ELSEVIER

Discrete Mathematics 200 (1999) 61–77

DISCRETE MATHEMATICS

# Induced subgraphs of given sizes

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Received 8 June 1998; revised 16 October 1998; accepted 16 October 1998

**Abstract**

We say $(n,e)\to(m,f)$, an $(m,f)$ subgraph is forced, if every $n$-vertex graph of size $e$ has an $m$-vertex spanned subgraph with $f$ edges. For example, as Turán proved, $(n,e)\to(k,\binom{k}{2})$ for $e>t_{k-1}(n)$ and $(n,e)\not\to(k,\binom{k}{2})$, otherwise. We give a number of constructions showing that forced pairs are rare. Using tools of extremal graph theory we also show infinitely many positive cases. Several problems remain open. 

*AMS classification:* 05C35; 05D10

*Keywords:* Turán's theorem; Density; Ramsey's theorem

## 1. Introduction, the Turán problem

Let $G=(V,\mathscr{E})$ be an $n$-vertex graph with vertex set $V$, edge set $\mathscr{E}$. Let $[n]$ denote the set of first $n$ integers, $[n]:=\{1,2,\ldots,n\}$. The *complete graph* on $p+1$ vertices is denoted by $K_{p+1}$, the *complete bipartite graph* with parts $A$ and $B$ is $K(A,B)$, while for integers $a,b\geqslant 1$, $K(a,b)$ denotes a $K(A,B)$ with $|A|=a$, $|B|=b$.

A graph $L$ is contained in $G$ if it is (a not necessarily induced) subgraph of it. Otherwise $G$ is called *L-free*. Let $\mathrm{ex}(n,L)$ denote the maximum number of edges of an $L$-free graph on $n$ vertices. This is frequently called the *Turán number* of $L$. Turán [21,22] proved that (for $n\geqslant p$) the only maximal $K_{p+1}$-free graph is the complete $p$-partite graph (also called the complete $p$-chromatic graph), i.e., a graph $G$ with its vertex set $[n]$ divided into $p$ almost equal parts, $[n]=V_1\cup\cdots\cup V_p$, where $|V_i|=\lfloor n/p\rfloor$

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0012-365X/99/$ – see front matter 
PII: S0012-365X(98)00387-2

or $|V_i| = \lceil n/p \rceil$, and its edge set $\mathscr{E}(G) := \{xy : x$ and $y$ belong to distinct $V_i$'s$\}$. The above graph is also called the $p$-partite *Turán graph* and its size is denoted by $t_p(n)$.

$$\mathrm{ex}(n, K_{p+1}) = t_p(n) = \frac{1}{2}\left(1 - \frac{1}{p}\right)n^2 - \mathrm{O}(1). \tag{1.1}$$

Kővári et al. [17] showed that

$$\mathrm{ex}(n, K(k,k)) \leqslant \tfrac{1}{2}(k-1)^{1/k} n^{2-1/k} + (k-1)n. \tag{1.2}$$

This inequality together with a random construction of Erdős implies the following. For every bipartite $F$ that is not a forest, there is a positive constant $c(F)$ such that $\Omega(n^{1+c}) \leqslant \mathrm{ex}(n, F) \leqslant \mathrm{O}(n^{2-c})$. Here we are going to use the following special case: there are relatively large graphs of girth at least $g$,

$$\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{g-1}\}) > \tfrac{1}{2} n^{1+(1/(g-2))}. \tag{1.3}$$

For graphs with chromatic number at least three we have the Erdős–Stone–Simonovits theorem [10,12] that says that for $\min_{L \in \mathscr{L}} \chi(L) = p + 1 \geqslant 3$ one has $\mathrm{ex}(n, \mathscr{L}) = (1 - 1/p)\binom{n}{2} + \mathrm{o}(n^2)$. Here we are going to use the following sharpening of the Erdős–Stone theorem due to Chvátal and Szemerédi [7]. Suppose that $\varepsilon$, $p \geqslant 2$ are fixed and $n$ is large enough. Let $t = \log n / (500 \log(1/\varepsilon))$, then

$$\mathrm{ex}(n, K_{p+1}(t)) \leqslant \left(1 - \frac{1}{p} + \varepsilon\right)\binom{n}{2}, \tag{1.4}$$

where $K_{p+1}(t)$ stands for the complete $(p+1)$-partite graph whose every vertex class has cardinality $t$. We remark that a randomized example of Bollobás [2] shows that (1.4) is best possible up to the constant 500. For latest developments see Bollobás and Kohayakawa [3].

## 2. Introduction, density questions

As the exact solutions of Turán-type problems, especially the hypergraph versions, seems to be so difficult, Erdős proposed a series of simpler looking, and important, questions. One of the natural generalizations of Turán's theorem is as follows, where the structure of forbidden subgraphs is reduced to one parameter, their size. What is the maximum number of edges of an $n$-vertex graph, $\mathrm{ex}(n; m, < f)$, if every $m$-element set spans less than $f$ edges? This was investigated by Erdős [8] in 1963. Some of his results were rediscovered and clarified by Gol'berg and Gurvich [14], for the latest developments see Griggs et al. [15]. The hypergraph version, i.e., the problem of

$$\mathrm{ex}_r(n; m, f) := \max_{\substack{\mathscr{F} \text{ is an } r \text{ uniform hypergraph on } n \text{ vertices,} \\ \text{every } m\text{-set spans less than } f \text{ edges}}} |\mathscr{F}|$$

was proposed by Brown et al. [4,5]. This problem, which contains Turán's hypergraph conjecture is even more difficult. For example, $\mathrm{ex}_3(n; 6, 3) = \mathrm{o}(n^2)$, by a celebrated result by Ruzsa and Szemerédi [20]. A concise proof and further problems can be found in Erdős et al. [9].

The above density problems seem to be related to such difficult number theory problems as to estimating $r_3(n)$. (Here $r_3(n)$ is the maximum size of a subset of $[n]$ containing no arithmetic progression of size 3.) In this note we deal with an easier topic. The aim of this paper is to illustrate the powerful methods of extremal graph theory by (partially) answering the following question of Erdős: what local edge densities are unavoidable in every $n$-vertex graph with $e$ edges?

