theorems 2–4

Proof of the lower bound for $\vartheta(k_1,\dots,k_r)$. Construction. There exists a correspondence between the coloured-weighted K_t's with distribution $\mathbf{u}$ and the graph sequences $\{B(h,t|n_1,\dots,n_t|\mu)\}$.

Given a coloured weighted K_t with a distribution (with a fixed r-colouring of the edges and vertices of K_t, with a distribution $u_1,\dots,u_t$ on the vertices of K_t and the weights $\frac{1}{2}$ and 1 on the edges), then there are canonically coloured generalized Bollobás-Erdős graphs $B(h,t|n_1,\dots,n_t|\mu)$ corresponding to these data. Namely, given the dimension h and $\mu > 0$, fix the sets $X_1,\dots,X_t$ on the h-dimensional sphere $\mathbf{S}$, so that

(a) $|X_i|=n_i=u_in+o(n)$ and

(b) the set $X=X_1\cup\ldots\cup X_t$ be distributed as described in Definition 3.

(c) take a "half" resp. a "full" connection between X_i and X_j if $w(i,j)=\frac{1}{2}$ resp. $w(i,j)=1$.
Obviously $e(B(h,t|n_1,\dots,n_t|\mu))=g(K_t(w),u)n^2+o(n^2)$.

We need also

Lemma 1. *The above $B(h,t|n_1,\dots,n_t|\mu)$ does not contain K_q in colour χ_ν iff the corresponding coloured weighted K_t does not contain a generalized K_q in colour χ_ν.*

Proof. If $x_1,\dots,x_q$ form a RED $K_q\subseteq B(h,t|n_1,\dots,n_t|\mu)$, then each X_i contains at most 1 vertex of this K_q, **unless** X_i is RED. Even so, it contains at most 2 vertices. Further, if X_i and X_j both contain 2 vertices of this K_q, then they must be RED and joined by RED edges completely. Hence the corresponding coloured, weighted K_t contains a RED generalized K_q. This proves half of Lemma 1. The other half is left to the reader. ∎

Now, given an arbitrary colouring ϕ of K_t satisfying the property $(*)$ in Definition 5 (of $\beta(k_1,\dots,k_r)$), build a $B(h,t|n_1,\dots,n_t|\mu)$ with the corresponding colours and weight function $\mathbf{w}$ and distribution $\mathbf{u}$. Then — by Lemma 1 and the Definitions 3, and 5 — $K_{k_\nu} \not\subseteq B(h,t|n_1,\dots,n_t|\mu)$ in colour χ_ν, and it will have

$$\beta(k_1,\dots,k_r)n^2 + o(n^2)$$

edges. (The error term comes from the fact that there is an error term in the BE-graph and that the vertices cannot be distributed exactly according to the distribution $\mathbf{u}$.) Hence

$$RT(n,k_1,\dots,k_r,o(n)) \geq \beta(k_1,\dots,k_r)n^2 + o(n^2). \tag{15}$$

∎

Proof of the upper bound

In the proof of the upper bound our first problem is that the condition $\alpha(G_n) = o(n)$ applies only to the union of the colours, and so G_n could contain cn independent vertices in each separate colour. We need a lemma to overcome this difficulty.

Lemma 2. *Fix an $\varepsilon > 0$. If G_m is an r-coloured graph with $\alpha(G_m) = \varepsilon m$, then one can partition the vertices of G_m into $r+1$ classes $C_0, C_1, \dots, C_r$ so that $|C_0| < 2\sqrt[r]{\varepsilon}m$ and for $1 \leq \nu \leq r$ for each C_ν every subset $Y \subset C_\nu$ of size $> \sqrt[r]{\varepsilon}m$ contains an edge of the νth colour χ_ν.*

Remark 5. One can ask if this partitioning is really necessary or under the condition of Lemma 2 there must always exist a colour χ and a constant η tending to 0 as $\varepsilon \to 0$, for which every set of size $> \eta m$ contains an edge of colour χ. The partitioning is needed. Let G_m be the disjoint union of two Erdős graphs, one of which is coloured in RED, the other in BLUE. Then (fixing any $\eta \in (0, \frac{1}{2})$), there is no fixed colour χ_0 (=RED or BLUE) such that every vertex set Y of size $> \eta m$ contains an edge of colour χ_0.

Proof of Lemma 2. Let us assume that G_m is coloured by r colours: $\chi_1, \dots, \chi_r$. For the sake of simplicity, we shall call χ_1 RED, χ_2 BLUE ...and χ_r BLACK. Let $\eta = \sqrt[r]{\varepsilon}$ and call an $X \subseteq V(G_m)$ RED if

(a) $|X| > \varepsilon m$ and

(b) every $X^* \subseteq X$ of size $> \eta|X|$ contains a RED edge.

Clearly, the disjoint union of RED sets is RED. (However, a subset of a RED set is not necessarily RED). Fix a maximal RED subset of $V(G_m)$. Denote it by V_1. In the remaining part fix a maximal BLUE subset $V_2, \dots$, and finally — in the rth step — fix a maximal BLACK subset. Put $U_0 = V(G_m) - \cup_i V_i$. If $|U_0| \leq \eta m$ then we are home.

Suppose that $|U_0| > \eta m$. If every ηm-tuple of U_0 contains a RED edge, then we are home.

