Combinatorics, Probability and Computing (1994) 3, 297–325

Turán–Ramsey Theorems and K_p-Independence Numbers

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Received 7 June 1993; revised 27 January 1994

Dedicated by the last four authors to Paul Erdős on his 80th birthday

Let the K_p-independence number $\alpha_p(G)$ of a graph G be the maximum order of an induced subgraph in G that contains no K_p. (So K_2-independence number is just the maximum size of an independent set.) For given integers $r, p, m > 0$ and graphs $L_1, \dots, L_r$, we define the corresponding Turán–Ramsey function $RT_p(n, L_1, \dots, L_r, m)$ to be the maximum number of edges in a graph G_n of order n such that $\alpha_p(G_n) \le m$ and there is an edge-colouring of G with r colours such that the j^{th} colour class contains no copy of L_j, for $j = 1, \dots, r$.
In this continuation of [11] and [12], we will investigate the problem where, instead of $\alpha(G_n) = o(n)$, we assume (for some fixed $p > 2$) the stronger condition that $\alpha_p(G_n) = o(n)$.
The first part of the paper contains multicoloured Turán–Ramsey theorems for graphs G_n of order n with small K_p-independence number $\alpha_p(G_n)$. Some structure theorems are given for the case $\alpha_p(G_n) = o(n)$, showing that there are graphs with fairly simple structure that are within $o(n^2)$ of the extremal size; the structure is described in terms of the edge densities between certain sets of vertices.
The second part of the paper is devoted to the case $r = 1$, *i.e.*, to the problem of determining the asymptotic value of

$$\theta_p(K_q) = \lim_{n\to\infty} \frac{RT_p(n, K_q, o(n))}{\binom{n}{2}},$$

[‡]for $p < q$. Several results are proved, and some other problems and conjectures are stated.

0. Notation

In this paper we will consider graphs without loops and multiple edges. Given a graph G, $e(G)$ will denote the number of edges, $v(G)$ the number of vertices, $\chi(G)$ the chromatic number, and $\alpha(G)$ the maximum cardinality of an independent set in G. More generally, given an integer $p > 1$, $\alpha_p(G)$ denotes the *p-independence number* of G: the maximum cardinality of a set S such that the subgraph of G spanned by S contains no K_p. Given a

[†] Supported by GRANT 'OTKA 1909'.
[‡] This notation, where we put $o(n)$ in place of $f(n)$ is slightly imprecise. It means that any function $f(n) = o(n)$ and will be clarified in Section 2.

graph, the (first) subscript will mostly denote the number of vertices: G_n, S_n, will always denote graphs on n vertices. For given graphs $L_1, \dots, L_r$, $R(L_1, \dots, L_r)$ will denote the usual Ramsey number, that is, the minimum t such that for every edge-colouring of K_t in r colours, for some v the v^{th} colour contains an L_v†. If we partition n vertices into q classes as equally as possible and join two vertices iff they belong to different classes, we obtain the so-called Turán graph on n vertices and k classes, denoted by $T_{n,k}$. This graph is the (unique) k-chromatic graph on n vertices with the maximum number of edges.

For a set Q, we will use $|Q|$ to denote its cardinality. Given two disjoint vertex sets, X and Y, in a graph G_n, we use $e(X, Y)$ to denote the number of edges in G_n joining X and Y, and $d(X, Y)$ to denote the edge-density between them:

$$d(X, Y) = \frac{e(X, Y)}{|X| \cdot |Y|}.$$

Given a graph G and a set U of vertices of G, we use $G[U]$ to denote the subgraph of G induced (spanned) by U. The number of edges in a subgraph spanned by a set U of vertices of G will be denoted by $e(U)$. We will say that X is *completely joined* to Y if every vertex of X is joined to every vertex of Y.

Given two points x, y in the Euclidean space $\mathbf{E}^h$, we use $\rho(x, y)$ to denote their ordinary distance.

1. Introduction

Ramsey's Theorem [23] and Turán's Extremal Theorem [33, 34] are both among the most well-known theorems of graph theory. Both served as starting points for whole branches of graph theory. (For Ramsey Theory, see the book by R. L. Graham, B. L. Rothschild and J. Spencer [21], and for Extremal Graph Theory, see the book by Bollobás [2], or the survey by Simonovits [29].) In the late 1960's a new theory emerged connecting these fields. Perhaps the first paper in this field is due to V. T. Sós [30], and quite a few results have been found since then.

The 'historical' part of the introduction of this paper is slightly condensed, to avoid too much repetition. For some further information see [12]. Some important references can be found at the end of the paper, see [3, 11, 12, 18, 20, 31].

In [11] P. Erdős, A. Hajnal, V. T. Sós, and E. Szemerédi investigated the following problem:

Suppose that a so-called forbidden graph L and a function $f(n) = o(n)$ are given. Determine

$$RT(n, L, f(n)) = \max\{e(G_n) \;:\; L \not\subseteq G_n \text{ and } \alpha(G_n) < f(n)\}.$$

They showed that this number depends (in some sense) primarily on the so-called Arboricity of L (which is a slight modification of the usual arboricity of L). In a continuation [12] of that paper, we started investigating the following problem:

† This is the only case when the (first) subscript is not the number of vertices: *i.e.* when we speak of the excluded graphs L_i.

Let G_n be a graph on n vertices the edges of which are coloured by r colours $\chi_1, \dots, \chi_r$, so that the subgraph of colour χ_ν contains no complete subgraph K_{p_ν}, ($\nu = 1, \dots, r$). Let a function $f(n)$ be given, (mostly $f(n) = o(n)$) and suppose that $\alpha(G_n) \le f(n)$. What is the maximum number of edges in G_n under these conditions?

In *this* continuation of [11] and [12] we will investigate the problem where, instead of $\alpha(G_n) = o(n)$, we assume a stronger independence condition: that the maximum cardinality of a K_p-free induced subgraph of G_n is $o(n)$:

$$\alpha_p(G_n) = o(n).$$

The concept of $\alpha_p(G)$ was introduced long ago by A. Hajnal, and also investigated by Erdős and Rogers, see [16]. (A similar 'independence notion' is investigated for random graphs in a paper of Eli Shamir [24], where he generalizes some results on the chromatic number of random graphs.)

The general problem

Assume that $L_1, \dots, L_r$ are given graphs, and G_n is a graph on n vertices, the edges of which are coloured by r colours $\chi_1, \dots, \chi_r$, and

$$(*) \qquad \begin{cases} \text{for } \nu = 1, \dots, r \text{ the subgraph of colour } \chi_\nu \text{ contains no } L_\nu \\ \text{and } \alpha_p(G_n) \le m. \end{cases}$$

What is the maximum of $e(G_n)$ under these conditions?

The maximum will be denoted by $RT_p(n, L_1, \dots, L_r, m)$. The graphs attaining the maximum in this problem will be called *extremal* graphs for $RT_p(n, L_1, \dots, L_r, m)$. It may happen that there exist no graphs satisfying our conditions. Then we will say that the maximum is 0.

Of course, for fixed m and large n – by Ramsey's theorem – there are no graphs with the above properties: the maximum is taken over the empty set. However, we are interested mainly in the case $m \to \infty$, $m = f(n) = o(n)$, but $m/n \to 0$ very slowly.

The existence of graphs satisfying (*) is far from being trivial. We will use a theorem of Erdős and Rogers to prove the existence of such graphs for the case of one colour and when the forbidden graph is a complete graph. We will sketch a constructive proof of the Erdős–Rogers theorem in Section 4, and return to this question in a more general setting in the Appendix, where we will characterize the cases when (*) can be satisfied (for 2-connected forbidden graphs). Among others, we will see that (*) can always be satisfied when all the forbidden graphs L_i are complete graphs of more than p vertices and $m = n^{1-c}$ for some small $c > 0$.

Some motivation Our problems are motivated by the classical Turán and Ramsey Theorems [33, 34, 23], and also (indirectly) by some applications of the Turán Theorem to geometry, analysis (in particular, potential theory) [35, 36, 37, 13, 14, 15], and probability theory (see, for example, Katona, [22], or Sidorenko, [25, 26]), (see also [38]).

In [12] we proved (among others), for the problem of $RT_2(n, K_{\ell_1}, \dots, K_{\ell_r}, o(n))$, the existence of a sequence of asymptotically extremal graph sequences of relatively simple structure†.

Assume now, that $\alpha(G_n)$ is *much smaller* than cn, for example $\alpha(G_n) \le \sqrt{n}$. Then we know (since $R(K_3, K_k) < k^2/\log k$) that for every fixed $c > 0$, every set of $> cn$ vertices of G_n will contain not only an edge, but also a K_3. Similarly, if we choose even smaller upper bounds for $\alpha(G_n)$, we can ensure the even stronger conditions that every induced subgraph of G_n of at least cn vertices contains a larger complete graph K_p. This also leads to the problems of the present paper, though apart from Theorem 2.1 we will deal only with the simplest case $f(n) = o(n)$.

Some basic definitions It is probably hopeless to give an exact description of the maximum in the general problem. Therefore we will try to find an asymptotically extremal sequence of graphs of relatively simple structure. The definitions listed here are needed to make precise what we consider 'relatively simple'.

Notation. For any given function f, let

$$\vartheta_\varepsilon = \vartheta_{\varepsilon,p,f}(L_1,\dots,L_r) = \limsup_{n\to\infty} \frac{e(G_n)}{\binom{n}{2}} \quad \text{and} \quad \vartheta_{p,f} = \lim_{\varepsilon\to 0} \vartheta_\varepsilon$$

where the limsup is taken for the r-coloured graphs G_n satisfying (*) with $m = \varepsilon f(n)$:

$$\begin{cases} \text{for } \nu = 1,\dots,r \text{ the subgraph of colour } \chi_\nu \text{ contains no } L_\nu \\ \text{and } \alpha_p(G_n) \le \varepsilon f(n). \end{cases}$$

(If the limsup is taken over the empty set (of graphs), it is defined to be 0.) Clearly, if $\varepsilon \to 0$, the lim sup above will converge, since it is monotone in ε. One can easily see the following claim.

Claim 1.1.

(a) If $\varepsilon_n \to 0$,

$$\limsup_{n\to\infty} \frac{RT_p(n, L_1,\dots,L_r, \varepsilon_n f(n))}{\binom{n}{2}} \le \vartheta_{p,f}(L_1,\dots,L_r).$$

(b) There exist a sequence $\varepsilon_n^* \to 0$ *and an infinite sequence* $(S_n : n \in \mathbb{N}_0)$ $(\mathbb{N}_0 \subseteq \mathbb{N})$ *of graphs with the property* (*) *for* $m = \varepsilon_n^* f(n)$ *where the equality holds in (a).*

(c) For every $\varepsilon_n \ge \varepsilon_n^*$, $\varepsilon_n \to 0$,

$$\lim_{\substack{n\in\mathbb{N}_0\\ n\to\infty}} \frac{RT_p(n, L_1,\dots,L_r, \varepsilon_n f(n))}{\binom{n}{2}} = \vartheta_{p,f}(L_1,\dots,L_r).$$

Proof. Here (a) is trivial from the definition, (c) is trivial from (a) and (b), by monotonicity, and (b) follows by an easy diagonalization.

