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Obituaries

Paul Erdős, January 1941

Paul Erdős, 1913–1996

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Paul Erdős was one of the greatest and most outstanding mathematicians of the twentieth century, a very special genius. An incredible number of obituaries has been published about him since his death, which is quite unusual in the world of

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science. It was moving to see in Budapest a crowd of friends and colleagues, celebrities from all over the world come to pay tribute to him at the cemetery and at the hastily arranged conference "*In memoriam Paul Erdős*". Dozens of recollections and spontaneous remembrances, both through e-mail and in other ways have been written about him in the past few months.

He died in Warsaw, on September 20th, 1996.

The circumstances of his death were profoundly moving. Quite suddenly he was taken to hospital from his hotel room, following an early morning heart attack. Unfortunately, his friends and colleagues received no word for about 12 hours, at which time they were informed that he died a short while ago.

Wherever he travelled in the world, during almost every moment of the day, he was surrounded by colleagues and friends who cared about him and for him which meant more and more to him. Unfortunately, his last hours were spent in an unfamiliar hospital, alone.

He had planned to attend a conference in Lithuania on Analytic Number Theory, then to go to the USA where he was to receive an honorary degree from Texas A&M. He then planned to go to Australia to visit many places, but first of all to visit Eszter Klein and George Szekeres, to work with them on the problem of convex k-gons (the topic of their second joint paper back in 1935).

His "brain was open" until the very last day. This is what he wished so much all the time.

He was born as a third child in 1913. His birth was overshadowed by a tragic event — his two sisters died of scarlet fever. His parents were both teachers of Mathematics and Physics. His extraordinary talents were recognized early in life. In part these abilities can be explained by the intellectual stimulation coming from his parents, who taught him at home, partly because they wanted to protect him. He completed only 3 years of his elementary and secondary studies at school, all the rest was done at home. His father was away for 6 years, until 1920, as a prisoner of war in Russia — and in his father's absence, Paul was brought up by his mother. She used to tell stories about her son how he could multiply four-digit numbers at age four. "Once I calculated how far the sun is, because my mother told me how long it would take a train to get to the sun if you could travel by train.... I amused myself by asking people how old they are and told them how many seconds they have lived" — as Paul himself related. The mother — "Onyuka" as he always called her — was fond of telling stories about how he discovered negative numbers, how open his mind was to Nature, History and languages. His openness, interest and moral sense can be seen from these stories.

A few words about his early years: The problem-solving competition of the monthly High School Mathematical Journal was the first public way to show his mathematical talent — as it was for most Hungarian mathematicians in this century.

By the time the best competitors met at the university at age 17–18, they had already known each other's face and name, had had an idea about each other's mathematical taste, strengths, and weak points. That was the start of life-long friendships and cooperation, sometimes tragically cut short by the Second World War. This group of students, Paul Erdős, Tibor Gallai, Géza Grünwald, Eszter Klein, Pál Turán, Endre Vázsonyi and some others from the Pázmány Péter University, Budapest (now called Eötvös Loránd University), joined by George Szekeres, a student in chemical engineering at the Technical University, was shuttling between these two universities to attend classes. According to their reminiscences, their regular meetings, day-long excursions around Budapest or in the City Park, had a great impact on their future lives in all respects. It was a kind of open, peripatetic university, brought about by the socio-political discrimination of the era which afflicted most of them.

Here are the words of G. Szekeres: "We had a very close circle of young mathematicians, foremost among them Erdős, Turán and Gallai, friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the thirties, a vicious world war and our scattering to the four corners of the world. Our discussions centered around mathematics, personal gossip and politics."

Erdős was not yet 18 when he solved a problem of Dénes König in graph-theory, the generalization of Menger's theorem to infinite graphs. (This was included in König's book, the first on graph theory – which remained the only one for decades.) He made his Ph.D. under the supervision of Leopold Fejér. Its topic was number theory, the distribution of primes in arithmetical sequences.