In the remaining case U_0 contains a U_1 of $> \eta m$ vertices but not containing RED edges. We shall use that — by the maximality of the V_i no subsets of U_1 can be BLUE,...or BLACK. Since U_1 is not BLUE, it contains a U_2 of $\eta^2 m$ vertices but not containing BLUE edges. ...Since U_{r-1} is not BLACK, it contains a U_r of

$>\eta^r m$ vertices but not containing BLACK edges. Clearly, U_{r-1} is an independent set of $>\varepsilon m$ vertices, a contradiction.

Remark 6. It is easy to see that $\sqrt[r]{\varepsilon}m$ cannot be replaced by any essentially smaller value. To see this fix an integer p, put $q=p^r$ and $\varepsilon=\frac{1}{q}$. Take a Turán graph $T_{m,q}$, enumerate its classes by r-tuples $(p_1,\dots,p_r)$, where $0\le p_i<p$. Colour the edges joining the class $C(a_1,\dots,a_r)$ and $C(b_1,\dots,b_r)$ with colour j if the two index-sequences $a_1,\dots,a_r$ and $b_1,\dots,b_r$ differ first in the jth position. In this graph each set of size $>\varepsilon m$ contains an edge, and one can easily see that with an $\eta<\frac{1}{(r+1)p}$ the lemma does not hold.

Symmetrization

One of the basic tools we shall use to prove our theorems is the symmetrization.

Let us modify Definition 5 by allowing (besides the weights $\frac{1}{2}$ and 1) the weight 0 as well. (This corresponds to allowing pairs of classes in $B(h,t|n_1,\dots,n_t|\mu)$ not joined at all.) If there exists such a 0 weight in our graph, call the graph deficient. (12*), namely $t<R(k_1,\dots,k_r)$ does not hold anymore for these deficient K_t's: the problem becomes infinite as soon as we allow deficient graphs. The next lemma tells us that for each deficient K_t there exists a non-deficient one, at least as good as the deficient one.

For a (coloured) graph G_n with the distribution $\mathbf{u}$ on the vertices and weights $\mathbf{w}$ on the edges let the **weighted degree** of a vertex x be

$$d(x)=n\sum_{(x,y)\in E(G_n)} w_{(x,y)}u_y\,,$$

where E is the set of edges.

Lemma 3. *Let K_t, w and $\mathbf{u}$ be as in Definition 5, with the only exception that some weight, e.g., the weight of the edge (x,y) be 0. Let $d(x)\ge d(y)$. Then $g((K_t-y)(w'))\ge g(K_t(w))$ if w' is the weight on K_t-y where the (original) weight of y (in K_t) is added to the weight of x (in K_t-y).*

Proof. Trivial.

Generalized Szemerédi Lemma.

Szemerédi Lemma [24] asserts — loosely speaking — that given an $\varepsilon>0$, the vertex-set of every G_n can be partitioned into a bounded number of classes so that almost all the pairs of classes will be ε -regular in the following sense.

Regularity condition. Given a graph G_n and two disjoint vertex sets in it, X and Y, we shall call the pair (X,Y) ε -regular if for every subset $X^*\subset X$ and $Y^*\subset Y$ satisfying $|X^*|>\varepsilon|X|$ and $|Y^*|>\varepsilon|Y|$,

$$|d(X^*,Y^*)-d(X,Y)|<\varepsilon.$$

The regularity condition means that the edges behave (in some weak sense) as if they were random. The following generalization asserts that if the edges of G_n are coloured by a bounded number of colours, then one can find a partition for which the assertion of Szemerédi Lemma holds for each colour simultaneously. Below we formulate this lemma in the form we shall need it. Let $d_\nu(X,Y)$ denote the density in colour χ_ν.

Generalized Regularity Lemma. *For every $\varepsilon > 0$, and integer λ_0 there exists a λ_ε such that for every r-coloured G_n $V(G_n)$ can be partitioned into sets $V_0, V_1, \dots, V_\lambda$ — for some $\lambda_0 < \lambda < \lambda_\varepsilon$ — so that each $|V_i| < \varepsilon n$, $|V_i| = m$ (is the same) for every $i > 0$, and for all but at most $\varepsilon\binom{\lambda}{2}$ pairs (i,j), for every $X \subseteq V_i$ and $Y \subseteq V_j$, satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|d_\nu(X,Y) - d_\nu(V_i, V_j)| < \varepsilon$$

for every colour χ_ν.

This generalization does not seem to follow from the original lemma, however, the original proof can easily be modified to yield it. The role of V_0 is to make possible that all the other classes be exactly of the same size, and the role of λ_0 is to make the classes V_i sufficiently small, so that — counting the edges — we could forget about the edges inside those classes.

Proof of the upper bound. (A) First we sketch the proof of the upper bound for $\vartheta(k_1, \dots, k_r)$, (or, equivalently, for $RT(n, k_1, \dots, k_r, o(n))$), and also fix some parameters.

Let us fix an arbitrary $\varepsilon > 0$. We shall also use a constant $\varepsilon_1 > 0$ which must be much smaller than ε, however, still fixed, and an $\varepsilon_2 > 0$ much smaller than ε but much bigger then ε_1. If $S = k_1 + \dots + k_r$, then

$$\varepsilon_1 = (\varepsilon/(16S))^{2S}, \qquad \varepsilon_2 = (\varepsilon/(16S))^S \tag{16}$$

is an appropriate — perhaps too cautious — choice.

Assume that a graph sequence (G_n) is given with $\alpha(G_n) =: \delta_n n = o(n)$. Let $\eta_n = \sqrt[r]{\delta_n}$. Here $\delta_n, \eta_n \to 0$. We shall fix an n_0 so that if $n > n_0$, then

$$\eta_n < (\varepsilon/16S)^{4S}. \tag{17}$$

Let G_n be coloured by r colours $\chi_1, \dots, \chi_r$, and let G_n^ν be the graph spanned by the edges of colour χ_ν. We shall apply the Generalized Szemerédi Lemma to G_n, with r, ε_1 and a lower bound $\lambda_0 = \frac{1}{\varepsilon}$ on the number of the classes.