## 3. The *m*-spectrum of graphs

For a given graph $G$ we say that the $(m,f)$ pair belongs to its *spectrum*, $(m,f)\in \mathrm{Sp}(G)$, if one can find an $m$-element subset of vertices $M\subset V(G)$, $|M|=m$ such that the induced subgraph $G\,|\,M$ has exactly $f$ edges. We also use the notation $G\to(m,f)$, and say that an $(m,f)$ subgraph is *forced*, or $f$ belongs to its $m$-spectrum. Otherwise, we say $G\not\to(m,f)$, or $G$ *avoids* $(m,f)$. Let $\mathscr{G}(n,e;m,f)$ denote the class of $n$-vertex graphs of $e$ edges avoiding $(m,f)$. Let $\mathscr{G}(n;m,f):=\bigcup_{0\leqslant e\leqslant\binom{n}{2}}\mathscr{G}(n,e;m,f)$. Our aim is to describe these graphs, or at least to prove a few basic properties of them.

As a first step, we would like to determine the $(n,e)$ pairs with $\mathscr{G}(n,e;m,f)=\emptyset$. We denote this by $(n,e)\to(m,f)$, every graph of $n$ vertices and $e$ edges contains an induced $(m,f)$-subgraph. Let $S(n;m,f)$ denote the set $\{e:(n,e)\to(m,f)\}$ and define

$$\sigma(m,f):=\limsup_{n\to\infty}\frac{|S(n;m,f)|}{\binom{n}{2}}.$$

As $S(n;m,f)\subset\{0,1,2,\ldots,\binom{n}{2}\}$ but 0 and $\binom{n}{2}$ cannot belong to it simultaneously, the fraction on the right-hand side is at most 1. We *conjecture* that the lim sup above is actually a limit for all fixed $m$ and $f$.

We also use the notation $\mathrm{Sp}_m(G)$ for the set of values $f$ for which $G\to(m,f)$. We also can write $\mathrm{Sp}_m(\mathscr{G})$ where $\mathscr{G}$ is a set of graphs, then, as usual, $\mathrm{Sp}_m(\mathscr{G}):=\bigcup_{G\in\mathscr{G}}\mathrm{Sp}_m(G)$. Some properties of $\mathrm{Sp}_m(G)$ were investigated in [11,13].

This paper is organized as follows. First, we consider a few special cases in Sections 4–7. In Section 8 we give a list of examples for $(m,f)$-free graphs. The constructions yield upper bounds for $\sigma(m,f)$ in various ranges of $m$ and $f$, we have also some overlapping which could be easily analyzed. Such a combination of constructions yields the main result of this paper (Theorem 1 in Section 9) where we prove that $\sigma(m,f)\leqslant\frac{2}{3}$ for all but 5 pairs $(m,f)$. For these five pairs we show $\sigma(m,f)=1$, we call them *unavoidable*. Probably, $\sigma(m,f)>\frac{1}{2}$ holds for only finitely many pairs, too. In Section 10 we summarize the negative examples showing that for fixed $m$ all but at most 300 pairs $\sigma(m,f)=0$ in the interval $0\leqslant f\leqslant O(m^{3/2})$. Finally, in Section 11, infinitely many pairs are given with $\sigma(m,f)=\frac{1}{12}$.

## 4. The case $f = 0$

The case $m = 2$ is trivial, $\mathscr{G}(n;2,0)$ consists of a single graph, $K_n$, thus $S(n;2,0) = \{0,1,\dots,\binom{n}{2}-1\}$, implying $\sigma(2,0) = 1$. Similarly, $\sigma(2,1) = 1$. From now on we suppose that $m \geqslant 3$.

Looking at the complements, it is obvious that $G \to (m,f)$ if and only if $\overline{G} \to \left(m,\binom{m}{2}-f\right)$. It follows that $\mathscr{G}(n,e;m,f) = \left\{\overline{G} : G \in \mathscr{G}\left(n,\binom{n}{2}-e;m,\binom{m}{2}-f\right)\right\}$, and $S(n;m,f) = \left\{\binom{n}{2}-e : e \in S\left(n;m,\binom{m}{2}-f\right)\right\}$, and

$$\sigma(m,f) = \sigma\left(m,\binom{m}{2}-f\right). \tag{4.1}$$

**Construction 1** ($p$-chromatic graphs). *Let $\mathscr{F}_1(p)$ be the class of p-chromatic graphs.*

We have that $\mathrm{Sp}_m(\mathscr{F}_1(p)) = \{0,1,\dots,t_p(m)\}$. As $K_{p+1}$ is never a subgraph, Turán's theorem (1.1) implies that $(n,e) \to (m,\binom{m}{2})$ if and only if $e > t_{m-1}(n)$. Hence,

$$\sigma\left(m,\binom{m}{2}\right) = \sigma(m,0) = \frac{1}{m-1}. \tag{4.2}$$

Using the notation introduced in (1.1) we have

$$f > t_p(m) \text{ implies } S(n;m,f) \subset \left\{t_p(n)+1,\dots,\binom{n}{2}\right\}. \tag{4.3a}$$

The length of the interval on the right-hand side is $(1+o(1))(1/p)\binom{n}{2}$, so we obtain $\sigma(m,f) \leqslant 1/p$. Considering complements for $\binom{m}{2} - t_{p+1}(m) \leqslant f < \binom{m}{2} - t_p(m)$ we have

$$S(n;m,f) \subset \left\{0,1,2,\dots,\binom{n}{2} - t_p(n) - 1\right\}, \tag{4.3b}$$

implying $\sigma(m,f) \leqslant 1/p$ again.

In the case $p = 2$, when we consider only bipartite graphs avoiding $(m,f)$, we have $\binom{m}{2} - \lfloor m^2/4 \rfloor = \lfloor (m-1)^2/4 \rfloor$. Thus (4.3a) and (4.3b) yield that

$$\sigma(m,f) > \tfrac{1}{2} \text{ is only possible if } \lfloor (m-1)^2/4 \rfloor \leqslant f \leqslant \lfloor m^2/4 \rfloor. \tag{4.4}$$

## 5. Union of cliques, the case $(m,f) = (3,2)$

The only $(3,2)$-graph is the induced path of two edges. Hence if $G \not\to (3,2)$, then any two vertices that are connected by a path must be connected by an edge, i.e., $G$ is a disjoint union of complete graphs. Let us denote the sizes of the cliques by $n_1, n_2, \dots, n_k$, then $n = \sum_{1 \leqslant i \leqslant k} n_i$ and $e = \sum_{1 \leqslant i \leqslant k} \binom{n_i}{2}$. We need to know when can $e$ be written as a sum of this form. This was a question of Erdős and (independently)

Winkler and was answered by Reznick [19] in the following way. Let

$$C(n) := \left\{ \sum \binom{n_i}{2} : \sum n_i = n, \text{ the } n_i\text{'s are non-negative integers} \right\}$$

and let $a(n)$ denote the largest integer so that for the interval $\{0,1,2,\ldots,a(n)-1,a(n)\} \subset C(n)$, then $a(n) = \binom{n}{2} - \sqrt{2}n^{3/2} + O(n^{5/4})$. We apply this as follows.