In 1934 Erdős was invited to Manchester; this was arranged by Mordell. He stayed there for 4 years, but visited his family and friends in Budapest quite regularly. In 1938 he was among the first who concluded that it became necessary to leave Hungary. He left for the USA and returned home only 10 years later in 1948 to visit his beloved mother and friends. From 1955 on his visits to Hungary, some longer, others shorter, became more and more frequent. From 1962 on he had a position at the Mathematical Institute of the Hungarian Academy of Sciences. In 1956 he became corresponding member, and in 1962 member of the Hungarian Academy of Sciences.

After 1938 he had no permanent home. He could not stay for a long period in the same place: in the same apartment/city/country/not even on a continent.

He was member of 8 academies in 4 continents, and received honorary degrees from about 15 universities. Erdős was awarded the Cole prize in 1951 which is given for the best paper in algebra or number theory published in an American journal and he received the Wolf Prize in 1984. He lectured at hundreds of universities, hundreds of mathematicians became his co-authors, thousands of mathematicians were inspired by him.

Before he was 25 he had already written joint papers with G. Szekeres, P. Turán, T. Gallai, E. Vázsonyi, G. Grüwald from Hungary, and H. Davenport, C. Ko, M. Kac, K. Mahler, A. Wintner were also his co-authors at that time. Already his early papers in number theory, geometry, interpolation theory, polynomials, combinatorics, probability theory, group theory had tremendous influence in these fields, and for 60 years they inspired an unbelievable number of mathematicians. All these areas were present in his work all the time — though with varying intensity.

Paul Turán — in his paper written on the occasion of Erdős's 50th birthday [50] — called him the Western Ramanujan. On another occasion, when Erdős was 43, Turán compared in a way Erdős's performance in mathematics to that of Mozart in music, also mentioning that Erdős was barely 20 when he was called the magician of Budapest (der Zauberer von Budapest) by I. Schur. Many called Erdős the Euler of the twentieth century. Will anybody ever be called the Erdős of the twenty-first century?

The significance and impact of his results cannot be separated from his ars mathematica.

His friend and coauthor Ernst Straus (who was also a coauthor of Albert Einstein), wrote about him: "The prince of problem-solvers and the absolute monarch of problem posers . . . ". Erdős describes this as follows: "Problems have always been an essential part of my mathematical life. A well chosen problem can isolate an essential difficulty in a particular area, serving as a benchmark against which progress in this area can be measured. An innocent looking problem often gives no hint as to its true nature. It might be a "marsh-mallow", serving as a tasty tid-bit supplying a few moments of fleeting enjoyment. Or it might be like an "acorn" requiring deep and subtle insights from which a mighty oak can develop." His problems typically have a very simple formulation, and very often even the experts of the field can realize only years later that Erdős posed the first, non-particular question of a basic, general theory (which can be extremely difficult to answer).

Here are a few examples to illustrate this. He asked 50 years ago the following question ([14]; see also below under Combinatorial Geometry). At most how many unit distances can occur among n points in the plane? This question is related also to number theory. It is still unsolved (there are some partial results). In the last 50 years the distribution of distances (also in general metric spaces) became a difficult and extensively investigated topic. Rather surprisingly, these kinds of problems became relevant in computer science.

In connection with the celebrated theorem of Van der Waerden about arithmetic progressions, in the early 1940s Erdős and Turán asked: What is the maximum length of a sequence of integers in $[1, n]$ not containing k-term arithmetic progressions? They conjectured that this maximum is $o(n)$. K. F. Roth proved this in 1954 for $k = 3$, then in 1973 E. Szemerédi proved it for the general case. H. Fürstenberg and I. Katznelson and others proved many generalizations of this result using tools from ergodic theory. In the last decade a whole new area developed around this innocent looking question.

Papers, lectures, correspondence, personal conversations were all important means for Erdős for presenting problems. From the 1950s on he developed a specific genre in his papers and lectures. He wrote about 200 problem-papers that formulate dozens of problems connected to a topic, also providing the background and remarks in connection to partial results of the moment. The paper "Problems and results in additive number theory" written in 1955 can be considered as his first "problem-paper".