(B) We shall prove that if $K_{k_\nu} \not\subseteq G_n^\nu$, $(\nu = 1, \dots, r)$ then

$$e(G_n) < \beta(k_1, \dots, k_r)n^2 + 3r\varepsilon n^2.$$

Below we sketch the proof and the details will be given in (C) and (D).

For each G_n — for $n > n_0$, in paragraph (C_5) below — we shall define a weighted, coloured graph $H(w_0)$ (with a weight function w_0 and a distribution $\mathbf{u}_0$, and allowing also multiple edges) so that

$$e(G_n) \le g(H(w_0), \mathbf{u}_0)n^2 + 3r\varepsilon n^2. \tag{18}$$

We shall call a weighted graph complete if any pair of vertices is joined by an edge of positive weight. $H(w_0)$ is not necessarily complete.

We shall apply symmetrization to $H(w_0)$ to obtain a weighted, coloured **complete** graph $K_t(w)$ satisfying

(1*) $g(K_t(w), \mathbf{u}) \ge g(H(w_0), \mathbf{u}_0)$.

(2*) If $K_t(w)$ contains a generalized complete K_q in colour χ_ν, then $H(w_0)$ also contains a generalized complete K_q in that colour and also an ordinary $K_q \subseteq G_n^\nu$.

(3*) Each pair of vertices of $K_t(w)$ is joined in exactly one of the colours $\chi_1, \ldots, \chi_r$ and with weight $\frac{1}{2}$ or 1. Since G_n^ν contains no K_{k_ν}, therefore $K_t(w)$ contains no generalized K_{k_ν} of colour χ_ν.

By (3*) $K_t(w)$ occurs in the set of weighted coloured graphs in the definition of $\beta(k_1, \ldots, k_r)$. Hence $g(K_t(w)) \le \beta(k_1, \ldots, k_r)$. By (1*), and (18),

$$e(G_n) \le g(K_t(w))n^2 + 2r\varepsilon n^2 \le \beta(k_1, \ldots, k_r)n^2 + 3r\varepsilon n^2.$$

Obviously, this (together with the already obtained lower bounds (15)) will prove Theorem 2".

(C) Now we define $H(w_0)$.

(C_1) Using Lemma 2 for n large enough we take a partition $V(G_n) = \bigcup_{\nu=0}^{r} U_\nu$ so that $|U_0| < \eta_n n$ and for $1 \le \nu \le r$ every $Y \subseteq U_\nu$, with $|Y| > \eta_n n$ contains an edge of colour χ_ν.

(C_2) Applying the Generalized Regularity Lemma to G_n, with the $\varepsilon_1 > 0$ and λ_0 (fixed above) we obtain the classes $V_0, V_1, \ldots, V_T$, with $|V_i| = m$, $1 \le i \le T$, $|V_0| < \varepsilon_1 n$, (where $T \ge \lambda_0$). We delete V_0.

(C_3) Consider the "union" of the two partitions; the classes defined by $V_{i,j} := V_i \cap U_j$, $1 \le i \le T$, $1 \le j \le r$. Keep only the graph G^* spanned by the vertex set

$$V^* = \bigcup_{|V_{i,j}| > \varepsilon_2 m} V_{i,j}$$

We shall use the notation $W_1, \ldots, W_M$ for these classes $V_{i,j}$ in V^*.

Observe that we deleted at most $(\varepsilon_2 + \eta_n)n$ points; $|V^*| > (1 - \varepsilon_2 - \eta_n)n$.

(C_4) For all but $r^2 \varepsilon_1 \binom{M}{2}$ pairs $((i,j),\ (i',j'))$, $i \ne i'$, for every χ_j

$$|d_{\chi_j}(X,Y) - d_{\chi_j}(V_{i,\nu},\ V_{i',j'})| < 2\varepsilon_1,$$

if $X \subseteq V_{i,j}$, $Y \subseteq V_{i',j'}$, $|X|, |Y| > \varepsilon_1 m$. Indeed, if $(V_i, V_{i'})$ is a regular pair, then

$$\begin{aligned}|d_\chi(X,Y) - d_\chi(V_{i,j}, V_{i'j'})| &\le |d_\chi(X,Y) - d_\chi(V_i, V_{i'})| \\ &\quad + |d_\chi(V_{i,j}, V_{i',j'}) - d_\chi(V_i, V_{i'})| < 2\varepsilon_1.\end{aligned}$$

(C_5) Now we define a weighted, coloured graph $H(w_0)$ whose vertices are the classes $V_{i,j}$, $V_{i,j} \subset V^*$.

(i) We colour a vertex $W_\ell = V_{i,j}$ by colour χ_{ν_j}; (here $V_{i,j} \subset U_j$).

(ii) We assign the weight $u_\ell := \frac{|W_\ell|}{|V^*|}$ to the vertex W_ℓ, $1 \le \ell \le M$.