**Construction 2** (Union of cliques). *Let $\mathscr{F}_2$ be the class of graphs where each connected component is a clique.*

We have that $\mathrm{Sp}_n(\mathscr{F}_2) = C(n)$,

$$(n,e) \to (3,2) \text{ if and only if } e \notin C(n).$$

This implies, e.g., that $S(n;3,2) \subset [\binom{n}{2} - O(n^{3/2}), \binom{n}{2}]$, hence

$$\sigma(3,2) = \sigma(3,1) = 0. \tag{5.1}$$

As $\max\{C(m)\backslash\binom{m}{2}\} = \binom{m-1}{2}$ we get $\{i : \binom{m-1}{2} < i < \binom{m}{2}\} \cap C(m) = \emptyset$. Therefore, as a generalization of (5.1), we have $\sigma(m,i) = 0$ for these values. Considering the complements, (4.1) implies for $m \geqslant 3$

$$\sigma(m,1) = \sigma(m,2) = \cdots = \sigma(m,m-2) = 0. \tag{5.2}$$

With a little more consideration one can see that Construction 2 implies that $\sigma(m,i) = 0$ for all $3 \leqslant m \leqslant 7$, $0 < i < \binom{m}{2}$ with the possible exceptions $(m,i) \in \{(4,3), (5,4), (7,6), (7,10)\}$ (and their complements $\{(5,6), (7,15), (7,11)\}$, the pair $(4,3)$ is self-complementary). We continue to investigate these cases in the next sections.

## 6. An unavoidable pair, the case $(m,f) = (4,3)$

**Construction 3** (Union of trees and cycles). *Let $\mathscr{F}_3(p)$ be the class of graphs where each connected component is either a tree of at most $p-1$ vertices, or a cycle $C_p$.*

If we have $G \in \mathscr{F}_3(p)$, $|V(G)| = n$, then $G$ has at most $n$ edges. Even more, if $n/p$ is not an integer, it has at most $n-1$ edges. If $n/p$ is an integer, then it cannot have $n-1$ edges either. Moreover, if $(p,i) \in \mathrm{Sp}(\mathscr{F}_3(p))$, then $i < p-1$ or $i = p$. We claim that for the case $(m,f) = (4,3)$ essentially there are no more examples avoiding it.

**Claim 6.1.** *Suppose that $G \in \mathscr{G}(n,e;4,3)$, i.e., $G \not\to (4,3)$, and $n \geqslant 5$. Then either $G$ or $\overline{G} \in \mathscr{F}_3(4)$.*

This implies

$$(n,e) \to (4,3) \text{ if and only if } n < e < \binom{n}{2} - n, \text{ or } e = n - \delta, \text{ or } e = \binom{n}{2} - n + \delta,$$

where $\delta = 1$ when 4 divides $n$, and $\delta = 0$ otherwise. We obtain that $S(n;4,3)$ contains a very long interval

$$\sigma(4,3) = 1. \tag{6.1}$$

**Proof of Claim 6.1.** Assume $G$ is not in $\mathscr{F}_3(4)$. Then one can find a vertex $x \in V(G)$ with $\deg_G(x) \geqslant 3$. The neighborhood of $x$ cannot induce an independent set, otherwise $K(1,3)$ is an induced subgraph with center $x$. Hence $G$ contains a triangle $X = \{x,y,z\}$. Applying the above argument for $\overline{G}$ we obtain that $G$ contains three independent vertices $Y = \{a,b,c\}$, too. Consider, first, the case when these two sets are disjoint, $\{x,y,z\} \cap \{a,b,c\} = \emptyset$. If one can find two vertices from $Y$, say $a$ and $b$, with incomparable neighborhoods in $X$ (this means that there exists a vertex in $X$ connected to $a$ but not connected to $b$, and there exists another vertex in $X$ from $N(b) \backslash N(a)$), then one can find an induced path $P_4$, a contradiction. We obtain that the neighborhoods of $Y$ in $X$ contain each other. These neighborhoods are non-empty, therefore there is an $x \in X$ joined to all of them, inducing again a $K(1,3)$. The case $X \cap Y \neq \emptyset$ is even simpler. □

## 7. Another unavoidable pair, the case $(m,f) = (5,4)$

**Construction 4** (Graphs avoiding $(5,4)$). *Let $\mathscr{F}_4$ be the class of graphs consisting of $n-k$ isolated vertices ($4 \leqslant k \leqslant n$) and a copy of either $K_k$ or $K_k$ with one edge deleted, or $K_k$ with 3 edges of a triangle deleted.*

Beside the above construction we also have that $(5,4) \notin \mathrm{Sp}(\mathscr{F}_3(5))$, hence the set $\{0,1,2,\ldots,n-2,n-\varepsilon\}$ is missing from $S(n;5,4)$. Here $\varepsilon = 1$ except if 5 divides $n$, then $\varepsilon = 0$. One can exclude another interval of length $\mathrm{O}(n^{3/2})$ from the other end by the next construction.

**Construction 5** (Very dense graphs). *Let $\mathscr{F}_5(p)$ be the class of graphs whose complement has girth at least $p$.*

If $G \in \mathscr{F}_5(5)$, then every 5-subset induces at least five edges, so $G \not\to (5,4)$. It is known [18] that there are graphs of girth 5 on $n$ vertices of size $(1/\sqrt{8} - \mathrm{o}(1))n^{3/2}$, therefore $(n,e) \to (5,4)$ is only possible if $e$ is not too close to $\binom{n}{2}$.