In the past decades the characteristic form of his lectures was always like this, whether in plenary lectures on international congresses or in educational lectures for teachers or students, interspersed with his characteristic stories, "erdősisms". It was a pleasure for him to engage in a mathematical conversation, whether his partner was a high school student or a famous mathematician.

He had a legendary memory. He could recall results of his own and of others with the place and date of their publications, conversations with hundreds of mathematicians several decades ago, like one recalls yesterday's events.

He was astonishingly well versed in literature, biology, history and politics.

His correspondence, which was part of his daily routine, of his life-style, should also be mentioned. Some may have kept several hundreds, others might have dozens of his letters, which he wrote in his special characteristic style, switching without transition from mathematics, old and new problems, to politics, friends and colleagues and back. The letter proved to be the most efficient means for his apostle-like, large-scale activities. He formulated thousands of problems and described related results, passed them on in his letters, between countries, through geographical and political borders, stimulating close friends or casual acquaintances to work with him. (Many results were achieved exclusively through correspondence.)

Another remarkable aspect of his letters is that one can reconstruct the events of world politics by reading his observations and predictions, worries and warnings in his war letters. After the war he passed on information about survivors and destruction between continents. In more peaceful years news, marriages, family events, trips and dinners, music and history all found their way into his letters.

The first two decades of his mathematics were dominated primarily by number theory, and also by analysis. Both Number Theory and Interpolation Theory remained present in his mathematics all the time. Now we tend to think that the majority of his work is combinatorial, discrete mathematics: graph theory, combinatorial number theory, combinatorial geometry, combinatorial set theory. As a matter of fact, his earlier work has definite combinatorial flavor too.

Combinatorics in close interaction with computer science have had an explosive development in the past 2 or 3 decades. Though Erdős himself had never been directly involved in computer science, the indirect influence of computer science, of course, can be found in his mathematics. Still, the other direction, the influence of Erdős's mathematics in computer science, is much stronger. Among others, his "random method" became one of the very important tools in Theoretical Computer Science, e.g. in giving lower bounds on algorithms.

In this paper I cannot cover all topics Erdős worked in, neither can I give a survey of any of these fields, and, of course, by no means will I try here to analyze his tremendous impact on mathematics. These areas all deserve one or more separate surveys, which will be written or have been written by various authors. It is my intention with the examples of results selected below, while putting some emphasis on Combinatorics, to illustrate also the wide scope and importance of Erdős's achievements in many other branches of mathematics.

We asked him to write a paper for the volume to be published in honour of his 80th birthday with the title "On my favourite theorems" instead of his usual "On my favourite problems", [16]. This also helps us to select some of his most important results in agreement with his own opinion.

Number Theory

Erdős could find elementary or simpler proofs for many classical theorems in prime number theory.

One of the most celebrated contributions of Erdős to prime number theory is the long awaited elementary proof of the prime number theorem. Erdős [11] and Selberg [47] gave such proofs in 1948. Both proofs start off from Selberg's "fundamental inequality"

$$\sum_{p<x} (\log p)^2 + \sum_{pq<x} \log p \log q = x \log x + O(x).$$

Behind Erdős's proof, as he observed immediately afterwards, is the following new type of general Tauberian theorem: Let $a_k > 0$, $s_m = \Sigma_{k=1}^{m} a_k$. Assume

$$\sum_{k=1}^{n} a_k (s_{n-k} + k) = n^2 + O(n).$$

Then

$$s_n = n + O(1).$$

Erdős's result on the difference of consecutive primes which is a classical, basic problem in number theory, related also to the Riemann hypothesis, was a breakthrough [10]. On one hand he gave an upper bound on $p_{n+1} - p_n$ (where p_n denotes the nth prime) showing the existence of a $c \in (0, 1)$ for which

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} \leq c.$$

On the other hand he proved the existence of a positive constant c s.t.

$$\limsup_{n \to \infty} \frac{(p_{n+1} - p_n)(\log \log \log p_n)^2}{\log p_n \log \log p_n} \geq c.$$