Indeed, assume that for $k = 1,\dots,t-1$ we have already fixed S_{n_k}. Now we fix $\varepsilon = \varepsilon_t = 1/t$ and find an S_{n_t} with the following properties: $n_t > n_{t-1}$,

$$e(S_{n_t}) \ge \left(\vartheta_\varepsilon - \frac{1}{t}\right)\binom{n_t}{2},$$

and $\alpha_p(S_{n_t}) \le (1/t) f(n_t)$. □

† The definitions can be found below.

Unfortunately, we cannot prove the corresponding assertions for *all* $n \geq n_0$: we cannot exclude the possibility that

$$\frac{RT_p(n, L_1, \ldots, L_r, \varepsilon_n f(n))}{\binom{n}{2}}$$

jumps up and down as $n \to \infty$.

We will often speak of the problem of determining $RT_p(n, L_1, \ldots, L_r, o(n))$, meaning the determination of $\vartheta_{p,f}(L_1, \ldots, L_r)$, for $f(n) = n$. This slightly imprecise notation will cause no problems. Similarly, if $f(n) = n$, we will often use the notation $\vartheta_p(L_1, \ldots, L_r)$ instead of $\vartheta_{p,f}(L_1, \ldots, L_r)$. Observe that ϑ is monotone: if we replace L_1 by an $L_1^* \supseteq L_i$, then $\vartheta_{p,f}(L_1, \ldots, L_r) \leq \vartheta_{p,f}(L_1^*, \ldots, L_r)$. In particular, $\vartheta_p(K_q)$ is monotone increasing in q.

Definition 1.2. (Asymptotically extremal graphs) Suppose that the forbidden graphs $L_1, \ldots, L_r$, and the function f are given. An infinite sequence of graphs, (S_n), will be called an *asymptotically extremal sequence* (for $L_1, \ldots, L_r$ and f) if the edges of each S_n can be r-coloured so that the v^{th} colour contains no L_v, $(v = 1, \ldots, r)$, $\alpha_p(G_n) = o(f(n))$, and

$$\frac{e(S_n)}{\binom{n}{2}} = \vartheta_{p,f}(L_1, \ldots, L_r) + o(1).$$

In Section 2 we will formulate some theorems asserting that, for any r, there are always asymptotically extremal graph sequences of *fairly simple* structure. To formulate these theorems, we have to introduce the notion of matrix graphs, and matrix graph sequences.

We will say that two disjoint vertex sets X and Y *are joined ε-regularly* in the graph G if for every subset $X^* \subseteq X$ and $Y^* \subseteq Y$ satisfying $|X^*| > \varepsilon|X|$ and $|Y^*| > \varepsilon|Y|$, we have

$$|d(X^*, Y^*) - d(X, Y)| < \varepsilon.$$

In the following $A = (a_{ij})$ will always be a *symmetric* matrix with all $a_{ij} \in [0, 1]$.

Definition 1.3. (A-matrix graph sequences) Given a $t \times t$ symmetric matrix $A = (a_{ij})$, a graph sequence (S_n) – defined for infinitely many n but not necessarily defined for every $n > n_0$ – is said to be an *A-matrix graph sequence* if the vertices of S_n can be partitioned into t classes $V_{1,n}, \ldots, V_{t,n}$ so that in S_n

— $e(V_{i,n}) = o(n^2)$, for every $i = 1, \ldots, t$,
— $d(V_{i,n}, V_{j,n}) = a_{ij} + o(1)$ for every $1 \leq i < j \leq t$ and
— the classes $V_{i,n}$ and $V_{j,n}$ are joined ε_n-regularly for every $1 \leq i < j \leq t$ for some $\varepsilon_n \to 0$.

We will associate a quadratic form $\mathbf{u}A\mathbf{u}^T$ to A and maximize it over the simplex $\sum u_i = 1, u_1, \ldots, u_i \geq 0$:

$$g(A) := \max\{\mathbf{u}A\mathbf{u}^T : \sum u_i = 1, u_i \geq 0\}.$$

The quadratic form will be used to measure the number of edges in the corresponding matrix graph sequence. The vectors attaining the maximum will be called *optimum vectors*. (Optimum below will always mean maximum.)

Definition 1.4. (Dense matrices) A matrix A is *dense*, if for any i deleting the i^{th} row and the i^{th} column of the matrix A we get an A' with $g(A') < g(A)$.

One can easily see [4] that if A is dense, it has a *unique* optimum vector and all the coordinates of this optimum vector are positive. The uniqueness implies that the symmetries of the matrix leave the optimum vector invariant: the corresponding coordinates are equal. This means that if a permutation π of $1,\dots,t$ applied to the rows and to the columns of A leaves A invariant, then π applied to the optimum vector also leaves it unchanged. Further, if $g(A') < g(A)$ for some symmetric minor A' of A, there exists an A'' obtained from A by deleting just one row and the corresponding column and satisfying $g(A'') < g(A)$. For a more detailed description of this function $g(A)$ see [4, 7].

Definition 1.5. (Asymptotically optimal A-matrix-graph sequences) Let A be a fixed matrix and $\mathbf{u} = (u_1,\dots,u_t)$ be an optimum vector for A. We will call an A-matrix graph sequence (S_n) *asymptotically optimal* if the classes $V_{i,n}$ can be chosen so that $|V_{i,n}|/n = u_i + o(1)$, for $i = 1,\dots,t$.

Clearly, an optimal matrix graph has

$$\frac{1}{2}g(A)n^2 + o(n^2)$$

edges. If the matrix A has a submatrix A' such that $g(A') = g(A)$, we can always replace the matrix graph sequence corresponding to A by the simpler matrix graph sequence corresponding to A'. This is why we are interested only in *dense* matrices.

2. Main results

We start with the existence of the limit.

Theorem 2.1. *For any $p_1,\dots,p_r$ and for $f(n) = n$, for any $\varepsilon_n \to 0$:*
(a) Let (S_n) be an extremal graph sequence for $RT_p(n, K_{p_1},\dots,K_{p_r},\varepsilon_n n)$. Then

$$\limsup_{n\to\infty} \frac{e(S_n)}{\binom{n}{2}} \le \vartheta_{p,f}(K_{p_1},\dots,K_{p_r}). \tag{1a}$$

(b) There exists an $\varepsilon_n^* \to 0$ for which on the left-hand side of (1a) the limit exists and

$$\lim_{n\to\infty} \frac{e(S_n)}{\binom{n}{2}} = \vartheta_{p,f}(K_{p_1},\dots,K_{p_r}). \tag{1b}$$

(c) For every $\varepsilon_n \to 0$ with $\varepsilon_n \ge \varepsilon_n^*$ the same – namely, (1b) – holds.

Here $f(n) = n$ means that we consider the case $\alpha_p(G_n) = o(n)$. The difference between this theorem and Claim 1.1 is that there we regard *all possible forbidden graphs*, here only complete graphs, and there we assert only the existence of a *sparse subset of integers* along which a limit exists, (*i.e.*, we assert that the limsup can be obtained in some specific way) here we assert that the actual limit exists.

The meaning of Theorems 2.2 and 2.3 below is that in the general case there are asymptotically extremal graph sequences of fairly simple structure, where 'simple' means that the structure depends on n weakly. This is a weak generalization of the Erdős–Stone–Simonovits Theorem (from ordinary extremal graph theory) [17, 19]. The optimal matrix graph sequences – in some sense – generalize the Turán graphs, while the matrix graphs generalize the complete t-partite graphs. (See also [8], and [28]).

Theorem 2.2. *For any $p_1, \dots, p_r$ let $\varepsilon_n \to 0$ sufficiently slowly (which means that the condition of (c) of Theorem 2.1 is satisfied). Then there exists a dense $\Omega \times \Omega$ matrix A with $\Omega < R(K_{p_1}, \dots, K_{p_r})$ and an asymptotically extremal sequence (S_n) for $RT_p(n, K_{p_1}, \dots, K_{p_r}, \varepsilon_n n)$ that is an asymptotically optimal A-matrix graph sequence.*

For general $L_1, \dots, L_r$ we have the following theorem.

Theorem 2.3. *Let r forbidden graphs $L_1, \dots, L_r$ be fixed. Let $f(n) \to \infty$ ($f(n) = O(n)$) be an arbitrary function for which for every $c \in (0, 1)$ there exists an $\eta = \eta_{f,c} > 0$ such that*

$$f(cn) > \eta_{f,c} f(n).$$

Then there exist a dense matrix $A = (a_{ij})_{\Omega \times \Omega}$ – for some $\Omega < R(L_1, \dots, L_r)$, and an asymptotically extremal sequence (S_{n_ℓ}) (for $L_1, \dots, L_r$ and f, for some subsequence of integers) that is an asymptotically optimal A-matrix graph sequence.

This means that the structure of some asymptotically extremal sequences is simple. The matrix A depends on the function f: for different f's we get different extremal densities. The matrix depends primarily on the sample graphs and on f. However, we are unable to exclude the possibility that A must, even in the simplest case $f = n$, depend on the actual subsequence of integers as well: that there is no common A for all $n > n_0$. The condition $f(cn) > \eta_{f,c} f(n)$ is a 'smoothness' condition, which is satisfied in 'all the reasonable cases'.

Remark 2.4. We are primarily interested in functions of type $f(n) = n^\gamma$. By the quantitative Ramsey Theorem, for every family $L_1, \dots, L_r$ we can fix a $\Gamma > 0$ so that if $\alpha(G_n) < f(n) = n^\Gamma$, then every r-colouring of G_n contains an L_ν of colour ν for some $\nu \le r$ (since it contains a large clique of colour ν): no graphs satisfy (*).

Remark 2.5. When we assert the existence of a matrix A in Theorems 2.2 and 2.3, we do not know too much about this A. The only thing we know is that it is dense and (therefore, by Lemma 3.3) its off-diagonal entries are all positive.

Unfortunately, most of the non-trivial results for the K_p-free case ($p > 2$) are related to the special case when all the forbidden subgraphs L_ν are complete graphs. So in Sections 4-6 we will assume that the graphs L_i are complete graphs. In Section 4 we will prove some general upper and lower bounds for the case of one colour ($r = 1$). The following result is a direct generalization of the Erdős–Sós Theorem from [18].

Theorem 2.6.