**Claim 7.1.** *There exists a constant $c$ $(1/(\sqrt{8}) \leqslant c < 12)$ such that the following holds. If $n - \varepsilon < e < \binom{n}{2} - cn^{3/2}$, then the cases described in Construction 4 are the only graphs with $G \not\to (5,4)$. (Here $\varepsilon = 0$ or 1 according to 5 divides $n$ or not.)*

Hence for $n - \varepsilon < e < \binom{n}{2} - cn^{3/2}$, we have

$$(n,e) \to (5,4) \text{ unless } e = \binom{k}{2}, \binom{k}{2} - 1 \text{ or } \binom{k}{2} - 3 \quad \text{for some integer } k.$$

Thus $S(n;5,4)$ contains almost all integers from $\{0,1,2,\dots,\binom{n}{2}\}$, implying

$$\sigma(5,4)=\sigma(5,6)=1. \tag{7.1}$$

**Proof of Claim 7.1.** Let $F$ be a *connected* graph on $n$ vertices with $F \not\to (5,4)$. First, we show that for $|V(F)|>6$ we have

$$\text{girth}(F)=3, \tag{7.2}$$

i.e., $F$ contains a triangle. Indeed, if $F$ has a vertex of degree at least 4, say $xx_i \in \mathscr{E}(G)$, $(1 \leqslant i \leqslant 4)$, then $\{x,x_1,x_2,x_3,x_4\}$ must contain a further edge, otherwise it spans exactly four edges. If the diameter of $G$ exceeds 3, and $y_0y_1y_2y_3y_4$ is a spanned path, we obtain a contradiction. Then diam $(G) \leqslant 3$ and max $\deg(G) \leqslant 3$ imply $|V(F)| \leqslant 18$ by [16]. The cases $6<n\leqslant 18$ can be eliminated by considering the shortest cycle. Let us note that as $K(3,3)$ shows, the condition $|V(F)|>6$ is necessary in (7.2).

Second, we prove that if in addition $F \not\to (4,4)$, then

$$F \in \mathscr{F}_4, \tag{7.3}$$

i.e., as $F$ is connected, it is almost a complete graph. Indeed, let $x \in V(F)$ and consider $F|X$ the graph induced by $X=\{x\} \cup N_F(x)$, the closed neighborhood of $x$. Then $F|X \not\to (3,1)$, so by the results of Section 4, we have that $\overline{F\,|\,X} \in \mathscr{F}_2$. But $\overline{F\,|\,X}$ cannot contain 2 disjoint cliques (as $F \not\to (4,4)$), neither a clique of size larger than 3. Then we have that $F\,|\,X \in \mathscr{F}_4$, it is almost a complete graph, it has all the edges except for the pairs in a set $Y \subset X$ of size at most 3. If $X=V(F)$ we are done. Suppose that $x$ has maximum degree in $F$. Then all vertices of $X \setminus Y$ have maximum degrees. Suppose that $y \in Y$ is connected to a further vertex $y' \notin X$. Then $y$ is the only neighbor of $y'$ from $X$, therefore it is easy to find a $(5,4)$ or a $(4,4)$ set from $X \cup \{y'\}$ unless $|X|=3$. For $|X|=3$ $F$ is a path or cycle, and we are done.

Now we are ready to prove by induction on $n$ the following statement: If $F \not\to (5,4)$ and $F$ is connected on $n$ vertices, then

$$|\mathscr{E}(F)| > \binom{n}{2} - cn^{3/2}. \tag{7.4}$$

This is obviously true for $n<4c^2$. Consider a vertex $x$ of $F$ of minimum degree $d$, and let $H$ be the graph induced by the vertices not adjacent to $x$, $V(H)=V(F)\setminus(\{x\} \cup N_F(x))$. If $n-d \leqslant c\sqrt{n}$, then (7.4) obviously holds, so from now on we may suppose that $|V(H)|>15$. If there are more than two connected components of $H$ with edges, then all of its components have size at most 6, otherwise there is a triangle in $H$ (by (7.2)), and a disjoint edge in another component, a $(5,4)$-graph. So in this case the number of components is at least $(n-d-1)/6$, one can find three vertices $x_1,x_2,x_3 \in V(H)$ such that they span no edge. By the minimality of $d$, $\deg_F(x_i) \geqslant d$, hence $x_i$ has at least $(d-5)$ neighbors in $N(x)$. For $d>15$ they have a joint neighbor $y \in N(x)$, so $\{y,x,x_1,x_2,x_3\}$ spans a star, a $(5,4)$-graph. For $d \leqslant 15$, we use a similar argument. Choose an edge from every component of $H$ connected to $N(x)$. Considering these (at least $(n-d-1)/6)$ edges one can find a spanned star. We conclude that $H$ has a single non-empty component, and might have some isolated vertices. In the same way we obtain that $H$ cannot have more than 1 isolated vertex, (for such a vertex $y$ one has $N_F(y)=N_F(x)$), so the only non-empty component in $H$ (by (7.3)) is an almost complete graph of size $n-d-2$ or $n-d-1$. To finish the proof of (7.4) count the number of edges of $F$ by using the fact that for any $xy$ edge the vertex $y$ must be adjacent to all triangles in $H$, thus $y$ is connected to at least $n-d-5$ vertices of $H$. Thus

$$\begin{aligned}|E(F)| &= |E(F|X)| + |E(N(x) \text{ to } H)| + |E(H)| \\ &\geqslant \binom{d+1}{2} - c(d+1)^{3/2} + d(n-d-5) + \binom{n-d-2}{2} - 3 \\ &= \binom{n}{2} - c(d+1)^{3/2} - (2n+2d).\end{aligned}$$

We have

$$\begin{aligned}&cn^{3/2} - c(d+1)^{3/2} - (2n+2d) \\ &\quad = c(n+\sqrt{n(d+1)}+d+1)(\sqrt{n}-\sqrt{d+1}) - 2(n+d)\end{aligned}$$

which is positive for $c(\sqrt{n}-\sqrt{d+1})\geqslant 2$, for example for $d<n-(2/c)\sqrt{n}$. This completes the induction for (7.4).

Finally, to finish the proof of Claim 7.1 let us consider an arbitrary graph $G\in\mathscr{G}(n,e;5,4)$, i.e., $G\not\to(5,4)$ and $e$ is in the range given in the Claim. If $G$ is connected, then by (7.4), we are done. If $G$ is not connected, then for every component $C$ we have $C\not\to(4,4)$ so (7.2) and (7.3) can be applied. So as above, we obtain that if $|V(C)|>6$ than it is the only non-empty component and (7.3) implies that $G\in\mathscr{F}_4$, as claimed. In the remaining case all components have at most six vertices. As $K(3,3)$ is excluded, we can obtain that all of them have at most five vertices. As $|\mathscr{E}(G)|>n-1$, there must be more than one non-nempty component, hence $G$ is triangle-free, we obtain that $G\in\mathscr{F}_3(5)$, a final contradiction. □

## 8. More constructions

In this section we give five more constructions providing a huge number of graphs avoiding certain $(m,f)$ pairs.