In spite of important improvements even in the last years, for further improvement Paul offered \$10,000 (which was the highest of his frequent monetary offers for solution of important mathematical problems). As one of his last offers, for a proof of

$$d_n = p_{n+1} - p_n > (\log n)^{1+\varepsilon} \qquad (*)$$

(for some $\varepsilon > 0$) for infinitely many n's, he offered \$10,000 and still offered \$5,000 for the previous weaker version. (Implicitly \$15,000 for (*).) We could list his results endlessly. Here is one more. Erdős and Turán proved in 1948 that the density of integers n for which $d_{n+1} > (1 + \varepsilon)\, d_n$ holds (for some $\varepsilon > 0$), is positive [34]. The same is true if we assume $d_n > (1 + \varepsilon)\, d_{n+1}$.

Combinatorial number theory was one of Paul's favourite topics. Let me quote the Acknowledgement of the book "Sequences" of Halberstam and Roth [39].

"Anyone who turns the pages of this book will immediately notice the predominance of results due to Paul Erdős. In so far as the substance of this book may be said to define a distinct branch of number theory — and its relevance to a wide range of topics in classical number theory appears to justify this claim — Erdős is certainly its founder." We mention just one example. Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers and let $f(n)$ denote the number of solutions $a_i + a_j = n$ (f is called the representation function of the sequence a_i). Classical problems in additive number theory, like Goldbach's problem, Waring's problem, the number of lattice points in a disc or Sidon's problem led to the investigation of the representation function in general.

Hardy and Landau proved that, if $F(n)$ is the number of lattice points in the circle $x^2+y^2<n$, then $F(n)-\pi n$ can not be "too small":

$$F(n)-\pi n=o(n^{1/4}(\log n)^{1/4})$$

is impossible. Erdős and Fuchs [18] proved the beautiful result, that almost the same holds for an arbitrary sequence of integers. Let $F(n)$ denote the number of solutions of $a_i+a_j<n$. Then

$$F(n)=cn+o\left(\frac{n^{1/4}}{(\log n)^{1/2}}\right)$$

cannot hold. This was the first result about the phenomenon that the representation function cannot be too nice, it cannot even be close to a nice function.

Erdős and Turán [32] proved the following theorem, which can be considered as a prototype for extremal problems in number theory. Let $f(n)$ denote the maximum number k of integers $1\le a_1<\cdots<a_k\le n$ for which a_i+a_j are all distinct (these are the so called Sidon sequences). Then

$$\frac{1}{2}n^{1/2}-o(n^{1/2})\le f(n)\le n^{1/2}+O(n^{1/4}).$$

(Chowla eliminated the factor 1/2 in the lower bound.) How dense an infinite Sidon sequence can be is still, after 50 years, an unsolved problem. The above theorem was one of the first theorems in Combinatorial Number Theory, which became in the last few decades a large area of Number Theory. We refer the reader to the survey paper of Pomerance-Sárközy [43].

Erdős and Turán proved the following basic and often used inequality in the Theory of Uniform Distribution, which was also generalized in many directions, [33]. Let z_v, $1\le v<\infty$, be a sequence of numbers in (0, 1). Then for every m and $0<\alpha<\beta<1$

$$\left|\sum_{\substack{v<n\\ \alpha<z_v<\beta}} 1-(\beta-\alpha)n\right|<c\frac{n}{m}+1+\sum_{j=1}^{m}\frac{1}{j}\sum_{k=1}^{n}\left|e^{2\pi ij\alpha k}\right|.$$

(For a survey on Erdős's work in Number Theory see [44] and [45].)

Probability

Here we should mention four different aspects: (a) Pure probability theory, (b) random like behaviour in different structures (e.g. in number theory), (c) random methods (application of probability theory in different fields) and (d) investigation of random structures.

Erdős writes [15]: "Heuristic probability arguments can often be used to make plausible but often hopeless conjectures on primes and other branches of number theory."