(a) For any integers $p > 1$ and $q > p$ we have

$$\vartheta_p(K_q) \leq \frac{1}{2}\left(1 - \frac{p}{q-1}\right).$$

(b) For every k, for $q = pk + 1$ this is sharp:

$$\vartheta_p(K_q) = \frac{1}{2}\left(1 - \frac{p}{q-1}\right) = \frac{1}{2}\left(1 - \frac{1}{k}\right).$$

To get the lower bound in Theorem 2.6 (*i.e.*, Theorem 2.6(b)) we will use a geometric construction of Erdős and Rogers [16]. Here we formulate their theorem, but the verification is postponed to Section 4.

Erdős–Rogers Theorem. *Let $p \geq 2$ be an integer. There are a constant $c = c_p > 0$ and an $n_0(p, c)$, such that for every $n > n_0(p, c)$, there exists a graph Q_n not containing K_{p+1}, but satisfying $\alpha_p(Q_n) \leq n^{1-c}$.*

Construction 2.7. Let $q = pk + 1$. Take k vertex-disjoint Erdős–Rogers graphs of size $(n/k) + o(n)$ (described in the previous theorem) and join each vertex to all the vertices in the other graphs. (We will sometimes describe this as putting (p, ε)-Erdős–Rogers graphs into each class of a $T_{n,k}$.) Thus we get a graph sequence (S_n) with $\alpha_p(S_n) \leq kn^{1-c}$ for some $c > 0$ and $K_{pk+1} \not\subseteq S_n$.

This proves the lower bound in Theorem 2.6. For $q = p + \ell$, $\ell = 2, 3, 4, 5$ we can improve the upper bound of Theorem 2.6, see Theorem 2.11.

Remark 2.8. Now, for $p \geq 2$ fixed, we know the value of every p^{th} $\vartheta_p(K_q)$. Perhaps the other values have a 'pseudo-periodical' behaviour similar to that of $\vartheta_2(K_q)$: the extremal structure is determined by the residue of q mod p. The situation is analogous to that in the Erdős–Hajnal–Sós–Szemerédi [11] Theorem, where the case of odd values of q was much simpler (and also much simpler to prove) than the case of even q's.

In Section 5, we investigate some special cases that seem to be interesting, because they suggest some conjectures for the general case. Perhaps the following conjecture holds.

Conjecture 2.9. *The asymptotically extremal graphs for $RT_p(n, K_q, o(n))$ have the following structure: Let $q = pk + \ell$, $(\ell = 1, 2, \ldots, p)$. Then n vertices are partitioned into $k + 1$ classes $V_{0,n}, \ldots, V_{k,n}$. For each pair $\{i, j\} \neq \{0, 1\}$, $V_{i,n}$ is almost completely joined to $V_{j,n}$ in the sense that every $x \in V_{i,n}$ is joined to every $y \in V_{j,n}$ with a possible exception of $o(n^2)$ pairs xy. Further, $d(V_{0,n}, V_{1,n}) = ((\ell - 1)/p) + o(1)$ (as $n \to \infty$), and $V_{0,n}, V_{1,n}$ are joined $o(1)$-regularly. Finally, $e(V_{i,n}) = o(n^2)$, $i = 1, \ldots, k$.*

Remark 2.10. For graphs of this kind the optimal sizes of the classes V_i can easily be computed: the optimal class-sizes are as follows. The edges in $G[V_i]$ can be neglected,

$$|V_i| = \frac{1}{2+(k-1)(2-\frac{\ell-1}{p})}n + o(n) \text{ for } i = 0,1$$

and

$$|V_i| = \frac{(2-\frac{\ell-1}{p})}{2+(k-1)(2-\frac{\ell-1}{p})}n + o(n) \text{ for } 2 \le i \le k.$$

From this, $e(S_n)$ can easily be calculated: if S_n is the graph described in the conjecture, it is almost regular, and the degrees in V_2 are $n - |V_2|$. Hence

$$e(S_n) \approx \frac{1}{2}\left(n - |V_2|\right) n \approx \left(1 - \frac{(2p-\ell+1)}{k(2p-\ell+1)-\ell+1}\right)\binom{n}{2}.$$

We describe some cases below, where we can prove the upper bound in Conjecture 2.9.

Theorem 2.11. *Let $\ell = 2,3,4$ or 5 and $\ell \le p+1$. If $K_{p+\ell} \not\subseteq G_n$ and $\alpha_p(G_n) = o(n)$, then*

$$e(G_n) \le \frac{\ell-1}{4p}n^2 + o(n^2).$$

By Theorem 2.6, we know that $\vartheta_p(K_{p+1}) = 0$ and $\vartheta_p(K_{2p+1}) > 0$. Here one of the main problems is:

Problem 2.12. *For fixed p determine the minimum ℓ for which*

$$\vartheta_p(K_{p+\ell}) > 0.$$

In particular, is $\vartheta_p(K_{p+2}) > 0$ or not? If $\vartheta_p(K_{p+\ell}) > 0$, how large is it?

Theorem 2.13. *For any $p \ge 2$,* $\quad \vartheta_p(K_{2p}) \ge \frac{1}{8}$.

It is worth observing that replacing K_{2p} by K_{2p+1} we get by Theorem 2.6(b), for any $p \ge 2$, $\quad \vartheta_p(K_{2p+1}) = 1/4$.

For $p = 2$ Theorem 2.13 is sharp: $\vartheta_2(K_4) \le 1/8$ was proved by Szemerédi [31] and $\vartheta_2(K_4) \ge 1/8$ was settled by Bollobás and Erdős in [3], via a high-dimensional geometric construction. In a slightly different form, Bollobás and Erdős did the following. Fix a high-dimensional sphere S^h and partition it into $n/2$ domains $D_1,\dots,D_{n/2}$, of equal measure and diameter $(1/2)\mu$, with $\mu = \varepsilon/\sqrt{h}$. Choose a vertex $x_i \in D_i$ and an $y_i \in D_i$ for $i = 1,\dots,n/2$ and put $X = \{x_1,\dots,x_{n/2}\}$ and $Y = \{y_1,\dots,y_{n/2}\}$. Let $X \cup Y$ be the vertex-set of our S_n, and

join an $x \in X$ to a $y \in Y$ if $\rho(x,y) < \sqrt{2} - \mu$;
join an $x \in X$ to a $x' \in X$ if $\rho(x,x') > 2 - \mu$;
join a $y \in Y$ to a $y' \in Y$ if $\rho(y,y') > 2 - \mu$.

For $p \geq 3$, our result follows from a generalization of this construction. Theorem 2.13 may also be sharp for $p \geq 3$, but we cannot prove it. Let $p = 3$. Our results show only that

$$0 \leq \vartheta_3(K_5) \leq \frac{1}{12}$$

and

$$\frac{1}{8} \leq \vartheta_3(K_6) \leq \frac{1}{6}.$$

One of the most intriguing problems is to determine the values and some asymptotically extremal graphs for $RT_3(n, K_5, o(n))$ and $RT_3(n, K_6, o(n))$. Unfortunately, this task seems to be too difficult. We do not know the answer to the simplest subproblem if $\vartheta_3(K_5) > 0$.

The last section contains some further open problems.

The basic proof techniques include primarily the application of Szemerédi's Regularity Lemma, [32], a modification of Zykov's symmetrization method, [39] and multigraph extremal-graph results [4, 5, 6, 7] (in the background).

Remark 2.14. It is difficult to find the places in this paper that would distinguish between the conditions '(+) S_n contains no L_i' and '(++) S_n can be coloured in r colours so that the ν^{th} colour contains no L_ν'. The reason for this is that the limit constants are the ones that are different: we have the existence theorems in the same generality for the more general case (++).

3. Proofs of Theorems 2.1–2.3

The aim of this section is to prove Theorems 2.1–2.3. We will start with the simpler Theorem 2.1, move on to the proof of Theorem 2.3 and then return to the proof of Theorem 2.2.

Proof of Theorem 2.1. Again, as in the 'proof' of Claim 1.1, (a) is trivial, (c) follows from (a) and (b) and the only thing to be proved is that if the forbidden graphs are complete graphs and we have an infinite sequence (S_{m_t}), as described in Claim 1.1(b), then we can extend this sequence to every $n > n_0$.

First fix an $\varepsilon > 0$. Assume that S_{m_t} is an extremal graph for $RT_p(m_t, K_{p_1}, \ldots, K_{p_r}, \varepsilon m_t)$. We may choose this sequence so that

$$e(S_{m_t}) \geq (\vartheta_\varepsilon - \varepsilon)\binom{m_t}{2}.$$

So S_{m_t} has an r-colouring in which the ν^{th} colour contains no K_{p_ν} and $\alpha_p(S_{m_t}) \leq \varepsilon m_t$ if $t > t_0(\varepsilon)$. Below, we will sometimes abbreviate m_t to m. Let h be an arbitrary integer and put $Z_{mh} = S_m \otimes I_h$, that is, let Z_{mh} be the graph obtained from S_m by replacing each vertex by h independent vertices and joining two new vertices in colour ν iff the original vertices have been joined in colour ν†. Since the forbidden graphs are complete graphs,

† Here I_h means the complementary graph of K_h.

the r-coloured Z_{mh} will contain no K_{p_ν} (either) in the ν^{th} colour. Further, trivially,

$$\frac{e(Z_{mh})}{(mh)^2} = \frac{e(S_m)}{m^2},$$

and

$$\frac{\alpha_p(Z_{mh})}{mh} = \frac{\alpha_p(S_m)}{m}.$$

(Indeed, each K_p-independent set increases by a factor h, and each K_p-independent set X of Z_{mh} induces a K_p-independent set of S_m of size at least $(1/h)|X|$.)

As described in the proof of Claim 1.1, we may choose a sequence S_{m_t} with $\varepsilon_t = (1/t)$, $\alpha_p(S_{m_t}) \le \varepsilon_t m_t$, and (for $f = n$ and $\vartheta = \vartheta_{p,f} = \lim \vartheta_\varepsilon$)

$$e(S_{m_t}) \ge \left(\vartheta - \frac{1}{t}\right)\binom{m_t}{2}.$$

Now, for every $n > m_1^2$ choose the largest $m_t \leqq \sqrt{n}$. Then choose $h = \lceil n/m_t \rceil$ and delete $n - m_t h$ vertices of $Z_{m_t h}$ to get a graph S_n^*. Clearly, $m_t \to \infty$. Since we have deleted at most $m_t h - n = o(n)$ vertices from $Z_{m_t h}$, we obtain a sequence S_n^* with $\alpha_p(S_n^*) = o(n)$ and

$$e(S_{m_t}) = \vartheta\binom{m_t}{2} + o(m_t^2).$$

□

(As $m_t \to \infty$, we cannot get all the integers in the form hm_t. Therefore we must approximate some n's by $hm_t > n$: to delete $\le h = o(m_t h)$ vertices from some of the $Z_{m_t h}$'s.)

One of the basic methods we use to handle Turán–Ramsey type problems is the Regularity Lemma [32].