**Construction 6** (Clique minus trees). *Let $\mathscr{F}_6(p)$ be defined as follows. Take a clique of size k, subtract any number of independent (i.e., vertex disjoint) trees where each tree has at most p vertices, and add any number of isolated vertices.*

Let us consider an $(m,f)$ pair, and write it in the form $f=\binom{b}{2}-b'$ where $0\leqslant b'<b-1$. Then for $((p-1)/p)b<b'<b-1$ the above construction does not contain an $(m,f)$-subgraph. Notice that $m$ is unimportant. We have that $|\mathrm{Sp}_n(\mathscr{F}_6(p))|=\sum_{3\leqslant k\leqslant n}(1+\lfloor((p-1)/p)k\rfloor)=(p-1)/p\binom{n}{2}+\mathrm{O}(n)$. Therefore,

$$\sigma(m,f)\leqslant\frac{1}{p}\quad\text{for } f=\binom{b}{2}-b',\quad \frac{p-1}{p}b<b'<b-1. \tag{8.1}$$

For example, $\sigma(7,11)$ (and therefore its complement, $\sigma(7,10)$) is at most $\frac{1}{2}$ (here $b=6$, $b'=4$), and $\sigma(8,16)$ (and hence $\sigma(8,12)$, too) is at most $\frac{1}{3}$ (here $b=7$, $b'=5$).

**Construction 7** (Clique plus trees). *Let $\mathscr{F}_7(p)$ be defined as follows. Take a clique of size $k$ and add any number of independent trees where each tree has at most $p$ vertices.*

Let us consider an $(m,f)$ pair, and write it in the form $f=\binom{l}{2}+l'$ where $0\leqslant l'<l$. For $l+(p/(p-1))l'>m$ this construction does not contain an $(m,f)$-subgraph. We have that $\mathrm{Sp}_n(\mathscr{F}_7(p))$ contains an initial interval of size of $((p-1)/(2p-1))^2\binom{n}{2}+\mathrm{O}(n)$. Above that bound $\mathrm{Sp}_n$ consists of intervals $[\binom{k}{2},\binom{k}{2}+(n-k)((p-1)/p)+\mathrm{O}(1)]$, hence $|\mathrm{Sp}_n(\mathscr{F}_7(p))|=(p-1)/(2p-1)\binom{n}{2}+\mathrm{O}(n)$.

$$\sigma(m,f)\leqslant\frac{p}{2p-1}\quad\text{for } f=\binom{l}{2}+l'<\binom{l+1}{2},\quad l+\frac{p}{p-1}l'>m. \tag{8.2}$$

**Construction 8** (Clique and large girth). *Let $\mathscr{F}_8(p)$ be defined as follows. Take a clique of size $k$, and add a graph of girth at least $p$ on $n-k$ vertices.*

It is known (see (1.3), for a proof see [2]) that there are graphs of girth $p$ on $v$ vertices and more than $v^{1+(1/(p-2))}$ edges. This implies that for $(n-k)^{(p-1)/(p-2)}>k$ the interval $[\binom{k}{2},\binom{k+1}{2}]\subset\mathrm{Sp}_n(\mathscr{F}_8(p))$, i.e., $|\mathrm{Sp}_n|>\binom{n}{2}-\mathrm{O}(n^{2-(1/(p-1))})$. On the other hand, every $m$-subset of a $G\in\mathscr{F}_8(m+1)$ spans a graph consisting of a clique and a forest. If there is no clique plus forest with exactly $m$ vertices and $f$ edges, then $S(n;m,f)\cap\mathrm{Sp}_n(\mathscr{F}_8(m+1))=\emptyset$. Then $\sigma(m,f)=0$. Thus if $f$ is in the form

$$f=\binom{l}{2}+l'\quad\text{with } 0\leqslant l'<l, \tag{8.3}$$

then

$$l'\geqslant m-l \text{ implies } \sigma(m,f)=0. \tag{8.4}$$

This implies, for example that $\sigma(8,13)$ (and therefore $\sigma(8,15)$) is 0.

Actually, Construction 8 shows that most of the $\sigma$'s are 0. Let $B(m)$ be the set $\{f\colon 0<f<\binom{m}{2}$ such that $l+l'\geqslant m$ when $f$ is written in the form of (8.3)$\}$. Then

$B(m)$ is the union of the disjoint intervals

$$\left[\binom{l}{2}+(m-l),\binom{l+1}{2}-1\right] \quad \text{for } l\geqslant(m+1)/2.$$

Therefore $|B(m)|=\sum(m-2l)=(\frac{1}{4})m^2+O(m)$, thus 50% of the $\sigma(m,f)$ are 0's. One can obtain more 0's by considering the union of $B(m)$ and its complement $B'(m):=\{\binom{m}{2}-f\colon f\in B(m)\}$. This gives more than 82%.

**Remark.** Consider $B(m)$ and the set $S(m;3,2)$ (our previous examples for $\sigma=0$). We obtain that they are incomparable. Namely, one can prove that for $(m,f)\notin B(m)$, i.e., $f$ is in the form (8.3) with $l'<m-l$, then $f$ can be written as sums of the binomial coefficients ($f\in C(m)$) except if $m-l=3$ or 6 and $l'=2$ or 5, respectively. In other words, for $m>m_0$

$$B(m)\backslash S(m;3,2)=\left\{\binom{m-6}{2}-5,\binom{m-3}{2}+2\right\}. \tag{8.5}$$

The main observation for the proof of (8.5) is that $(x,i)\in \mathrm{Sp}(\mathscr{F}_2)$ for all $0\leqslant i\leqslant x-1$ except for the pairs $(3,2)$ and $(6,5)$.

**Construction 9** (Complete bipartite graphs and large girth). *Let $\mathscr{F}_9(p)$ be defined as follows. Take a complete bipartite graph of size k, and add a graph of girth at least $p$ on $n-k$ vertices.*

Again using (1.3) about the maximum number of edges of a graph of given girth, one can see that $\mathrm{Sp}_n(\mathscr{F}_9(p))$ (for fixed $p$ as $n\to\infty$) almost covers the interval $[0,\lfloor n^2/4\rfloor]$. On the other hand, every $m$-subset of a $G\in\mathscr{F}_9(m+1)$ spans a graph consisting of a complete bipartite graph and a forest. Therefore, if for $m$ vertices there is no such graph with exactly $f$ edges, then $S(n;m,f)\cap \mathrm{Sp}_n(\mathscr{F}_8(m+1))=\emptyset$. In other words, let $D(m):=\{ab+c\}$ where $a,b,c$ are nonnegative integers, $a+b\leqslant n$ and for $c\geqslant 1$ we have $a+b+c\leqslant n-1$.

$$\text{If } f\notin D(m) \text{ then } |S(n;m,f)\cap[0,n^2/4]|=o(n^2). \tag{8.6}$$

This obviously implies $\sigma(m,f)\leqslant\frac{1}{2}$.