Erdős has many important results related to the law of the iterated logarithm. The first one (in 1942, [13]) sharpened the law of the iterated logarithm by giving an asymptotic series expansion of the "threshold" function.

Dvoretzky, Erdős and Kakutani [6] proved a surprising result about multiple points of random walks, which show different behaviours in dimension 2 and in higher dimensions.

The systematic application of random methods in number theory, graph theory, and in many other fields is one of the most important contributions of Erdős.

The deliberate and systematic application of probability theory to number theory started with the celebrated Erdős-Kac theorem. As a special case it gives the central limit theorem for the function $\omega(n)$, the number of distinct prime divisors of n.

As Elliott writes in his book on Probabilistic Number Theory [8] about the Erdős-Kac theorem: "This result, of immediate appeal, was the archetype of many results to follow. It firmly established the application of the theory of probability to the study of a fairly wide class of additive and multiplicative arithmetic functions."

Erdős describes the history of this as follows [15]: "In 1934 Turán gave a simple proof of

$$\sum_{n=1}^{x} (\omega(n) - \log\log n)^2 < cx \log\log x.$$

. . . He really used a classical theorem of Chebishev but we did not realize this connection . . . In the late 30's I often applied successfully the method of Turán and Brunn." Probabilistic Number Theory was born with the Erdős-Kac theorem.

Here is the Erdős-Kac theorem. Suppose that $f(n)$ is an additive function for which $|f(p)| < 1$, $f(p^\alpha) = f(p)$ and

$$\sum_{p<x} \frac{f(p)}{p} = A(x), \qquad \sum_{p<x} \frac{f^2(p)}{p} = B(x), \qquad B(x) \to \infty.$$

Then the density of integers n for which

$$f(n) < A(n) + c\sqrt{B(n)}$$

is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-x^2/2}\,dx.$$

Erdős and Turán have several papers on Statistical Group theory. As a first result they proved that for almost all permutations P the order $O(P)$ satisfies

$$\frac{1}{2}-\varepsilon<\frac{\log O(P)}{\log^2 n}<\frac{1}{2}+\varepsilon,$$

moreover it has a normal distribution. Their last paper on that subject is [35], for a survey see also [29].

Combinatorics

A remarkable feature of "Erdős type" combinatorial problems is that they are related to and the results can be applied in many other fields (coding theory, designs, information theory, theoretical computer science, geometry, number theory). In fact there is an interaction between those fields.

In Erdős's mathematics extremal problems were present all the time. The character of a general (combinatorial) extremal type problem is the following: In a certain subset of a structure we would like to know the existence of some specific substructure. The problem is to find conditions — e.g. in terms of some parameters — which guarantee that.

The most developed topic of that sort is Extremal Graph Theory. Extremal Set Theory has a wide, extended literature. There are lots of results (and problems) in combinatorial number theory which could be called Extremal Number Theory. A large part of Erdős's work belongs to these fields, moreover he was the founder or one of the founders of these fields.

We illustrate each of these. Let $f(n, G)$ denote the maximum number of edges a graph on n vertices can have, if it does not contain a graph isomorphic to a given graph G. Turán determined $f(n, G)$ if G is a complete graph. Erdős and Stone [28] proved that for a large complete t-partite graph the function $f(n, G)$ is asymptotically the same as for the complete graph on t vertices. A surprising consequence of this theorem is that basically the chromatic number determines $f(n, G)$: if G is r-chromatic then

$$f(n, G)=(1+o(1))\frac{n^2}{2}\left(1-\frac{1}{r}\right),$$

i.e. is asymptotically the same as for the complete graph on r vertices. This is formulated in Erdős and Simonovits [26]. As we mentioned, extremal graph theory is a widely developed subject. See [3], [48], and about the importance of Erdős's work in that field see the survey [49].