The Regularity Lemma The regularity condition means that the edges behave (in some weak sense) as if they were random. The Regularity Lemma asserts that the vertices of the graph can be partitioned into a bounded number of classes $V_0, \dots, V_k$ such that almost every pair is ε-regular.

The Regularity Lemma. (See, for example, [32].) *For every $\varepsilon > 0$ and integer κ there exists a $k_0(\varepsilon, \kappa)$ such that every graph G_n, the vertex set $V(G_n)$ can be partitioned into sets $V_0, V_1, \dots, V_k$ – for some $\kappa < k < k_0(\varepsilon, \kappa)$ – so that $|V_0| < \varepsilon n$, $|V_i| = m$ (is the same) for every $i > 0$, and for all but at most $\varepsilon\binom{k}{2}$ pairs (i, j), for every $X \subseteq V_i$ and $Y \subseteq V_j$ satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|d(X, Y) - d(V_i, V_j)| < \varepsilon.$$

Remark 3.1. The role of V_0 is purely technical: it makes it possible for all the other classes to have *exactly* the same cardinality. Indeed, having a κ and choosing $\kappa' > \kappa, \varepsilon^{-2}$ and applying the Regularity Lemma with this κ, one can distribute the vertices of V_0 evenly among the other classes so that $|V_i| \approx |V_j|$ and the ε-regularity will be preserved with a slightly larger ε. So from now on (for the sake of simplicity) we will assume that $V_0 = \emptyset$. The role of κ is to make the classes V_i sufficiently small, so that the number of edges inside those classes are negligible. The partitions described in the Regularity Lemma, or here, will be called *Regular Partitions* of G_n.

Now we turn to the second tool used in our proof: the application of matrix graph sequences.

Dense matrices, matrix graph sequences

Lemma 3.2. *Let A be a symmetric matrix, τ and π, $\tau \neq \pi$ be given integers, and let $a_{\tau,\pi} = 0$. Then deleting either*

— the τ^{th} row and column, or

— the π^{th} row and column

we get a matrix A' with

$$g(A') = g(A).$$

This implies

Lemma 3.3. *If a symmetric matrix A is dense, then all its off-diagonal entries are positive.*

The lemma is a variant of Zykov's symmetrization [39], and its proof can be found, for example, in [4]. Hence we only sketch its proof here†.

Proof of Lemma 3.2. (Sketched) Let $\mathbf{u}$ be an optimum vector for A, *i.e.*,

$$g(A) = \max\left\{\mathbf{u}A\mathbf{u}^T \ : \ u_i \geq 0 \quad (i = 1, \dots, \ell) \text{ and } \sum u_i = 1\right\}.$$

We define $\mathbf{u}(h)$ to be the vector where the τ^{th} coordinate of the optimum vector $\mathbf{u}$ is decreased by h and the π^{th} is increased by h. Clearly,

$$\varphi(h) = \mathbf{u}(h)A\mathbf{u}(h)^T = (a_{\pi,\pi} + a_{\tau,\tau})h^2 + c_1 h + c_2$$

for some constants c_1, c_2 (because $a_{\tau,\pi} = a_{\pi,\tau} = 0$). For any interval, such functions attain their maximum at some end of the interval (and maybe, inside as well). Hence we may choose either $h = u_\tau$ or $h = -u_\pi$ and still get the same maximum $g(A)$. But now one of the coordinates is 0, therefore the value of $g(A)$ is the same as if we had deleted the τ^{th} or π^{th} row and column: $g(A) = g(A')$. □

Lemma 3.4. *Assume that $f(n)$ satisfies the condition of Theorem 2.3. Then for every sequence $\varepsilon_n \to 0$ we can find a sequence $\beta_n \to 0$ such that*

$$f(\beta_n n) \geq \sqrt{\varepsilon_n} f(n). \tag{2}$$

† A. Sidorenko [27] has found a generalization of this lemma, providing a necessary and sufficient condition for being dense.

Proof. Let t be an integer and $\beta = 1/t$. Then $f(\beta n) \geq \eta_{f,\beta} f(n)$. If $n > n_t$ then $\varepsilon_n \leq \eta^2_{f,\beta}$. Thus $f(\beta n) \geq \sqrt{\varepsilon_n} f(n)$. We may assume $n_{t+1} > n_t$. Define $\beta_n = 1/t$ for $n \in [n_t, n_{t+1})$, $t := 1, 2, 3 \ldots$ Then $\beta_n \to 0$ and (2) holds. □

Proof of Theorem 2.3. For every fixed $\varepsilon > 0$, for some infinite set of integers $\mathbb{N}_\varepsilon$, for every $n \in \mathbb{N}_\varepsilon$, we may fix an S_n satisfying

(i) $\begin{cases} \text{for } \nu = 1, \ldots, r \text{ the subgraph of colour } \chi_\nu \text{ contains no } L_\nu \\ \text{and } \alpha_p(S_n) \leq \varepsilon f(n), \end{cases}$

and

(ii) $\vartheta_\varepsilon = \lim_{n \in \mathbb{N}_\varepsilon} \frac{e(S_n)}{\binom{n}{2}}$.

Apply the Regularity Lemma to this sequence (S_n) with this ε and $\kappa = 1/\varepsilon$ (where κ is the lower bound on the number of classes). Thus we get a $k_0 = k_0(\varepsilon)$ such that the vertices of S_n can be partitioned into the classes $V_{1,n}, \ldots, V_{k,n}$ for some $\kappa < k < k_0$ so that

(iii) all but $\varepsilon\binom{k}{2}$ pairs are ε-regular, ($k = k(n)$.) †

Using a diagonalization, we may find an infinite set of integers $\mathbb{N}^*$ and for each $n \in \mathbb{N}^*$ an r-coloured graph S_n, with a Regular Partition $\{V_{1,n}, \ldots, V_{k(n),n}\}$, satisfying

(i*) $\begin{cases} \text{for } \nu = 1, \ldots, r \text{ the subgraph of colour } \chi_\nu \text{ contains no } L_\nu \\ \text{and } \alpha_p(S_n) \leq \varepsilon_n f(n), \end{cases}$

with some $\varepsilon_n \to 0$, and

(ii*) $\vartheta = \lim_{n \in \mathbb{N}^*} \frac{e(S_n)}{\binom{n}{2}}$, and

(iii*) all but $\varepsilon_n\binom{k}{2}$ pairs are ε_n-regular in the corresponding Regular Partition.

Here ε_n usually tends to 0 very slowly, but still it tends to 0! We may assume that $\vartheta > 0$.

Next, delete the edges $(x, y)\ : x \in V_{i,n}, y \in V_{j,n}$ if

(a) either $(V_{i,n}, V_{j,n})$ is nonregular, or
(b) $d(V_{i,n}, V_{j,n}) < 2\varepsilon_n$.

Thus we have deleted by (a) at most $\varepsilon_n\binom{k}{2}(n/k)^2 < (1/2)\varepsilon_n n^2$ edges and by (b) at most $2\varepsilon_n(n/k)^2\binom{k}{2}$ edges. In this way we have ensured that all the pairs $(V_{i,n}, V_{j,n})$ are ε_n-regular. The number of edges has been changed by at most $(3/2)\varepsilon_n n^2$. Denote the resulting graph by T_n.

There is a matrix $A = A_n$ of $k < k_0(\varepsilon_n)$ rows (and columns), corresponding to this graph T_n (and its ε_n-regular partition), where $a_{ij} = d(V_{i,n}, V_{j,n})$ (this value being the density in T_n). Clearly, if **e** is the k-dimensional vector each coordinate of which is n/k, then

† $V_0 = \emptyset$ is assumed, by Remark 3.1.

$(1/2)\mathbf{e}A\mathbf{e}^T$ counts the edges between the classes (but it does not count the edges within the classes) and

$$e(T_n) < \frac{1}{2}\mathbf{e}A\mathbf{e}^T + \varepsilon_n n^2 \le \frac{1}{2}g(A)n^2 + \varepsilon_n n^2.$$

Thus

$$\vartheta\binom{n}{2} - \varepsilon_n n^2 \le e(S_n) < \frac{1}{2}g(A)n^2 + 3\varepsilon_n n^2,$$

and therefore

$$g(A) \ge 2\vartheta - 8\varepsilon_n.$$

In the following $m = |V_1|$, $M = |\cup_{i\in I} V_i|$. We will find a subgraph H_M of T_n, equally dense (but possibly much smaller), spanned by the union of some $\Omega = \Omega_n \le R(L_1, \dots, L_r)$ classes $V_{i,n}$. (This makes the problem bounded in some sense.) For any subset $\{V_{i,n} : i \in I\}$ of $\{V_{i,n}\}$ we have a symmetrical minor (submatrix) A' of A and a corresponding number $g(A')$. We will choose an I for which $g(A') \ge g(A)$ and $|I|$ is the minimum. (Since $g(A') \le g(A)$, we will actually have $g(A') = g(A)$.) By Lemma 3.2, all the densities between these classes are positive in T_n, and therefore are at least 2ε. Further, the resulting matrix A' is dense.

So, if we end up with Ω classes, any two of which are joined by density $> 2\varepsilon_n$, then, by a very standard application of the Regularity Lemma, $T_n \supset K_\Omega$†. (See, for example, [11]) Hence $\Omega < R = R(L_1, \dots, L_r)$. In other words, we end up with a bounded number of classes (independently of n and ε).

Originally, when $n \to \infty$, we have $\varepsilon_n \to 0$, and the number of classes in the Regular Partition could have tended to ∞ and the entries a_{ij} to 0. Now the situation is nicer, the numbers of rows and columns in the matrices A' are bounded, independently of ε and n. So we can take a convergent subsequence of these matrices, while $n \to \infty$: we may assume that the matrices A'_n converge to a matrix A^*. Still, it can happen that A^* is not dense. In that case we can take a dense submatrix A_0 of A^*. (Otherwise $A_0 = A^*$.)

Now we have a (mostly very sparse) sequence of integers n_t and the corresponding graphs S_{n_t} with their Regular Partitions (described in the Regularity Lemma) and the corresponding matrices A_{n_t} with their dense submatrices A'_{n_t} converging to A^*. We consider only the dense submatrix A_0 of A^*. Let A_0 be an $\Omega \times \Omega$ matrix. It has an optimum vector $\mathbf{u}$ and each coordinate of $\mathbf{u}$ is positive, say at least $\gamma > 0$. So we can fix the corresponding $\Omega \le R(L_1, \dots, L_r)$ classes, say $V_1, \dots, V_\Omega$, and the corresponding $u_i m$ vertices in them, thus getting an optimal A_0-matrix graph sequence

$$H_m \subseteq S\left[\bigcup_{i\le\Omega} V_{i,n_t}\right].$$

Since each class of $W_i := V_i \cap V(H_m)$ of H_m has at least γm vertices, the W_i's will be joined to each other $(1/\gamma)\varepsilon_n$-regularly: they will induce an optimal A_0-matrix graph sequence.