One can strengthen (8.6) combining it with the complement of Construction $\mathscr{F}_2$ (complement of bipartite graphs). Inequality (8.6) and (4.3b) imply that

$$\text{if } f\notin D(m) \text{ and } f<\lfloor(m-1)^2/4\rfloor \text{ then } \sigma(m,f)=0. \tag{8.7}$$

This implies, for example, that $\sigma(12,29)$, $\sigma(15,39)$, $\sigma(15,47)$ (and therefore their complements, too) are 0. These cases of $\sigma=0$, and probably infinitely many more, are not implied by our previous examples. We will show a 4th kind of $\sigma(m,f)=0$ in (8.10). For a fixed $m$ all of them give only $O(m^{3/2})$ 0's, except Construction 8.

**Construction 10** (Partition into cliques). *Let $\mathscr{F}_{10}(p)$ be denote the class of graphs consisting of the vertex disjoint union of at most p cliques.*

Let

$$C(n,p):=\left\{\sum_{1\leqslant i\leqslant p}\binom{n_i}{2}:\ \sum n_i=n,\ \text{the } n_i\text{'s are non-negative integers}\right\}. \quad (8.8)$$

The Cauchy–Schwarz inequality implies that $n^2/(2p)-(n/2)\leqslant \min C(n,p)$. Brueggeman and Hildebrand [6] showed that for $p\geqslant 9$ there exists a constant $c_p$ such that

$$\left[\frac{n^2}{2p}+c_p n,\ \binom{n}{2}-c_p n^{3/2}\right]\subset C(n,p). \quad (8.9)$$

In other words, for almost all numbers $(1/p)\binom{n}{2}<e<\binom{n}{2}$ there exists an $n$ vertex graph with $e$ edges consisting of exactly $p$ cliques. We are going to use only the case $p=9$. For $m>m_0$ we have that $m^2/(2p)+c_p m<m^2/4$. The following corollary is obtained by using bipartite graphs, for $e\leqslant n^2/4$, and $\mathscr{F}_{10}(9)$ for larger $e$'s.

$$\text{If } f\notin C(m,9) \text{ and } f>m^2/4,\ m>m_0,\ \text{ then } \sigma(m,f)=0. \quad (8.10)$$

Let us remark that originally, [6] contains only the proof when $p$ is odd, but this easily implies the statement for larger $p$. Indeed, defining $n_{p+1}=n/(p+1)+O(1)$ we obtain that

$$\left(\binom{n_{p+1}}{2}+C(n-n_{p+1},p)\right)\cup C(n,p)\subset C(n,p+1),$$

implying (8.9) for $p+1$, too.

## 9. Unavoidable pairs, the end

**Theorem 1** (Unavoidable pairs). *Suppose that $\sigma(m,f)>\frac{2}{3}$. Then we have $(m,f)\in\{(2,0),(2,1),(4,3),(5,4),(5,6)\}$. In these cases $\sigma(m,f)=1$.*

Consider the equation

$$2\binom{l}{2}=\binom{z^2}{2}. \quad (9.1)$$

Bennett [1] proved that it has only the solution $l=3$, $z=2$ in positive integers, as we have suspected earlier. Let us note that our conjecture was proved for $z\leqslant 10^{1000}$ by M. Simonovits using a computer. Bennett's proof used linear forms in elliptic logarithms and the $L^3$ lattice basis reduction algorithm. He has also pointed out for us that it is immediate that (9.1) has only finitely many solutions, since the equation defines an elliptic curve of genus one of the form quadratic = quartic.

Concerning the simpler equation

$$2\binom{l}{2} = \binom{m}{2} \tag{9.2}$$

one can easily convert it to a Pell-equation of the form $2x^2 - y^2 = 1$. Thus, it is a standard argument that the $(l_i, m_i)$ pairs give all solutions of (9.2) where the sequences $(l_0, l_1, \ldots)$ and $(m_0, m_1, \ldots)$ are defined by the following recurrences. $l_0 = m_0 = 1$, $l_1 = 3$, $m_1 = 4$, and then

$$l_{i+2} = 6l_{i+1} - l_i - 2 \quad \text{and} \quad m_{i+2} = 6m_{i+1} - m_i - 2.$$

The next few terms are (15,21), (85,120), (493,697) and, in general, we have $m_k = \frac{1}{4}((\sqrt{2}+1)^{2k+1} - (\sqrt{2}-1)^{2k+1} + 2)$. In the proof of Theorem 1 we are interested in the cases when $m$ is a perfect square.

**Proof of Theorem 1.** $\sigma(m,f) = 1$ follows for the claimed pairs from Claims 6.1 and 7.1. Here our aim is to show that for all other pairs we have $\sigma(m,f) \leqslant \frac{2}{3}$. This can be easily checked for $m \leqslant 7$ using the above constructions, hence we suppose that $m > 7$. We show that for $m > 7$, $\sigma(m,f) > \frac{2}{3}$ implies that $f = m(m-1)/4$, $f = \binom{l}{2}$ for some integer $l$, and that $\sqrt{m}$ is an integer. Then Bennett's theorem, (9.1), can be applied to conclude that no further solution exists.

Formula (4.4) implies that we may suppose that

$$\left\lfloor \frac{(m-1)^2}{4} \right\rfloor \leqslant f \leqslant \left\lfloor \frac{m^2}{4} \right\rfloor. \tag{9.3}$$

For these values, if $f$ is written in the form (8.3) where $f = \binom{l}{2} + l'$, then for $m > 7$

$$l \geqslant \frac{m}{2} + 1. \tag{9.4}$$

If $l + 2l' > m$, then Construction $\mathscr{F}_7(2)$ (i.e. (8.2)) implies that $\sigma \leqslant \frac{2}{3}$. From now on we may suppose that $l + 2l' \leqslant m$. This and (9.4) imply

$$l - l' > \frac{l+1}{2}. \tag{9.5}$$

If $l' \neq 0$, then we can write $f$ in the form $f = \binom{l+1}{2} - (l - l')$. Then (9.5) and (8.3) imply that $\sigma \leqslant \frac{1}{2}$. The only missing case is $l' = 0$. We conclude that for $m > 7$, $\sigma(m,f) \leqslant \frac{2}{3}$ except if $f$ is in the range of (9.3) and is of the form $f = \binom{l}{2}$.