Extremal set theory has many applications (just as extremal graph theory has). One of the most quoted theorems of Erdős is the Erdős-Ko-Rado [21] theorem. Let $|S| = n$, and $A_i \subset S$, $1 \le i \le t$, be a family of subsets of S for which $A_i \cap A_j \neq \emptyset$, $|A_i| \le k < n/2$ and $A_i \nsubseteq A_j$. Then $t \le \binom{n-1}{k-1}$. Equality holds if and only if all the A_i have k elements and all contain the same element. This is one of the starting points of extremal set theory.

A family of sets $\{A_i\}$ is called a strong Δ-system if $A_i \cap A_j = B$ for $1 \le i < j \le t$. Let $g(n)$ be the smallest integer for which every family of $g(n)$ sets of size n contains a Δ-system of 3 elements. Erdős and Rado [23] proved that

$$2^n \le g(n) \le 2^n n!.$$

One of the conjectures most often mentioned by Paul is that $g(n) < c^n$ with some $c > 1$, see [23].

Ramsey theory is also an area related to many fields in mathematics. Ramsey's theorem was formulated first for graphs and for regular hypergraphs. Let $r(k, \ell)$ be the usual Ramsey number: the minimum integer t such that whenever we colour the edges of K_t in RED and BLUE, there will always be either a RED K_k or a BLUE K_ℓ. As a first result Erdős and Szekeres proved [30] that

$$r(k, \ell) \le \binom{k+\ell-2}{k-1}.$$

To determine the Ramsey numbers even for the most special case, namely for graphs, seems to be hopeless. There are many interesting results, good estimates. But even the existence of $\lim_{k\to\infty} r(k, k)^{1/k}$ is not known. This was one of the favorite problems of Erdős. For the important contribution of Erdős to that field, see [36].

Infinite Combinatorics

Partition Calculus is the topic in Combinatorial Set Theory which was initiated by Erdős and Rado. Hajnal writes in [38]:

"The partition calculus was developed by Erdős and Rado in the early 1950s. The long paper [22] contains all the results they had proved till then. Their first discovery was that the partition relation $\kappa \rightarrow (\lambda_\nu)^r_{\nu<\gamma}$ made sense for order types as well as cardinals, or even for a mixture of these. This led them to a great variety of new problems, some simple and some difficult, but requiring different methods."

This area (including about fifty papers of Erdős and Hajnal) became perhaps the widest and deepest part of Infinite Combinatorics. The scope of this short paper does not allow to explain even the simplest results. (See also the book [20] by Erdős, Hajnal, Máté and Rado, cf. [38].)

Random Graphs

Erdős and Rényi founded the systematic study of random graphs, random graph theory in the early 1960s. The significance of this field keeps growing. Random graph theory (now one of the most extensively investigated subjects in graph theory) started with the paper [24]. The general question, which they considered, was the following. What is the typical structure of graphs with n vertices and $M(n)$ edges? They call a function $F(n, P)$ the threshold function of property P, if for any $\varepsilon > 0$ a.a. graphs on n vertices and $(1+\varepsilon)F(n, P)$ edges have Property P and almost no graphs with $(1-\varepsilon)F(n, P)$ edges have property P.

One of the first and most surprising results they proved is that the threshold functions for connectedness and for not having isolated points are the same: $\frac{1}{2}n \log n$. It is even more surprising, that it is the same for having a Hamiltonian path (as they conjectured). See the monographs [4], [27], [1] and, about the "origin" of this theory, see [40].

Combinatorial Geometry

The first paper of Erdős where he considered the problem of distribution of distances determined by points in the plane is [14]. He considered the problems how many different distances must occur at least and how many times a given distance can occur at most. There are many results and problems on distance distribution, also in general metric spaces.

Erdős and Szekeres [30] proved that, if $\binom{2n-4}{n-2}$ points are given in a plane, no three on a line, then there are always n points among them which form a convex n-gon. The conjecture is, that $2^{n-2}+1$ is the correct value. (This is the problem on which, as I mentioned in the introduction, after 50 years Erdős and Szekeres intended to work in Australia last fall). Combinatorial geometry became a rich area in the last few decades. For references see [25], [7], [42].