We have to prove four things:

† Here we need that ε is small in terms of the Ramsey number $R(L_1, \dots, L_r)$.

(α) The corresponding graphs can be coloured in r colours so that the ν^{th} colour contains no L_ν;
(β) $\alpha_p(H_m) = o(f(m))$,
(γ) this matrix graph sequence has enough edges to be asymptotically extremal.
(δ) $e(W_i) = o(m^2)$.

(α) This is trivial, since $H_{m_t} \subseteq S_{n_t}$ and the S_n's have this colouring property.
(β) Up to this point we have used one fixed sequence ε_n. Replacing this sequence by another $\varepsilon'_n > \varepsilon_n$ tending to 0, everything above remains valid (with the same regular partition). Given the original sequence ε_n, we fix a sequence β_n as described in Lemma 3.4. For any fixed ε the upper bound k_0 of the Regularity Lemma is a constant. So we may find an $\varepsilon''_n \to 0$ (very slowly) for which, for ε''_n and $\kappa = 1/\varepsilon''_n$ we have $k_0(\kappa, \varepsilon''_n) < 1/\beta_n$. If $\tilde{\varepsilon}_n = \max\{\sqrt{\varepsilon_n}, \beta_n, \varepsilon''_n\}$, then with this $\tilde{\varepsilon}_n \to 0$ we have for every induced subgraph $H_m \subseteq S_n$ of at least n/k vertices

$$\alpha_p(H_m) \leq \sqrt{\varepsilon_n} f(n) \leq f(n/k) \leq f(m).$$

(γ) This follows by a simple computation: we have $g(A'_{n_t}) \geq \vartheta - 8\varepsilon_t$. Hence $g(A_0) \geq \vartheta$. So for an A_0-graph H_m, we would know that $e(H_m) \geq (1/2)g(A_0)m^2$. Now the subgraph of S_{n_t} spanned by the selected classes $V_{i,n_t}\ :\ i \in I_{n_t}$ is only a 'nearly'-A_0-graph: the entries in A'_{n_t} tend to the corresponding entries of A_0, but they are not equal. Thus we have only

$$e(H_m) \geq \vartheta \binom{m}{2} - o(m^2).$$

However, this is enough to ensure that (H_m) is an asymptotically extremal graph sequence.
(δ) In principle, some classes of H_m could contain too many edges (in terms of m). Now we exclude this. By the construction, $g(A_0) = \vartheta_{p,f}(L_1, \ldots, L_r) = \vartheta$. Hence, on the one hand, for $W_i = V_{i,n} \cap V(H_m)$,

$$e(H_m) \geq \frac{1}{2}(g(A_0) - o(1))\, m^2 + \sum e(W_i) = \frac{1}{2}(\vartheta - o(1))\, m^2 + \sum e(W_i).$$

On the other hand,

$$e(H_m) \leq \frac{1}{2}\vartheta m^2 + o(m^2).$$

Thus $\sum e(W_i) = o(m^2)$.

□

Remark 3.5. This remark is aimed primarily at those who know the Zykov symmetrization. Here we try to explain something of the background of the above proof. In constructing (finding) the 'good" subgraph $H_m \subseteq S_n$, we have basically used a modification of Zykov's 'symmetrization' method [39]. The original Zykov type symmetrization means that (instead of deleting vertices) we change the edges incident with some vertices, obtaining a graph with the same number of vertices, but of simpler, more symmetric structure. This method breaks down because the symmetrization may increase the independence number

($\alpha(G_n)$ or $\alpha_p(G_n)$), and that is not allowed here. Further, symmetrization can introduce unwanted subgraphs: it may happen for example that G_n contains no $K_3(10,10,10)$ but after several symmetrizations it will. Deleting vertices, we can replace the original method of symmetrization: unless we delete too many of them, $\alpha_p(G_n) = o(f(n))$ will be preserved, and of course, no new subgraphs occur. At the same time, the structure becomes simpler and, in some very vague sense, more symmetric.

Proof of Theorem 2.2. We know that there is a sequence of graphs (described in the proof of Theorem 2.3) that is for some *fixed* matrix A an optimal A-matrix graph sequence. We need to show that for each $n > n_0$ the *same* matrix A can be used. As in the proof of Theorem 2.1, we will blow up some good graphs S_{m_t}.

If we have an infinite sequence (S_{m_t}) and a fixed matrix A such that (S_{m_t}) is an optimal A-matrix graph sequence, and asymptotically extremal for some $\varepsilon_t \to 0$, for $RT_p(m_t, K_{p_1}, \dots, K_{p_r}, \varepsilon_t m_t)$, then $Z_{m_t h} = S_{m_t} \otimes I_h$ will also be optimal A-matrix graphs.

Hence, fix the matrix A obtained in the proof of Theorem 2.3 for a sequence $\varepsilon_t \to 0$ and some sequence m_t. For every n, take the largest $m_t \le \sqrt{n}$, then put $h = \lceil n/m_t \rceil$ and delete $(hm_t - n)$ vertices of $Z_{m_t h} = S_{m_t} \otimes I_h$. The resulting A-matrix graph sequence (S_n^*) proves Theorem 2.2. □

4. Quantitative results for one colour

In this section we obtain various estimates for $\vartheta_p(K_q)$.

Proof of Theorem 2.6. In the following, the constants $c_0, c_1, c_2, \dots$ are positive and independent of n, m. Assume indirectly that there exist a constant $c_0 > 0$ and infinitely many graphs G_n not containing K_q, satisfying $\alpha_p(G_n) = o(n)$ and yet having many edges:

$$e(G_n) > \left(1 - \frac{p}{q-1} + c_0\right)\binom{n}{2}.$$

By a standard argument, for some constants c_1, $c_2 > 0$, there exist subgraphs $H_m \subseteq G_n$ with minimum degree

$$d_{\min}(H_m) > \left(1 - \frac{p}{q-1} + c_1\right)m, \qquad m > c_2 n \text{ and } \alpha_p(H_m) = o(m). \tag{3}$$

By a 'saturation argument', we may assume that $H_m \supset K_{q-1}$: if not, add edges to it one by one, until it does. Clearly, (3) remains valid. Fix a $K_{q-1} \subseteq H_m$. Now

$$e(K_{q-1}, H_m - K_{q-1}) > (q - p - 1 + c_1)(m - q + 1).$$

Therefore, for some $c_3 > 0$, there exists a set U of $c_3 m$ vertices of $H_m - K_{q-1}$, each joined to the *same* $q - p$ vertices of this fixed K_{q-1}. By the assumption, $\alpha_p(G_n) = o(n)$, if n (and therefore m) is sufficiently large, then there is a $K_p \subset U$. This K_p, together with the fixed $q - p$ vertices of K_{q-1} forms a $K_q \subseteq H_m \subseteq G_n$. This contradiction proves (a). As we have mentioned, Construction 2.7 provides the lower bound, *i.e.* (b). □

For $q = p + 1$, Theorem 2.6 reduces to the following claim.

Claim 4.1. *For any $p > 1$, $\vartheta_p(K_{p+1}) = 0$.*

This also has a trivial direct proof.

Proof. (Direct) Suppose that (G_n) is a graph sequence with

$$K_{p+1} \not\subseteq G_n \text{ and } \alpha_p(G_n) = o(n).$$

If x is an arbitrary vertex, then its neighbourhood $N(x)$ contains no K_p. Therefore $d(x) = |N(x)| \le \alpha_p(G_n) = o(n)$. Hence $e(G_n) \le n\alpha_p(G_n) = o(n^2)$. □

Now we can return to the proof of Theorem 2.11, which improves Theorem 2.6 in some special cases. We will need the following two lemmas.

Lemma 4.2. *For any integers $p \ge 2$ and $0 \le \gamma < p$, and constant $c > 0$, there exists a constant $M_{p,c}$ with the following properties. Let $\varepsilon > 0$ be fixed and $\eta \ge M_{p,c}\varepsilon$. Suppose $\alpha_p(H_n) = o(n)$ and $B_e \subseteq H_n$ be a bipartite graph with colour classes V_1 and V_2 that are joined ε-regularly. Let $|V_1| = |V_2| > cn$ and $d(V_1, V_2) \ge (\gamma/p) + \eta$ and $n > n_0(p, c, \eta)$. Then $H_n \supset K_{p+\gamma+1}$.*

Obviously, we are thinking of the case when we apply the Regularity Lemma to a large graph and V_1, V_2 are two classes in the resulting partition connected to each other regularly and with a sufficiently high density.

Proof. For n large enough, all but at most εn vertices of V_1 are joined to at least $((\gamma/p) + (1/2)\eta)|V_2|$ vertices of V_2. Hence V_1 contains a K_p joined with at least $(\gamma + (1/2)p\eta)|V_2|$ edges to $|V_2|$. Thus (for some fixed constant $c_1 > 0$) V_2 contains at least $c_1 n$ vertices joined to the same $\gamma + 1$ vertices of *this* $K_p \subset V_1$. They form a $K_{\gamma+1} \subseteq K_p \subseteq V_1$. The $c_1 n$ vertices in V_2 contain a K_p completely joined to $K_{\gamma+1} \subseteq V_1$: $K_{p+\gamma+1} \subseteq H_n$. □

Lemma 4.3. *For any integers $p, k \ge 2$, and $0 \le \gamma < p$ and constant $c > 0$ there exists a constant $M_{p,c,k}$ with the following properties. Let $\varepsilon > 0$ be fixed and $\eta \ge M_{p,c,k}\varepsilon$. Let $\alpha_p(H_n) = o(n)$ and $V_1, \dots, V_k \subseteq V(H_n)$, $V_i \cap V_j = \emptyset$, $|V_i| > cn$. Assume that for every $1 \le i < j \le k$ the pairs of classes (V_i, V_j) are ε-regular, and $d(V_i, V_j) > \eta$. If $d(V_1, V_2) \ge (\gamma/p) + \eta$ and $n > n_0(p, c, \eta)$, then $H_n \supset K_{p+\gamma+k-1}$.*

Proof. For $j = k, k-1, \dots, 3$ we fix, recursively, a vertex $x_j \in V_j$, so that they form a complete $k - j + 1$-graph and are joined completely to some sets $V_{ij} \subseteq V_i$ $(i < j)$ and $|V_{ij}| > c_j^* n$ for some constant $c_j^* > 0$. For $j = 3$ we get a complete $(k-2)$-graph joined completely to some sets $V_1^* \subseteq V_1$ and $V_2^* \subseteq V_2$, $|V_1^*|, |V_2^*| > c^* n$, for some constant $c^* > 0$. (We use $\eta \ge M_{p,c,k}\varepsilon$ to ensure that all the sets V_{ij} above are large enough to apply the ε-regularity of the Regularity Lemma iteratively.) Applying Lemma 4.2

to the corresponding bipartite graph $H(V_1^*, V_2^*)$ (with $\eta - k\varepsilon$ instead of η), we get a $K_{p+\gamma+1+k-2} = K_{p+\gamma+k-1} \subseteq H(V_1 \cup V_2 \cup \ldots \cup V_k)$. □

Proof of Theorem 2.11.