(iii) We join two classes $W_\ell = V_{i,j}$, $W_{\ell'} = V_{i',j'}$ in colour χ_ν with weight

$$w_{\chi_\nu}(\ell,\ell') = \begin{cases} 0, & \text{if } d_{\chi_\nu}(W_\ell,W_{\ell'}) < 2\varepsilon \text{ or } (V_i,V_{i'}) \text{ is not } \varepsilon_1\text{-regular or } i = i'; \\ \frac{1}{2}, & \text{if } 2\varepsilon \le d_{\chi_\nu}(W_\ell,W_{\ell'}) < \frac{1}{2}+2\varepsilon,\ i \ne i' \text{ and } (V_i,V_{i'}) \text{ is } \varepsilon_1\text{-regular}; \\ 1, & \text{if } \frac{1}{2}+2\varepsilon \le d_{\chi_\nu}(W_\ell,W_{\ell'}),\ i \ne i' \text{ and } (V_i,V_{i'}) \text{ is } \varepsilon_1\text{-regular.} \end{cases}$$

Below, comparing $e(G_n)$ and $g(H(w_0))n^2$, we encounter 4 types of errors (typical in the applications of the Regularity Lemma):

— We have discarded a small number of vertices, $V - V^*$;

— We have discarded all the edges joining pairs in the same V_i's;

— We have replaced $e(V_i, V_{i'})$ by 0 for the irregular pairs;

— We had a "rounding" error 2ε while counting the edges between a regular pair $(V_i, V_{i'})$.

There is still a fifth type of error, coming from the fact that we have many colours. Namely, we shall estimate $e(W_\ell, W_{\ell'})$ by

$$\sum_{\ell,\ell'}\sum_{\nu} w_{\chi_\nu}(\ell,\ell')|W_\ell||W_{\ell'}| \le \sum_{\ell,\ell'}\sum_{\nu} w_{\chi_\nu}(\ell,\ell')u_\ell u_{\ell'}n^2,$$

where the "total density" $\sum_\nu w_{\chi_\nu}(\ell,\ell') > 1$ can occur. In such cases we agree to replace some $w_{\chi_\nu}(\ell,\ell')$'s by 0 to get $\sum_\nu w_{\chi_\nu}(\ell,\ell') = 1$. Since $e(W_\ell, W_{\ell'}) \le |W_\ell||W_{\ell'}|$, the estimates below will still hold.

Clearly,

$$e(G^*) \le \sum_{\ell,\ell'}\sum_{\nu} w_{\chi_\nu}(\ell,\ell')u_\ell u_{\ell'}n^2 + r\varepsilon n^2 + r^2\varepsilon_1 n^2 + \frac{n^2}{T},$$

where the term $r\varepsilon n^2$ comes from the "rounding" in the definition of $w_{\chi_\nu}(\ell,\ell')$, $r^2\varepsilon_1 n^2$ estimates the number of edges corresponding to irregular pairs. The term $\frac{n^2}{T} < \varepsilon n^2$ represents the edges the endvertices of which belong to the same groups V_i.

From

$$g(H(w_0),\mathbf{u_0}) = \sum_{\ell,\ell'}\sum_{\nu} w_{\chi_\nu}(\ell,\ell')u_\ell u_{\ell'}$$

we have

$$e(G^*) \le g(H(w_0),\mathbf{u_0})n^2 + 2r\varepsilon n^2.$$

By $|V - V^*| < (\eta_n + (r+2)\varepsilon_2)n$,

$$e(G_n) - e(G^*) \le |V - V^*|n \le (\eta_n + (r+2)\varepsilon_2)n^2 < 2\varepsilon n^2.$$

Therefore

$$e(G_n) \le g(H(w_0))n^2 + 3r\varepsilon n^2.$$

(D) Now we define a weighted, coloured K_t which satisfies (1*), (2*) and (3*).

(D_1) Assume that W_ℓ and $W_{\ell'}$ are independent in $H(w)$, i.e., $w_{\chi_\nu}(W_\ell, W_{\ell'}) = 0$ for $1 \le i \le r$. Let $d(W_\ell) \ge d(W_{\ell'})$. Provide W_ℓ with the weight $u_\ell + u_{\ell'}$ and delete $W_{\ell'}$. Obviously this contraction will not decrease the edge-density and will not increase the maximum size of a generalized complete subgraph in any colour. (Actually, this is the symmetrization lemma.) We repeat this step until all the 0 densities disappear. And even then we repeat this deletion and weight shifting until we get a dense K_t.

(D_2) We show that if for some ℓ, ℓ' $w_{\chi_\nu}(\ell,\ell') > 0$ for more than one ν, then without increasing the maximum size of a generalized complete subgraph in any colour, we can replace all these coloured edges with one appropriately coloured edge of weight 1.

Indeed, if $w_{\chi_\nu}(\ell,\ell') > 0$ e.g. for ν, ν', corresponding to the RED and BLUE edges between W_ℓ and $W_{\ell'}$, then we may assume that W_ℓ is not BLUE. We replace all the edges between W_ℓ and $W_{\ell'}$ with a BLUE edge of multiplicity 1. Obviously this cannot increase the maximum size of a generalized complete subgraph in any colour but BLUE. (As a matter of fact, there may be at most 2 such colours and both must have weight $\frac{1}{2}$.)

We show that it does not increase the maximum size of a K_q in BLUE either.

Suppose that after this step the sets X, Y define a BLUE generalized K_q. Then $W_\ell, W_{\ell'} \in X$, and $W_\ell \notin Y$ because W_ℓ is not BLUE, (see Definition 4). Hence the contribution of W_ℓ to $|X|+|Y|$ is 1, independently of the weight of $(W_\ell, W_{\ell'})$.

Clearly the edge-density will not increase at this step.

The only thing left to be proven is (2*): if we choose the parameters as described in (A) and if a generalized RED $K_q \subseteq K_t(w)$, then (i) a generalized RED $K_q \subseteq H(w_0)$, and (ii) (an ordinary) $K_q \subseteq G_n^{RED}$. Here (i) is trivial.