Suppose that $\sigma(m,f) > \frac{2}{3}$ and apply the above argument for $f' = \binom{m}{2} - f$. It is in the same range of (9.3) and is of the form $f' = \binom{k}{2}$. If $f < \binom{m}{2} - f$, then $l < k$ therefore $\binom{k}{2} - \binom{l}{2} \geqslant l$. On the other hand, it is at most $\lfloor m^2/4 \rfloor - \lfloor (m-1)^2/4 \rfloor = \lfloor (m+1)/2 \rfloor$. This contradicts (9.4). We obtain that $f = \binom{l}{2} = \frac{1}{2}\binom{m}{2}$, as claimed.

To prove that $m$ is a perfect square, consider Construction $\mathscr{F}_9(m+1)$. We obtain from (8.6) that $f \in D(m)$ and it is in the form of $f = ab + c$. If $c > 0$, then $ab + c \leqslant 1 + (m-2)^2/4$ which is less than $\lfloor (m-1)^2/4 \rfloor$ (for $m > 7$.) Thus $f$ is in the form $f = ab$, $a + b \leqslant m$. For $a + b < m$ we again $ab < \lfloor (m-1)^2/4 \rfloor$, thus we have $f = a(m-a)$. Solving $a(m-a) = \frac{1}{2}\binom{m}{2}$ we get $a = \frac{1}{2}(m \pm \sqrt{m})$, and we are done. □

Let us note that the argument in the above paragraph gives that $\sigma(m,f) > \frac{1}{2}$ implies that either

— $f = \lfloor (m-1)^2/4 \rfloor$, or
— $f = \lfloor m^2/4 \rfloor$, or
— $\lfloor (m-1)^2/4 \rfloor < f < \lfloor m^2/4 \rfloor$ and $f = a(m-a)$, $\binom{m}{2} - f = b(m-b)$.

The last case is equivalent to $m = (m-2a)^2 + (m-2b)^2$ so it is again rather rare.

## 10. A long interval of zeros

In Section 7 Construction 8 (more exactly (8.3) and (8.4)) show that the majority of the $\sigma$'s are 0. Here we prove that if $f$ is small compared to $m^2$ then $\sigma(m,f)$ is almost always 0.

**Theorem 2** (A long interval of zeros). *There exists a constant c such that for* $23m < f < cm^{3/2}$ *one has* $\sigma(m,f) = 0$. *Also for* $0 < f \leqslant 23m$ *(and* $m > m_0$*)* $\sigma(m,f) > 0$ *is only possible if f is of the form* $am - \binom{a+1}{2} - b$ *with positive integers* $0 \leqslant b < a \leqslant 23$.

Thus, for fixed $m$, all but at most $c_0$ $(c_0 < 300)$ $\sigma(m,f) = 0$ in the interval $0 \leqslant f \leqslant cm^{3/2}$.

**Proof of Theorem 2.** We prove the complement of the statement considering $(m,f)$ pairs where $f$ is close to $\binom{m}{2}$. More precisely, consider a pair $m, f$ with $m > m_0$, where $m_0$, comes from (8.10). Write it in the form (8.3), i.e., $f = \binom{l}{2} + l'$, where $0 \leqslant l' < l$ and suppose that $l > m - c\sqrt{m}$, where $c > 0$ is a suitable constant defined by $c = \min\{c_9, 3\}$, where $c_9$ comes from (8.9). For $l' \geqslant m - l$ Construction 8, i.e., (8.4) implies that $\sigma(m,f) = 0$. So from now on we suppose that

$$0 \leqslant l' < m - l < c\sqrt{m} \leqslant 3\sqrt{m}. \tag{10.1}$$

If $f \notin C(m,9)$, then (8.10) implies $\sigma = 0$, so we may suppose that one can find integers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_9 \geqslant 0$, $\sum n_i = m$ such that $f = \sum \binom{n_i}{2}$. Here $n_1 \leqslant l$. We claim that $n_1 = l$. Indeed, for $n_1 \leqslant l$ we have

$$\binom{l}{2} \leqslant f \leqslant \binom{n_1}{2} + 8\binom{(m-n_1)/8}{2} \leqslant \binom{l-1}{2} + 8\binom{(m-l+1)/8}{2}$$

which implies $l-1 \leqslant (m-l+1)(m-l-7)/16$ contradicting (10.1). Thus $f \in C(m,9)$ and $n_1 = l$ implies that

$$m-l > l' = \sum_{2\leqslant i\leqslant 9} \binom{n_i}{2} \geqslant 8\binom{(m-l)/8}{2}.$$

This yields $m-l \leqslant 23$. Writing $m-l =: a$, $l' =: b$ we get $m-f = am - \binom{a+1}{2} - b$, $0 \leqslant b < a$. Finally, we note that using the condition $b \geqslant 8\binom{a/8}{2}$ we can further narrow the possible exceptions in the range $0 < f < 23m$. □

## 11. Positive results

One can think that almost all $\sigma(m,f)=0$, or at least $\lim_{m\to\infty}(\max_{0\leqslant f\leqslant\binom{m}{2}} \sigma(m,f)) = 0$. The next theorem shows that this is not true. Using the tools of extremal graph theory we show infinitely many positive cases.

**Theorem 3** (Non-zeros). *Let $f = \binom{a}{2} = c(m-c) = \binom{m}{2} - \binom{b}{2}$, where $a,b,c$ are positive integers. Suppose that $q$ is the smallest integer such that $f$ can be written in the following form*:

$$f = \sum_{1\leqslant i\leqslant q+1} \binom{x_i}{2}, \quad \sum x_i = m, \quad x_i > 0 \text{ integers.} \tag{11.1}$$

*Then $\sigma(m,f) \geqslant 1/q$. Moreover, for $q \geqslant 9$ we have $\sigma(m,f) = 1/q$.*

Theorem 3 gives infinitely many pairs with $\sigma(m,f) \geqslant \frac{1}{8}$. For example, considering the identity

$$\binom{3t}{2} + \binom{4t-1}{2} = \binom{5t-1}{2},$$

set $a=3t$, $b=4t-1$, $m=5t-1$. Hence we have $c = \frac{1}{2}(5t-1-\sqrt{7t^2-4t+1})$. We get an infinite sequence of integer solutions $(0,1,12,187,2976,\ldots)$ defined by the recurrence $t_i = 16t_{i-1} - t_{i-2} - 4$. For all these values $\frac{1}{2} \geqslant \sigma(m,f) \geqslant \frac{1}{8}$.