(a) Let $\alpha_p(G_n) = o(n)$ and $K_{p+\ell} \not\subseteq G_n$. Fix an $\varepsilon > 0$ and put $\eta = M_{p,c,k}\varepsilon$. We apply the Regularity Lemma to G_n, with this ε. Thus we get a partition $V_1, \ldots, V_k$ of the vertices into $k \leq k_0(\varepsilon, \kappa)$ sets of size $\approx n/k$ (see Remark 3.1 on V_0).

(b) For any graph G let

$$\Phi(G) = e(G) \Big/ \binom{v(G)}{2}.$$

We apply symmetrization in the sense described in the proof of Theorem 2.3: we find a subset of the classes V_i, say $V_1, \ldots, V_t$ so that the density between any two of them is at least 2η and the density for the obtained $G_M = G\left[\cup_{i \leq t} V_i\right]$ is high:

$$\Phi(G_n) < \Phi(G_M) + 2\eta.$$

There is a unique integer γ such that for these t classes the largest density occuring is $\geq (\gamma/p) + \eta$ but $\leq ((\gamma + 1)/p) + \eta$. The density $\Phi(G_M) = e(G_M)/\binom{M}{2}$ can be estimated as follows:

$$\Phi(G_M) \leq \frac{1}{2}\left(1 - \frac{1}{t}\right)\left(\frac{\gamma + 1}{p} + \eta\right).$$

Here $G_M \supseteq K_{p+t+\gamma-1}$ and $G_M \not\supseteq K_{p+\ell}$. Therefore $\gamma \leq \ell - t$, so

$$\Phi(G_M) \leq \frac{1}{2}\left(1 - \frac{1}{t}\right)\left(\frac{\ell - t + 1}{p} + \eta\right). \tag{4}$$

Put

$$h(t, \ell) = \left(1 - \frac{1}{t}\right)\frac{\ell - t + 1}{2p}.$$

For $t = 2$ we get the conjectured density: $h(2, \ell) = (\ell - 1)/4p$. What we have to prove is that for $\ell = 2, 3, 4$ and 5, $h(t, \ell) \leq h(2, \ell)$:

$$\frac{1}{2}\left(1 - \frac{1}{t}\right)\frac{\ell - t + 1}{p} \leq \frac{1}{4}\frac{\ell - 1}{p},$$

which follows from

$$h(2, \ell) - h(t, \ell) = \frac{(t - 2)(2t - \ell - 1)}{2tp} \geq 0.$$

□

Proof of Theorem 2.13 In proving the lower bound on $RT_2(n, K_4, o(n))$, Bollobás and Erdős used a geometric, or more precisely, an 'isoperimetric' theorem. Theorem 2.13 is a generalization of the Bollobás–Erdős result. So it is natural to prove Theorem 2.13 using a generalization of the original Isoperimetric Inequality. This generalization was conjectured by Erdős and proved by Bollobás [1].

We need the following definition.

Definition 4.4. ([1]) For $k \geq 2$ define the *k-diameter* of a set A in a metric space by

$$d_k(A) = \sup_{x_1,\dots,x_k \in A} \min_{i<j} \rho(x_i, x_j).$$

(In other words, this is the k^{th} 'packing constant' of A.)

A spherical cap is the intersection of an h-dimensional sphere $\mathbf{S}^h$ and a halfspace Π.

Bollobás Theorem. ([1])*Let A be a nonempty subset of the h-dimensional sphere $\mathbf{S}^h$ of outer measure $\mu^*(A)$*†*, and let C be a spherical cap of the same measure. Then $d_k(A) \geq d_k(C)$ for every $k \geq 2$.*

In the following, whenever we speak of 'measure', we will always consider relative measure, which is the measure of the set on the sphere $\mathbf{S}^h$ divided by the measure of the whole sphere.

Denote by $\delta = \delta_p$ the diameter of a p-simplex. ($\delta_2 = 2$, $\delta_3 = \sqrt{3}, \dots$)

Corollary 1 of Bollobás Theorem. *Let the integer p and two small constants ε and $\eta > 0$ be fixed. Then for $h > h_0(p, \varepsilon, \eta)$, if A is a measurable subset of $\mathbf{S}^h$ of relative measure $> \varepsilon$, there exist p points $x_1, \dots, x_p \in A$ such that all $d(x_i, x_j) > \delta_p - \eta$.*

Proof. Indeed, if A does not contain such a p-tuple, its p-diameter is at most $\delta_p - \eta$. Hence – by the Bollobás theorem – the outer measure of A is at most as large as that of a spherical cap of p-diameter $\delta_p - \eta$. For some constant $c_{p,\eta} > 0$ the ordinary diameter of such a cap is at most $2 - c_{p,\eta}$, independent of the dimension h. Hence the relative measure of such a spherical cap is at most $(Q_{p,\eta})^h$ for some constant $0 < Q_{p,\eta} < 1$ and so the relative measure of A is at most $(Q_{p,\eta})^h \leq \varepsilon$ if $h > h_0(p, \varepsilon, \eta)$, a contradiction. □

Corollary 2 of Bollobás Theorem. (Erdős–Rogers Theorem) *For any integer p, there exists a sequence (S_n) of graphs with $K_{p+1} \not\subseteq S_n$ but $\alpha_p(S_n) = O(n^{1-c})$ for some $c > 0$.*

Proof of the Erdős–Rogers Theorem. Let δ_p be the edge-length of the regular p-simplex in $\mathbf{S}^{p-1} \subseteq \mathbf{R}^{p-1}$:

$$\delta_p := \sqrt{\frac{2p}{p-1}}. \tag{5}$$

‡Clearly, $\delta_p \searrow \sqrt{2}$.

For a given $\varepsilon > 0$, we fix a sufficiently high-dimensional sphere $\mathbf{S}^h$ and fix an $n \gg h$. We partition the surface of $\mathbf{S}^h$ into n domains D_i ($i = 1, \dots, n$) of equal measure and of diameter

$$\leq \frac{\delta_p - \delta_{p+1}}{4}.$$

(This can be done if n is sufficiently large.) Then we choose n vertices $x_i \in D_i$ ($i = 1, \dots, n$).

† We will only use 'nice sets', but Bollobás formulated his result in this generality. The reader can replace 'outer measure' by 'measure'.

‡ (5) is taken from [16], and will be obtained (as a by-product) in the proof of Theorem 2.13.

They will be the vertices of our graph Q_n. We join x_i and x_j if

$$\rho(x_i, x_j) > \frac{\delta_{p+1} + \delta_p}{2} > \delta_{p+1}.$$

Trivially, $K_{p+1} \not\subseteq Q_n$. If we choose εn vertices x_ℓ of Q_n and A is the union of the corresponding D_i's, then the relative measure of A is at least ε, and – by Bollobás Theorem – A contains some $w_1, \dots, w_p$ with $\rho(w_i, w_j) > \delta_p$, $(1 \le i < j \le p)$. Replacing each $w_i \in D_i$ by the corresponding vertex $x_i \in A_i$, we still have $\rho(x_i, x_j) > (1/2)(\delta_p + \delta_{p+1})$, *i.e.* we have found a K_p in the subgraph induced by these n^{1-c} vertices: $\alpha_p(Q_n) =\le \varepsilon n$. As $\varepsilon \to 0$, the dimension $h \to \infty$ and $\alpha_p(Q_n) = o(n)$. Using a more careful calculation, we get $\alpha_p(Q_n) = O(n^{1-c})$. □

Proof of Theorem 2.13. We will use a Bollobás–Erdős type construction (see [3]) to get a graph sequence (B_n) to prove Theorem 2.13. Fix a high-dimensional sphere $\mathbf{S}^h$ and partition it into $n/2$ domains $D_1, \dots, D_{n/2}$, of equal measure and diameter $(1/2)\mu$, with $\mu = \varepsilon/\sqrt{h}$. This can always be done if $\varepsilon > 0$ is first fixed, h is then chosen to be sufficiently large, and, finally, $n > n_0(\varepsilon, h)$.

Choose a vertex $x_i \in D_i$ and a $y_i \in D_i$ (for $i = 1, \dots, n/2$), and put $X = \{x_1, \dots, x_{n/2}\}$ and $Y = \{y_1, \dots, y_{n/2}\}$. Let $X \cup Y$ be the vertex-set of our B_n and

join an $x \in X$ to a $y \in Y$ if $\rho(x, y) < \sqrt{2} - \mu$;
join an $x \in X$ to a $x' \in X$ if $\rho(x, x') > \delta_p - \mu$;
join a $y \in Y$ to a $y' \in Y$ if $\rho(y, y') > \delta_p - \mu$.

(a) First we show that $\alpha_p(B_n) = o(n)$. To show this, choose εn vertices of B_n. At least $(1/2)\varepsilon n$ vertices belong to (say) X and the union of the corresponding D_i's has relative measure $\ge (1/2)\varepsilon$. Denote by A the union of the D_i's corresponding to these x_i's. By Bollobás Theorem, if $d_p(A) \le (1/2)(\delta_p + \delta_{p+1})$, then $\mu(A) < \varepsilon$, provided that $h > h_0$. So we may choose $w_1, \dots, w_p \in A$ such that for each $i \ne j$, $\rho(w_i, w_j) > \delta_p - (1/2)\mu$, and therefore $\rho(x_i, x_j) > \delta_p - \mu$, yielding a K_p in the subgraph of B_n spanned by these εn vertices.