Remember that the vertices of $H(w_0)$ represent classes of vertices of G_n of size $> \varepsilon_2 m$, and the pairs of positive weight w_0 have the regularity property on level ε_1. Let our generalized RED K_q be spanned by the vertex sets X and $Y \subset X$ (as in Definition 4.) Then, if $\varepsilon_1, \varepsilon_2$ and δ_n satisfy (16) and (17), then we can choose recursively 1 vertex from each class represented by the points in $X-Y$ and 2 vertices from each class represented by the points in Y which form the RED K_q, as follows. Assume that we wish to choose the vertices e.g. from the classes $W_1, \dots, W_{|X|}$ and have already chosen the corresponding vertices from ℓ classes $W_1, \dots, W_\ell$ so that these vertices have at least $\varepsilon^\ell |W_i|$ common neighbours in W_i, $i = \ell+1, \dots, |X|$. Denote the class of these common neighbours by W_i^ℓ. Now we pick 1 or 2 vertices (according to the "plan") from $W_{\ell+1}$ so that they be joined to at least $\varepsilon^{\ell+1}|W_i|$ common neighbours in W_i, $i = \ell+2, \dots, |X|$. This can be done because

— all the sets W_i^ℓ in this argument have $> \varepsilon_1 m$ elements and are in some V_j. Thus the regularity lemma is applicable to them with ε_1.

— By the regularity, all but $S \cdot \varepsilon_1 m$ vertices of $W_{\ell+1}$ are joined to each W_i^ℓ by their "typical" densities: $> 2\varepsilon$ when we wish to pick 1 vertex; by $> \frac{1}{2} + 2\varepsilon$ when we have to pick two vertices. Denote the set of these vertices by $W_{\ell+1}^*$. Now the first case is trivial, we may pick any of these vertices.

— In the second case $W_{\ell+1}^*$ is in W_{RED} and is too large to be independent in RED. Hence it contains a RED edge uv such that at least $2\varepsilon|W_i^{\ell+1}|$ vertices of $W_i^{\ell+1}$ are joined in RED to both u and v. So we pick these u and v and go on with the recursion. ∎

Proof of Theorem 2'. It follows from the above argument and the construction in the proof of the lower bound.

Proof of Corollary 1, using Theorem 2". Basically we have to show that β of Definition 5 can algorithmically obtained. Since the coloured K_t we use to get $\vartheta(k_1, \dots, k_r)$ contains no K_{k_ν} of colour χ_ν, therefore

$$t < R(k_1, \dots, k_r)$$

(see (12*)). Further, given a t, and the weights w_{ij}, we can find the **optimum distribution**

$$\mathbf{u} = (u_1, \dots, u_t) \qquad (u_i \geq 0, \sum u_i = 1)$$

for which $g(K_t(w), u)$ is the maximum. This maximum was denoted by $g(K_t(w))$ and called the density of $K_t(w)$. To find β in the above definitions it is enough to regard finitely many values of t, finitely many colouring and weighting (χ, w) for each t, check if there is a "bad" colour (containing a K_{k_ν}). If not, then "$g(K_t(w))$ is good". Then we have to take the maximum of the good $g(K_t(w))$'s. Hence there is a finite algorithm to determine $\beta(k_1, \dots, k_r)$.

Proof of Theorem 3. Theorem 3(a) is generalized to Theorem 4, therefore its proof will follow from that of Theorem 4. The proof of the stability property in (a)–(d) is left to the reader.

Lemma 4. *Let B_n be a graph and A a set of vertices in B_n, $m = n - |A|$. If A contains $o(n^2)$ edges, and*

$$e(B_n - A) \leq \left(1 - \frac{1}{j}\right)\binom{m}{2},$$

then

$$e(B_n) \leq \left(1 - \frac{1}{j+1}\right)\binom{n}{2}.$$

Or equivalently, if for a vertex $x \in K_t(w)$

$$g(K_t(w) - x) \leq \left(1 - \frac{1}{k-1}\right),$$

then

$$g(K_t(w)) \leq \left(1 - \frac{1}{k}\right).$$

The meaning of this lemma is that if we can delete a set of independent or almost independent vertices of a graph G_n so that the resulting graph G_m has only as many edges as the "best" $k-1$-chromatic graph on m vertices, then G_n has only as many edges as the "best" k-chromatic graph on n vertices. The proof is trivial.

Lemma 5. [3] *The optimal weight distribution of vertices of a dense $K_t(w)$ is attained when all the weighted degrees are equal. Then the density equals the half of this weighted degree.*

Proof of Lemma 5. It follows easily using symmetrizations. See also [3].

Lemma 6. *If in a dense $K_t(w)$ each vertex is incident to at least λ half-edges, then*

$$2g(K_t(w)) \le \left(1 - \frac{1}{t} - \frac{\lambda}{2t}\right).$$

If there exist no λ-regular graph on t vertices, then we have strict inequality:

$$2g(K_t(w)) < \left(1 - \frac{1}{t} - \frac{\lambda}{2t}\right).$$

Proof of Lemma 6. Let $d(i)$, $1 \le i \le t$ be the weighted degree of the vertex i in $K_t(w)$. Let $i_1, \dots, i_{\lambda(i)}$ be the vertices joined to i with a "half"-edge. For $1 \le i \le t$

$$d(i) \;=\; \sum_{j \ne i} w(i,j) u_j \;\le\; 1 - u_i - \frac{1}{2}(u_{i_1} + \dots + u_{i_{\lambda(i)}})$$

Since each u_j occurs in at least λ inequalities, summing the above inequalities for $1 \le i \le t$ we get

$$\min_{1 \le i \le t} d(i) \;\le\; \frac{1}{t} \sum_i d(i) \;\le\; \frac{1}{t}\left(t - 1 - \frac{\lambda}{2}\right) \;=\; 1 - \frac{2+\lambda}{2t}$$

By Lemma 5 this proves Lemma 6.