Analysis

In addition to the above mentioned fields Erdős contributed to many others from his early youth till the end. An example of such an important area is interpolation theory. One of the first important results of Erdős and Turán [31] is the following: if the point group for an interpolation process consists of the roots of the Legendre polynomial then for every Riemann integrable function f

$$\int_{-1}^{+1} (f(x) - \mathscr{L}_n(f(x)))^2\,dx \to 0, \qquad \text{as } n \to \infty,$$

where $\mathscr{L}_n(f(x))$ denotes the interpolation polynomial of degree $n-1$ of f for the above mentioned point group. They also obtained very sharp results about the distribution of the roots of orthogonal polynomials and also about the distribution of the roots of polynomials defined by various interpolatory properties. For a survey on Erdős's work on interpolation and approximation theory see [9].

Besides interpolation and approximation theory, Erdős was also interested in various parts of analysis. Erdős and Gallai (called Grünwald at that time) proved [19] that, if the polynomial $f(x)$ has only real roots, $f(-1) = f(+1) = 0$, if f has no other roots in $(-1, +1)$, and $\max_{-1 \le x \le 1} f(x) = 1$, then

$$\int_{-1}^{+1} f(x)\,dx \le \frac{4}{3}.$$

Equality holds only for $1 - x^2$.

For other results of Erdős on polynomials we refer to the paper [46] of A. Schinzel.

Erdős was great in connecting distant fields. One example is the following theorem: Let $f_\alpha(z)$, $1 \le \alpha < \aleph_c$, be a family of entire functions and assume that for every z_0 there are only countably many different values of $f_\alpha(z_0)$. Then if $c = \aleph_1$ then the family $\{f_\alpha(z)\}$ can have power c but if $c > \aleph_1$ then the family must be countable, [12].

He has beautiful results about the points of divergence of power series, e.g. he proved with Herzog and Piranian that every set on the unit circle of logarithmic Hausdorff measure 0 is the set of divergence of a Taylor series.

Erdős has several papers on functional equations. We refer to L. Losonczi's paper in this issue.

There are many fields which we did not even touch, such as diophantine approximation, group theory, finite geometry — designs, topology, etc.

About 1500 papers were written by Erdős, the number of his co-authors is around 500. It will take a long time to summarize and to understand his work and the impact of his work on twentieth century's mathematics.

The life-style of Paul centered around three values — independence, search for truth, caring humanitarianism.

To secure his personal independence (free of convention, doctrines, property, all kinds of possessions, political power, family, job) was not possible without sacrifices, as he himself often remarked. He paid the price for it all his life. The search for truth in science, but also in politics and every-day life gave the motivation to his round-the-clock activity and productivity, ignoring pains, sickness, and aging. He always travelled around the world, wrote 20–30 papers and gave several dozens of lectures yearly, until the very end.

Notwithstanding the opinion of many people, he was not ascetic, even though he despised property. He liked going to a restaurant, he had a gourmet's taste (without a gourmand's appetite). However in India he refused to eat decent food, he would not go to a good restaurant. It was not for shortage of money — he thought that in a country where hundreds of millions go hungry he shouldn't eat like a gourmet. When he delivered a lecture about infant prodigies at the Tata Institute in Bombay, he made a remark to his audience, consisting mostly of the well-to-do, scholarly elite, that he did not understand how one could accept prosperity when poverty was conspicuous just by looking out of the window.

Fifty years of acquaintance, more than three decades of friendship and collaboration have made it hard for me to write these recollections. Paul Erdős had already been a living legend in his lifetime. Now that he is gone, his friends remember him differently. This cannot be helped. A lot of exaggeration and mistakes can be found in the stories about his eccentricity, which are cherished and quoted ever so often in articles about him. Some of these stories he did not like — some of them he himself liked to remember and retell, unintentionally contributing to the canonization of these anecdotes.

In spite of all these legendary stories, which were further embellished after his death, he will be remembered clearly and unalterably as a brilliant, unique mathematician, of pure character, and an emphatic, charitable man.

We hope this is the picture of Paul Erdős that will emerge in time and endure.

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