(b) Now we show that the resulting graph B_n contains no K_{2p}. Clearly, if $2p$ vertices form a $K_{2p} \subseteq B_n$, then p of them must be in X and the other p in Y, since – for sufficiently small ε – neither X nor Y contains a K_{p+1}. Suppose that $a_1, \dots, a_p \in X$ and $b_1, \dots, b_p \in Y$ form a K_{2p}. In the following, a_i's and b_j's are unit vectors and points of the sphere at the same time. The idea of the proof is as follows. We will show that the existence of such a K_{2p} implies that $(\sum a_i - \sum b_j)^2 < 0$, which is a contradiction. To get this, we will estimate $\sum a_i a_j$, and $\sum b_i b_j$ from above, and $\sum a_i b_j$ from below.
Let $d = \delta_p - \mu$ and $t = \sqrt{2} - \mu$. Now, $|a_i| = 1$, $|b_j| = 1$, and $|a_i - a_j| > d$. Therefore

$$2 \sum_{1 \le i < j \le p} a_i a_j = \sum_{1 \le i < j \le p} \left((a_i^2 + a_j^2) - (a_i - a_j)^2\right) < \binom{p}{2}(2 - d^2).$$

The same holds for the b_i's. Hence

$$2 \sum_{1 \le i < j \le p} \left(a_i a_j + b_i b_j\right) < (p^2 - p)(2 - d^2).$$

Let us now turn to the mixed terms. By $|a_i - b_j| < t$, we have

$$2\sum_{i=1}^{p}\sum_{j=1}^{p} a_i b_j = \sum_{i=1}^{p}\sum_{j=1}^{p}(a_i^2 + b_j^2) - \sum_{i=1}^{p}\sum_{j=1}^{p}(a_i - b_j)^2 > p^2(2 - t^2).$$

This implies that

$$\begin{aligned}
\left(\sum_{i=1}^{p} a_i - \sum_{j=1}^{p} b_j\right)^2 &= \sum_i a_i^2 + \sum_j b_j^2 + 2\sum_{1\le i<j\le p}(a_i a_j + b_i b_j) - 2\sum_{i=1}^{p}\sum_{j=1}^{p} a_i b_j \\
&< 2p + 2(p^2 - p) - (p^2 - p)d^2 - (2p^2 - p^2 t^2) \\
&= p^2 t^2 - (p^2 - p)d^2 = p^2(\sqrt{2} - \mu)^2 - (p^2 - p)(\delta_p - \mu)^2 \\
&= (2p^2 - (p^2 - p)\delta_p^2) - 2\left(\sqrt{2}p^2 - (p^2 - p)\delta_p\right)\mu + p\mu^2.
\end{aligned}$$

To avoid clumsy calculations involving δ_p, observe that in all the above formulas we have equality if $\varepsilon = 0$, $\mu = 0$, that is, a_i's are the vertices of a regular p-simplex and b_j's are the vertices of another. Indeed, in this case $\sum a_i = 0$ and $\sum b_j = 0$. Hence $2p^2 - (p^2 - p)\delta_p^2 = 0$, that is,

$$\delta_p = \sqrt{\frac{2p}{p-1}}.$$

Returning to the $\mu > 0$ case, we get

$$\begin{aligned}
0 \le \left(\sum a_i - \sum b_j\right)^2 &< -2\left(\sqrt{2}p^2 - (p^2 - p)\sqrt{\frac{2p}{p-1}}\right)\mu + p\mu^2 \\
&= -2\sqrt{2}p\left(p - \sqrt{p^2 - p}\right)\mu + p\mu^2 < 0,
\end{aligned}$$

provided that μ is sufficiently small, is a contradiction. This shows that $B_n \not\supseteq K_{2p}$.

(c) Each vertex has degree $(n/4)+o(n)$, since each a_i is joined to the b_i's on an 'approximate half-sphere' and thus the the surface considered has measure $\ge (1/2) - O(\varepsilon)$ and the number of vertices b_j is proportional to this measure. So

$$\frac{n^2}{8} - O(\varepsilon n^2) \le e(B_n) \le \frac{n^2}{8} + O(\varepsilon n^2).$$

This completes the proof.

□

5. Two special cases

The last problems we discuss here are:

How large are $\vartheta_3(K_8)$ and $\vartheta_3(K_9)$?

Conjecture 2.9 asserts that $\vartheta_3(K_8) = 3/11$ and $\vartheta_3(K_9) = 3/10$. The conjectured extremal structures (described in Conjecture 2.9) in both cases have 3 classes and are as follows. Put $x = (3n/11) + o(n)$ vertices in the classes V_1, V_2 and $y = (5n/11) + o(n)$ vertices into

V_3. Then join V_1 and V_2 with $d(V_1, V_2) = 1/3$, $o(1)$-regularly, and join V_3 completely to the other two classes. The classes V_i contain some edges to ensure $\alpha_3(G_n) = o(n)$. However, the problem is that we are unable to find such graphs.

One reason that we cannot prove Conjecture 2.9 (even for $p = 3$, $q = 8, 9$) is that we are unable to construct bipartite graphs analogous to the Bollobás–Erdős [3], or Erdős–Rogers graph [16], but with density 1/3 (or 2/3) instead of 1/2. Here the 'analogous' means that we fix, for some t, a $t \times t$ matrix $D = (d_{ij})$ of positive elements, and on a high-dimensional sphere $\mathbf{S}^h$, we choose some sets $X_1, \ldots, X_t$, each uniformly distributed on the sphere in some sense, and join two vertices $u \in X_i$,and $v \in X_j$ if their Euclidean distance $\rho(u,v) \approx d_{ij}$, or $\rho(u,v) \geq d_{ij}, \ldots$

So we have only an upper bound on the number of edges.

Theorem 5.1. $\vartheta_3(K_8) \leq \dfrac{3}{11}$.

In the proofs of this and the next theorem we need some case-distinction. In many cases we know that the graph structure considered is dense, and we can easily calculate the edge-densities by solving a small system of linear equations. Here we formulate a lemma, which covers most of the cases we need. (It has a more general form as well.)

Lemma 5.2. *Let $A = A_{h,k,\lambda,\varphi,\beta}$ be a symmetric $(h+k) \times (h+k)$ matrix satisfying*

$$a_{i,j} = \begin{cases} \lambda & \text{if} \quad 1 \leq i < j \leq h, \\ \varphi & \text{if} \quad h < i < j \leq h+k, \\ \beta & \text{else.} \end{cases}$$

If A is dense, its optimum vector $\mathbf{w}$ has coordinates

$$w_i = \frac{\beta k - \varphi(k-1)}{2\beta hk - \varphi h(k-1)) - \lambda k(h-1)} \qquad (i \leq h), \tag{6a}$$

and

$$w_i = \frac{\beta h - \lambda(h-1)}{2\beta hk - \varphi h(k-1)) - \lambda k(h-1)} \qquad (i > h). \tag{6b}$$

The density is

$$g(A) = \frac{\beta^2 hk - \lambda\varphi(h-1)(k-1)}{2\beta hk - \varphi h(k-1)) - \lambda k(h-1)} . \tag{7}$$

Proof. Assume that H_n is an optimal matrix graph corresponding to A. Let the classes of H_n be $V_1, \ldots, V_{h+k}$. Then $|V_i| \approx w_i n$. When counting the sizes of the classes in an optimal matrix graph, it is enough to take into account that the degrees must be asymptotically equal – provided that the matrix is dense† (see, for example [4]).Let the first h coordinates of the optimum vector be x, the others y‡. Now the vertices in the first h classes will have

† For dense matrices this condition is necessary and sufficient.

‡ Because of the symmetry, the first h class sizes will be asymptotically the same, and the same holds for the other k classes.

degree $(\lambda(h-1)x+\beta ky)n$, while in the last k classes the degrees will be $(\beta hx+\varphi(k-1)y)n$. Furthermore, $hx+ky=1$. Solving this system of linear equations, we get (6a) and (6b). Now, $g(A)$ is the common degree divided by n, (the edge-density is half of this). This proves (7)†. □

Remark 5.3. These formulas become much simpler if, for example, $h=1$ or $k=1$. For $k=1$, φ drops out and we get

$$g(A)=\frac{\beta^2 h}{2h\beta-(h-1)\lambda}. \tag{8}$$

Proof of Theorem 5.1. Let us fix an η as described in Lemma 4.3. Using the argument of the proof of Theorem 2.11, we get some sets $V_1,\ldots,V_t$, and we define γ to be an integer for which the largest density between these classes is between $(\gamma/p)+\eta$ and $((\gamma+1)/p)+\eta$. By (4), applied with $p=3$, $\ell=5$, we have

$$\Phi(G_M)\le\frac{1}{2}\left(1-\frac{1}{t}\right)\left(\frac{6-t}{3}+\eta\right)\le\frac{3}{11},$$

if $t>3$ and η is small enough. Therefore we may assume that $t\le 3$.

With $t=2$ the maximum density is $1/4<3/11$. So we may suppose that $t\ge 3$, that is, $t=3$.

(i) If the classes are V_1,V_2,V_3 and $d(V_1,V_2)\le(1/3)+\eta$, then the density is the maximum if the other two densities are 1, *i.e.* (by (8) applied with $\lambda=1/3$ and $\beta=1$, $h=2$) the maximum is at most $(3/11)+O(\eta)$ and we are home.

(ii) If, for example, $d(V_3,V_1)>(2/3)+\eta$, and $d(V_3,V_2)>(2/3)+\eta$, then we are home: we may choose a K_3 in V_3 and a subset $V_i'\subseteq V_i$ of c_1n vertices in both other classes, completely joined to this K_3. By Lemma 4.2, we find a K_5 in $V_1'\cup V_2'$, and we are home again.

(iii) In the remaining case there is a class adjacent to the other 2 classes with density $\le(2/3)+\eta$. We may assume that $d(V_3,V_1)\le(2/3)+\eta$, and $d(V_3,V_2)\le(2/3)+\eta$. By (8) (applied with $h=2$, $\beta=(2/3)+\eta$, $\lambda=1$) the edge-density is at most $(4/15)+O(\eta)<3/11$.

□

Theorem 5.4. $\vartheta_3(K_9)\le\dfrac{3}{10}$.

We know that $\vartheta_3(K_9)\ge 2/7$ because we may fix 3 classes V_1,V_2,V_3 of sizes $2n/7$, $2n/7$, $3n/7$, join V_3 to $V_1\cup V_2$ completely and build a graph on $V_1\cup V_2$ as described in the proof of Theorem 2.13. Put an Erdős–Rogers graph into V_3. The resulting graph contains no K_9, since V_3 contains no K_4 and $G[V_1\cup V_2]$ contains no K_6.

† Of course, the proof can be given entirely in the language of Linear Algebra without mentioning graphs.

Proof of Theorem 5.4. (Sketched.) Again, as above, we have to end up with at least $t \geq 3$ classes after the symmetrization, and if we have $t \geq 5$, then, by (4), the density is smaller than 3/10. So we may assume that $t \leq 4$.

The case of 3 classes is easy. Now at least one of the 3 densities is at most $(2/3) + 2\eta$, otherwise we have a $K_9 \subseteq G_n$. So the density is at most $(3/10) + O(\eta)$ (by (8), applied with $\lambda = (2/3) + \eta$, $\beta = 1$, $h = 2$), and we are home. Hence we may assume that $t = 4$.

We will distinguish 3 types of connections between V_i and V_j:

— if $d(V_i, V_j) < (1/3) + \eta$, we will call (V_i, V_j) a (1/3)-pair;
— if $(1/3) + \eta \leq d(V_i V_j) < (2/3) + \eta$, we will call (V_i, V_j) a (2/3)-pair;
— if $d(V_i V_j) > (2/3) + \eta$, we will call (V_i, V_j) a 1-pair.