Below we formulate a lemma to calculate the density of some graph structures.

Lemma 7. *Let — in $K_t(w)$ — q vertex-independent K_3's and r independent edges be given, where the edges and triangles are also pairwise vertex independent. Assume that the edges in this system (i.e. the edges and the pairs in the triangles) have weight $\frac{1}{2}$. Then*

$$2g(K_t(w)) \le \left(1 - \frac{6}{6t - 9q - 4r}\right).$$

(There may be some further half-edges as well!)

Proof of Lemma 7. If we replace a half-edge by a full edge, that cannot decrease the density. Therefore we may assume that $3q$ vertices are covered by K_3's; $2r$ vertices are covered by the independent half-edges, and all the other edges are full edges. There are $s = t - 3q - 2r$ other vertices. This structure will be denoted by $W(q,r,s)$.

One could easily prove this lemma, using the Lagrange method. Below we shall use an equivalent but more combinatorial technique. One can easily prove that if we have two dense structures A and B and join each vertex of A to each vertex of B with weight 1, then the obtained structure is dense again. Hence $W(q,r,s)$ is a dense structure.

Further, by Lemma 5, the optimum distribution $\mathbf{u}$ yielding $g(K_t(w))$ can be characterized by the fact that all the weighted degrees are equal. If the optimum distribution is $x,x,\dots,x,y,\dots,y,z,\dots,z$, the corresponding vertices of $K_t(w)$ satisfy the following equations:

$$1-2x=1-\frac{3}{2}y=1-z,$$

Hence, if $z=6m$, then $x=3m$, $y=4m$. Thus $1=9qm+8rm+6(t-3q-2r)m=(6t-9q-4r)m$, and consequently,

$$g(K_t(w))=\frac{1}{2}\left(1-\frac{6}{6t-9q-4r}\right).$$

In the proofs below we apply Theorem 2'. If $B_n=B(h,t|m,\dots,n_t|\mu)$ is an asymptotically extremal sequence then we consider the corresponding coloured, weighted graph $K_t(w)$ defined in its proof.

We agree to call a K_s RED (or BLUE) if its edges are RED (or BLUE) and this has nothing to do with the colours of the vertices.

Proof of Theorem 3(b). The lower bound is given by $U(n,3)$ of Construction 1(b). Let $B_n=B(h,t|n_1,\dots,n_t|\mu)$ be an asymptotically extremal structure of $RT(n,K_3,K_4,o(n))$, with its canonical extremal colouring. Let $K_t(w)$ be the corresponding weighted coloured complete graph.

Now, if in $K_t(w)$ there exists a BLUE point incident to a BLUE edge, then we have a BLUE $K_3\subset B_n$, and we are home. So we may assume that if there is a BLUE point $x_1\in K_t(w)$ at all, then all the edges joining it to the other vertices are RED. Let X_1 be the corresponding class of B_n. We may apply Lemma 4 to $B_m=B_n-X_1$: it contains neither a BLUE K_3, nor a RED K_3. Therefore $e(B_m)\le\frac{1}{4}m^2+o(m^2)$, and consequently, $e(B_n)\le\frac{1}{3}n^2+o(n^2)$. ∎

Proof of Theorem 3(c). Let $\{1,2,\dots,t\}$ be the vertices of $K_t(w)$.

If one of the vertices, say x is BLUE, then all the edges (x,y) of $K_t(w)$ are RED, otherwise we had a BLUE K_3 in G_n. Hence the remaining part contains neither a RED K_4, nor a BLUE K_3. By Theorem 4(b), its density is at most $\frac{1}{2}\left(1-\frac{1}{3}\right)$, hence — by Lemma 4 — $g(K_t(w))<\frac{1}{2}\left(1-\frac{1}{4}\right)<\frac{1}{2}\left(1-\frac{1}{5}\right)$.

So we may assume that all the vertices of $K_t(w)$ are RED. Hence

(*) $K_t(w)$ contains neither a BLUE K_3, nor a RED K_4 with edges of arbitrary weights. Hence $t\le R(3,4)=9$.

(**) All the edges of a RED K_3 have weight $\frac{1}{2}$.

In case of $t\le 5$ the density can be at most $\frac{1}{2}\left(1-\frac{1}{5}\right)$. This is achieved only when $t=5$ and all the edges are "full". Hence the colouring of K_5 is the colouring of $U(n,3,3)$, the PENTAGONLIKE colouring: otherwise (**) would be violated.

Assume now that $t=6$. Since — by (*) and Ramsey Theorem — there exists a RED K_3 in $K_t(w)$, therefore, by (**), its edges must be of weight $\frac{1}{2}$.

Apply Lemma 7 with $q=1$, $r=0$:

$$2g(K_t)\le 1-\frac{2}{2t-3}<1-\frac{1}{5}\quad\text{if}\quad t=6.$$

Finally let $7\le t\le 9$.

1. Observe that each x is joined to at most 3 other vertices in BLUE. Indeed, if x is joined to $y_1,y_2,\dots,y_i$, then — since the BLUE K_3's are excluded, — $y_1,\dots,y_i$ form a complete RED graph. Thus $i<4$.