Theorem 3 gives infinitely many exact values. With the notations of (8.8) we can write that $f \in C(m,q+1)\backslash C(m,q)$. Then for $q \geqslant 9$ (8.9) implies that $\sigma \leqslant 1/q$, thus equality holds in Theorem 3. For example, taking the identity

$$\binom{7t}{2} + \binom{24t-3}{2} = \binom{25t-3}{2},$$

set $a=7t$, $b=24t-3$, $m=25t-3$ and $c=\frac{1}{2}(25t-3-\sqrt{527t^2-136t+9})$. For the infinitely many integer solutions we get $\sigma = \frac{1}{12}$. (The smallest $t>1$ is 920.)

To get infinitely many more explicit numbers let us consider the case $m-1=\binom{l}{2}\geqslant 6$. Then

$$\sigma(m,m-1)=1/\lfloor m/3\rfloor$$

because for $m\neq 3,6$ one can write $m-1$ in the form (11.1) with $q=\lfloor m/3\rfloor+1$. This formula gives, e.g., $\sigma(7,6)=\sigma(7,15)=\frac{1}{2}$.

Theorem 3 also gives the following form of Theorem 1, which does not use Bennett's theorem. Suppose that $f=\frac{1}{2}\binom{m}{2}$ $m$ is a perfect square, then $\sigma(m,f)=1$. Indeed,

$$f=\frac{1}{2}\binom{m}{2}=\binom{(m+\sqrt{m})/2}{2}+\binom{(m-\sqrt{m})/2}{2},$$

therefore $q=1$.

We also *conjecture* that in Theorem 3 equality holds, i.e., in the cases not listed among the constructions the graph $G$ indeed contains an $(m,f)$ subgraph. We intend to continue the investigation of $\sigma(m,f)$ in a forthcoming paper.

**Proof of Theorem 3.** We start with three lemmas.

**Lemma 11.1.** *Suppose that* $f\in C(m,q+1), G\in\mathscr{G}(n,e;m,f)$ *with*

$$e<\binom{n}{2}-t_q(n)-\varepsilon n^2. \tag{11.2}$$

*Then, as* $n\to\infty$, *G contains arbitrarily large independent sets; we have* $\alpha(G)>\Omega((\log n)^{1/m})$.

**Proof.** Apply (1.4) to the complement of $G$. We get the disjoint $t$-element sets $V_1,\dots,V_{q+1}$, where $t>\log n/(500\log(1/\varepsilon))$, such that there is no edge of $G$ between these classes. If each $V_i$ contains a complete graph $K_{x_i}$, where $x_i$ comes from (11.1), then the union of these gives an $(m,f)$ subgraph, a contradiction. We get $\omega(G|V_j)<m$ for some $j$. Ramsey's theorem, more exactly the well-known Erdős–Szekeres upper bound $R(u,v)\leqslant\binom{u+v-2}{u-1}$, implies that the independence number $\alpha(G)\geqslant\alpha(G|V_j)\geqslant t^{1/m}$. □

**Lemma 11.2.** *Suppose that* $f=c(m-c)$, $G\in\mathscr{G}(n,e;m,f)$ *with*

$$n^{2-1/R}<e. \tag{11.3}$$

*Then, G contains a clique of size at least* $R^{1/m}$, *we have* $\omega(G)>R^{1/m}$.

**Proof.** Apply (1.2) to $G$. We get two disjoint $R$-element sets $V_1,V_2$ such that they induce a complete bipartite graph. If each $V_i$ contains an empty graph of size at least $m$, then $K(c,m-c)$ is an induced subgraph of $G$, a contradiction. We get $\alpha(G|V_j)<m$ for some $j$. Ramsey's theorem again, implies that the clique number $\omega(G)\geqslant\omega(G|V_j)\geqslant R^{1/m}$. □

**Lemma 11.3.** *Suppose that* $f = \binom{a}{2}, \binom{m}{2} - f = \binom{b}{2}$ *and* $G \in \mathscr{G}(n,e;m,f)$. *Then, either* $\alpha(G) \leqslant 4^m$ *or* $\omega(G) \leqslant 4^m$.

**Proof.** Suppose, on the contrary, that there are (disjoint) sets, $A$ and $B$, of size at least $4^m$ such that $G|A$ is a complete graph, $B$ is independent and $|A| = |B|$. Consider the induced bipartite graph $G[A,B]$. If it contains a bipartite graph $K(m-b,b)$ (with $m-b$ vertices in $A$), then it forms an induced $(m,f)$ graph, a contradiction. So again, (1.2) can be used to see that $|\mathscr{E}(G[A,B])| < \frac{1}{2}|A|^2$. Now consider the bipartite graph $H[A,B]$ formed by the non-edges between $A$ and $B$. This graph has more than $\frac{1}{2}|A|^2$ edges. However, $H$ does not contain a $K(a, m-a)$, so we get $|\mathscr{E}(H[A,B])| < \frac{1}{2}|A|^2$, a final contradiction. □

**The end of proof.** The above lemmas obviously imply that for the claimed values of $(m,f)$ in Theorem 3 one has that

$$G \rightarrow (m,f) \quad \text{if } n^{2-1/R} < e < \frac{1}{q}\binom{n}{2} - \varepsilon n^2,$$

where $R = 4^m$. □

## Acknowledgements

Part of this research was done while the authors were visiting ZiF, University of Bielefeld, whose hospitality is gratefully acknowledged. The research of the second author was supported in part by the Hungarian National Science Foundation under grants OTKA 016389, and by a National Security Agency grant No. MDA904-98-I-0022. The second author is also indebted to the hospitality of IME, Sao Paulo University where most of this manuscript was written. The research of the fourth author was supported in part by OTKA 016389.

We are also thankful M. Bennett, J.A. Brueggeman and A. Hildebrand (from UIUC, Urbana, IL) and A. Sidorenko (Courant Institute, NY) for fruitful conversations and for M. Simonovits (Math. Inst., Hungary) for helpful suggestions. We also would like to credit A.F. Sidorenko with finding Construction 6, implying $\sigma(7,10) \leqslant \frac{1}{2}$, and for showing that $\sigma(7,6) = \frac{1}{2}$.

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