We may assume that there is at least one 1-pair, otherwise the density could be estimated by

$$\frac{1}{2}\left(1 - \frac{1}{4}\right)\left(\frac{2}{3} + \eta\right) < \frac{3}{10}.$$

How many 1-pairs can we have on 4 classes? If we have two adjacent 1-pairs, (V_a, V_b) and (V_a, V_c), then (V_b, V_c) must be a (1/3)-pair: otherwise – by the proof of Theorem 5.1 – we could find $K_8 \subseteq V_a \cup V_b \cup V_c$, extendable into a K_9.

This immediately implies that we may have at most 4 1-pairs. If we have exactly 4 1-pairs, they form a 4-cycle and the remaining 2 densities are 1/3. Applying Lemma 5.2 with $h = k = 2$, $\lambda = \varphi = 1/3$ and $\beta = 1$ we get that the edge-density is at most $7/24 < 3/10$.

Here, unfortunately, we have to distinguish some cases.

(i) If $t = 4$ and there are 3 1-pairs meeting in one class, the other 3 pairs form a (1/3)-triangle. Applying (8) with $h = 3$, $\lambda = (1/3) + \eta$, $\beta = 1$ we get that the edge-density is at most $9/32 < 3/10$, and we are home again.

(ii) Suppose that we have on 4 classes 3 1-pairs that do not meet. Now they form a path, say $V_1 V_2 V_3 V_4$. The density is the highest when

$$d(V_1, V_3) = d(V_2, V_4) = \frac{1}{3} + \eta$$

and

$$d(V_1, V_4) = \frac{2}{3} + \eta.$$

An easy calculation shows that the optimal weights (for $\eta = 0$) are 1/6, 1/3, 1/3, 1/6, the density is $5/18 < 3/10$. (Or we can reduce this case to the case when the 1-edges form a C_4.)

(iii) We have settled the case when the number of 1-pairs is 4 or 3. The case of one 1-pair or when we have 2 independent 1-pairs can be majorized by the case when we have 2 independent 1-pairs and all the other pairs are (2/3)-pairs. By Lemma 5.2, applied with $k = h = 2$, $\lambda = \varphi = 1$, $\beta = 2/3$ we again get that the edge-density is smaller than $(7/24) + O(\eta) < 3/10$.

(iv) The only remaining case to be settled is when we have 2 adjacent 1-pairs, say (V_1, V_2) and (V_1, V_3). Now we know that we get the maximum density if $d(V_2, V_3) = (1/3) + \eta$

and $d(V_i, V_4) = (2/3) + \eta$. One can easily check (by determining the optimum vector of this structure) that the maximum density is $(11/39) + O(\eta) < 3/10$.

□

6. Open problems

Various open problems are stated in [12] and we have already stated the above Problem 2.12. Here we list some others. The first two of these are the simplest special cases of Conjecture 2.9, where we got stuck.

Problem 6.1. *How large is $\vartheta_3(K_{11})$?*

Problem 6.2. *How large is $\vartheta_3(K_{14})$?*

Conjecture 2.9 states that $\vartheta_3(K_{11}) = 11/32$ and $\vartheta_3(K_{14}) = 8/21$.

Problem 6.3. *Can one always find a matrix A such that one has a graph sequence $(S_n : n > n_0)$ obeying the partition rules of the matrix A and being asymptotically extremal for $RT_p(n, L_1, \ldots, L_r, o(n))$ (and not only for an infinite sequence of integers n_{k_ℓ})?*

The answer to this problem is very probably YES. (If it were not, it would probably mean that the extremal structure sharply depends on some parameters such as, for example, the divisibility properties of n, which are not really graph theoretic properties.)

Problem 6.4. *Is there a finite algorithm to find the limit*

$$\vartheta_p(L) = \lim \frac{RT_p(n, L, o(n))}{\binom{n}{2}}?$$

We have shown in our previous paper that there is a finite algorithm for finding $\vartheta_2(L_1, \ldots, L_r)$ if the sample graphs L_i are complete graphs. A paper of Brown, Erdős and Simonovits [7] shows that for the digraph extremal problems without parallel arcs (which seem to be very near to the Turán-Ramsey problems) there is an algorithmic solution, though far from being trivial. What is the situation in case of $\vartheta_p(L_1, \ldots, L_r)$?

Some hypergraph problems (and results) on Turán–Ramsey problems can be found in [18, 20].

Appendix A. Are there graphs satisfying (*)?

In the above, the forbidden graphs were complete graphs, here we discuss the general case, where $L_1, \ldots, L_r$ are arbitrary graphs.

We are interested in two strongly connected problems. Given either a family $\mathscr{L}$ or r families of excluded graphs, $\mathscr{L}_1, \ldots, \mathscr{L}_r$ and a graph sequence (G_n) with $\alpha_p(G_n) = o(n)$. Under what conditions on $\mathscr{L}$ or the families $\mathscr{L}_i$ can we assert that there exists a graph sequence (G_n) such that

(i) G_n contains an $L \in \mathcal{L}$ for $n > n_0$; or

(ii) there is an r-colouring of G_n so that for no colour v is there a v-coloured $L \in \mathcal{L}_v$?

The case $p = 2$ is easy. In both problems, if no L is a tree, such graphs exist. On the other hand, if (in each $\mathcal{L}_i$) some L is a tree, those graphs do not exist. Indeed, in [9] Erdős has proved that for every ℓ there exist a $c = c_\ell$ $(0 < c_\ell < 1)$ and an n_ℓ such that for every $n > n_\ell$ there exist graphs S_n with girth greater than ℓ and independence number $\alpha(S_n) < O(n^{1-c})$. This implies that if none of the graphs $L \in \mathcal{L}$ is a tree or a forest, and $\ell = \max_{L\in\mathcal{L}} v(L)$, the above graphs S_n will contain no L's and $\alpha(S_n) < O(n^{1-c})$. This answers (i) and (ii) also, since $\alpha(G_n) = o(n)$ implies that for all r-colourings of G_n some colours contain all the trees of at most ℓ vertices for $n > n_\ell$. For $p > 2$ the situation is similar, but somewhat more complicated. First we will solve the problem (i). We start with some definitions.

Definition A.1. A graph T is a *p-forest* if

(a) it is the union of complete graphs of order p, having no common edges and

(b) for every integer $t > 1$, the union of any t of these K_p's has at least $pt - t + 1$ vertices; or

(c) it is a subgraph of a graph described in (a) and (b).

Definition A.2. (Girth)

(1) We will say that the *girth* of a p-uniform hypergraph H is at least ℓ if the union of any $t < \ell$ hyperedges has at least $pt - t + 1$ vertices.

(2) We will say that the *p-girth* of a graph G is at least ℓ if every subgraph of G of fewer than ℓ vertices is a p-forest.

Clearly, the 2-forests are exactly the ordinary forests and the 2-girth of a graph is the ordinary girth.

Erdős–Hajnal Theorem. ([10, Theorem 13.3]) *For every given p, and ℓ and suitable constants $c_1, c > 0$ (for $n > n_0(p, \ell, c, c_1)$) there exist p-uniform hypergraphs H_n for which*

— *any two hyperedges intersect in at most one vertex (such hypergraphs are sometimes called* linear hypergraphs*),*

— *any set of $c_1 n^{1-c}$ vertices contains a hyperedge, and*

— *the union of any $t < \ell$ hyperedges has at least $pt - t + 1$ vertices. (In other words, the p-girth of H_n is at least ℓ.)*

The proof used random hypergraphs.

Let us call a graph U_n the *shadow* of a p-uniform hypergraph H_n if H_n and U_n have the same vertex-sets, and (x, y) is an edge of U_n iff there is a hyperedge in H_n containing both x and y. We will call the shadow S_n of H_n of [10, Theorem 13.3] the *Erdős–Hajnal Random Graph.*

As for the shadow, one can easily see that if the girth of H_n is at least 4, (which implies also that H_n is a 'linear hypergraph'), then H_n can easily and uniquely be reconstructed from U_n. The following claim is an immediate consequence of Theorem 13.3 of [10].

Claim A.3. *There exist a constant $c = c_{p,\ell} > 0$ and an integer $n_{p,\ell}$ such that for every $n > n_{p,\ell}$ there exist graphs S_n with p-girth greater than ℓ and independence number $\alpha_p(S_n) = O(n^{1-c})$.*

Indeed, the Erdős-Hajnal Random graph (S_n) proves Claim A.3. This implies the following claim.

Claim A.4. *If no $L \in \mathscr{L}$ is a p-forest, then there exist graph sequences (S_n) with $\alpha_p(S_n) = O(n^{1-c})$ (for some $c > 0$) and with $L \nsubseteq S_n$ $(L \in \mathscr{L})$.*

This is sharp:

Claim A.5. *If (S_n) is a graph sequence with the property that $\alpha_p(S_n) = o(n)$ and L is a p-forest, then $L \subseteq S_n$ for $n > n_0$.*

The case of many colours In the following, we will use the notation $R(\mathscr{L}_1, \dots, \mathscr{L}_r)$ in the obvious way. Clearly, if $\alpha_p(G_n) = o(n)$ and $n > n_0$, then $K_p \subseteq G_n$

Since $\alpha_p(S_n) = o(n)$ implies $K_p \subseteq S_n$, if $p \geq R(\mathscr{L}_1, \dots, \mathscr{L}_r)$, then any r-colouring of S_n has for some v an $L \in \mathscr{L}_v$ of colour v.

This trivial assertion is sharp for 2-connected excluded graphs.

Theorem A.6. *Assume that the excluded graphs in all the $\mathscr{L}_v$'s $(v = 1, \dots, r)$ are 2-connected and $p < R(\mathscr{L}_1, \dots, \mathscr{L}_r)$. Then there exist graph sequences (G_n) with $\alpha_p(G_n) = O(n^{1-c})$ (for some constant $c > 0$) such that the graphs G_n are r-colourable such that no monochromatic copies of any $L \in \mathscr{L}_v$ in the v^{th} colour occurs $(v = 1, \dots, r)$.*

Proof. Let

$$\ell > \max_{L \in \cup_v \mathscr{L}_v} v(L).$$

We can take the Erdős–Hajnal Random graph $G_n = S_n$ with p-girth larger than ℓ, and edge-colour each $K_p \subseteq S_n$ in r colours without monochromatic L's, since $p < R(\mathscr{L}_1, \dots, \mathscr{L}_r)$. If $L \subseteq S_n$ is 2-connected, $L \in \mathscr{L}_v$ is in a uniquely defined $K_p \subseteq S_n$ and therefore cannot be monochromatic, of colour v. □

Some similar results can be formulated for the case when the 2-connectedness of the graphs L_v is dropped. In fact, one can define a p-tree W_K of size $K(p, \ell)$ such that if all the excluded graphs are of order at most ℓ, then (*) can be satisfied iff W_K can be coloured in v colours without having $L \in \mathscr{L}_v$ in the v^{th} colour. The details are easy and omitted here.

Acknowledgement

We would like to thank the referee for many helpful suggestions.

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