2. We show that for $t\geq 7$ each x is incident to at least $t-5$ (RED) half edges. Indeed, if x is joined to $y_1,\dots,y_i$ in BLUE and to $z_1,\dots,z_j$ in RED, and, say, xz_1 is a full RED edge, then it is not contained in RED triangles. Therefore z_1 is joined to all the other z's in BLUE. If there is another full RED edge, say xy_2, then y_2 is also joined to all the other y's in BLUE. Since there is no BLUE triangle, thus $j<3$: there are at most 2 BLUE edges incident with x. Since there are at most 3 BLUE edges xy, — if 2 of these RED edges are full, then $t\leq 1+3+2=6$.

Since $t\geq 7$, x is incident to at least 2 RED edges (with weight $\frac{1}{2}$). So we can apply Lemma 6: $2g(K_t(w))\leq 1-\frac{2}{t}<1-\frac{1}{5}$ if $t\leq 9$.

Proof of Theorem 3(d). The following conditions must hold on the colouring of the extremal $B(h,t|n_1,\dots,n_t|\mu)$.

(a) There are no 4 classes C_i, C_j, C_k and C_m joined in the same colour (i.e. forming a monochromatic K_4 in the reduced graph).

(b) If $\Delta=(C_i,C_j,C_k)$ is RED, then all its "vertices" C_i,C_j and C_k are BLUE (where a RED triangle means that the edges between the different groups are RED!)

(c) If C_i and C_j are RED, then either they form a RED Bollobás–Erdős graph or they are joined, by a BLUE edge.

(a')–(c') If we swap BLUE and RED, (a)–(c) still hold.
(As a matter of fact, the converse statement is also true:
if for $B(h,t|n_1,\dots,n_t|\mu)$ (a–c') hold, then it contains no monochromatic K_4.)
(A) First we prove, that $t=6$.
1. Suppose that $t\leq 4$. Then

$$2g(K_t(w))\leq\left(1-\frac{1}{4}\right)<\frac{11}{14}.$$

2. Let $t=5$. If there is at least one "half" edge, then similarly as in the proof of 4(c) we get

$$2g(K_t(w))\leq\frac{10}{13}<\frac{11}{14}.$$

Consequently, we may suppose, that all the edges are "full". In that case we shall prove that in any colouring of $K_5(w)$ — satisfying (a)–(c') — there must be a monochromatic K_4 in B_n.

The PENTAGONLIKE colouring is excluded, since in that case we can always find two classes of the same colour and a full connection between them in the same colour. This would yield a monochromatic K_4.

If the vertices x_1,x_2,x_3 of $K_5(w)$ form a RED triangle, then x_1,x_2,x_3 must be BLUE by (b).

By (a) there exists a BLUE "full" connection between x_4 and x_1,x_2,x_3. Hence (by (c)) x_4 must be RED. Similarly, x_5 is also RED. Hence (by (c')) they are joined by BLUE.

There are at most two RED edges between $\{x_1,x_2,x_3\}$ and $\{x_4,x_5\}$. Otherwise say x_4 were joined to two of x_1,x_2,x_3 by RED edge and consequently, we would have a RED triangle, say (x_1,x_2,x_4) with one RED vertex x_4, contradicting (b). There are at most three BLUE edges between $\{x_1,x_2,x_3\}$ and $\{x_4,x_5\}$: otherwise say x_1 were joined to both x_4 and x_5 in BLUE, yielding a BLUE triangle (x_1,x_4,x_5) incident to a BLUE vertex.

Since there are 6 edges between $\{x_1,x_2,x_3\}$ and $\{x_4,x_5\}$, 5 vertices with all "full-connections" cannot be properly coloured.

(B) Next we consider the case when $t>6$.

First we prove, that in an admissible colouring of $K_t(w)$ all the monochromatic triangles have the same colour. Indeed, assume that x_1,x_2,x_3 is a RED, y_1,y_2,y_3 is a BLUE triangle Then x_i's are BLUE and y_j's are RED vertices. At least 5 of the 9 connections x_iy_j have the same colour, say RED. Thus we get a RED triangle x_i,x_k,y_j, contradicting (b).

Assume that there exists at least one BLUE vertex, say x_1, and **all the triangles** are also BLUE. Then there is no RED triangle. x_1 cannot be contained in a BLUE triangle, by (b'). This implies that x_1 has at most 2 BLUE connections: otherwise, say x_2, x_3 and x_4 were joined to x_1 in BLUE, and therefore x_2,x_3,x_4 were a RED triangle. Similarly, x_1 is joined to at most 3 other vertices in RED, otherwise we have a BLUE K_4 in $K_t(w)$. Hence x_1 can be joined to at most 5 other vertices, a contradiction.

So the only remaining case to be settled is when all the groups are RED, and all the triangles are BLUE.

First observe that in this case each vertex in $K_t(w)$ is joined to at least $t-6$ other vertices in RED: otherwise, say, x_1 were joined to the remaining 6 vertices in BLUE and 3 of these vertices would form a BLUE triangle, yielding with x_1 a BLUE K_4. Now (c) and Lemma 6 yields

$$2g \le \left(1 - \frac{1}{t} - \frac{t-6}{2t}\right) = \frac{1}{2}\frac{t+4}{t}.$$

Thus for $t>7$ we obtained that $2g<11/14$, for $t=7$ we use the second part of the lemma: $t-6=1$ and there is no 1-regular graph on 7 vertices.

(C) Assume now that $t=6$.

If there exists exactly one half-connection, or two half-connections incident to the same vertex, say x_6, then we can apply (A) to the remaining 5 vertices $x_1,\dots,x_5$. We have seen that in that case one of the conditions (a–c') must be violated: there exists a monochromatic K_4.

So we have proved that there exist at least two independent "half- connections" among the 6 vertices.

To finish the proof we have to give a "good" colouring of such a weighted K_6 with 2 independent half edges and determine the optimal weight distribution, showing that its density is $\frac{1}{2}\cdot\frac{11}{14}$.

This is given by Construction 4.

Construction 4. For $n=14t+o(t)$ fix 6 groups $W_1,\dots,W_6$ as follows:

$$|W_1|\approx 2t, |W_2|\approx 2t, |W_3|\approx 3t, |W_4|\approx 3t, |W_5|\approx 2t, |W_6|\approx 2t.$$

W_1 and W_2 form a RED Bollobás–Erdős graph, W_3, W_4, are also RED groups: we build RED Erdős graphs on them; however, they are joined completely, in BLUE. W_5, and W_6 form a BLUE Bollobás–Erdős graph. All the other pairs of classes are completely joined, as follows:

$$(W_1,W_3),(W_1,W_4),(W_2,W_3),(W_2,W_4),(W_1,W_5),(W_2,W_6)$$

are BLUE,

$$(W_3,W_5),(W_3,W_6),(W_4,W_5),(W_4,W_6),(W_1,W_6),(W_2,W_5)$$

are RED. One can easily check that there are two BLUE triangles but all their groups are RED and the shortest odd RED cycle is a pentagon, and each "full" RED edge has at least one BLUE endgroup, no "full" BLUE edge connects BLUE groups. Thus the resulting $B(h,6|n_1,\dots,n_6|\mu)$ contains no monochromatic K_4 in the above colouring and has only $o(n)$ independent vertices.

It is easy to check that in (C) we have the optimal distribution and that $e(B)=\frac{1}{2}\left(1-\frac{3}{14}\right)n^2+o(n^2)$. ∎

Proof of Theorem 4. We shall call the "odd-girth" of G the length of the shortest odd cycle in it. Put BLUE Erdős graphs of "odd-girth" $>p,q$ into the classes of a RED $T_{n,2}$. The obtained $U_{n,2}$ contains no short odd RED or BLUE cycles. This yields the (constructive) lower bound in Theorem 3.

Apply the proof methods of Theorem 2", (upper bound). If the reduced graph $H(w_0)$ contains a BLUE vertex and a BLUE edge incident to it, then G_n contains a BLUE C_p, assumed that $\varepsilon=\varepsilon(p,q)$ is chosen appropriately. The same holds in RED. Hence in the reduced coloured graph $K_t(w)$ all the vertices have colours different from the colour of any edge. Therefore all the vertices are of the same colour, say RED, and all the edges are of the other colour, say BLUE. If K_t contains a BLUE K_3, then G_n contains a BLUE $K(p,p,p)$, a contradiction. Hence there are at most 2 classes: $t=2$. ∎

5. Open problems

There are various intriguing open problems in connection with the above theorems. We list below some of them.

Problem 1. *How large can $t=t(q,s)$ be if $B(h,t|n_1,\dots,n_t|\mu)$ is asymptotically extremal for the problem $RT(n,q,s,o(n))$?*

It is easy to see, that t is at least exponentially large. Indeed, let $q=2k+1$, $t=R(k,s)$, and consider $U(n,t)$. Clearly, $U(n,t)$ contains neither RED K_q, nor BLUE K_s and has $\left(1-\frac{1}{t}\right)\binom{n}{2}$ edges. Further, all the GBE-graphs with fewer classes have fewer edges. Hence the extremal GBE-graph has $t\geq R\left(\left[\frac{q}{2}\right],s\right)$ classes, otherwise it has too few edges.

Problem 2. *Determine $\vartheta(K_q,K_3)$.*

Perhaps the following is true.

Conjecture 1. *$U(n,k,3)$ of Construction 2 is extremal for $RT(n,3,q,o(n))$ if $q = 2k+1$.*

One of the basic problems to be attacked is in general: Given a graph L, how the graph theoretic properties of L influence $\vartheta(L)$? In case of ordinary graph extremal problems the chromatic number is the most important factor in determining $\text{ext}(n,L)$. The previous paper [10] showed that the behaviour of $\vartheta(L)$ largely depends on the arboricity of L. Still, it did not give a complete solution of the question. The case of $K_3(2,2,2)$ seemed to be the first real difficulty. Since $K_3(2,2,2)\in Arb(4)$, therefore

$$\vartheta(K_3(2,2,2)) \le \vartheta(K_4) = \frac{1}{8},$$

(see [10]).

We cannot improve this bound.

Problem 3. *Decide if $\vartheta(K_3(2,2,2))=0$.*

One way to settle this question would be to show that the Bollobás–Erdős graph (or some slight modification of it) contains no $K(2,2,2)$. We cannot decide even this (seemingly simple) question.

Perhaps replacing $o(n)$ by a slightly smaller functions, say by $f(n)=\frac{n}{\log n}$ one could get smaller upper bounds.

Problem 4. *Is it true that for some $c>0$,*

$$RT\left(n, K_4, \frac{n}{\log n}\right) < \left(\frac{1}{8} - c\right) n^2?$$

Problem 5. *Is it true that for $RT(n,k_1,\dots,k_r,o(n))$ the asymptotically extremal structure is weakly stable?*

The same type of questions can be asked for hypergraphs.

Some hypergraph results on Turán–Ramsey problem can be found in [15] and in [17].

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