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GOOD APPROXIMATION AND CHARACTERIZATION OF SUBGROUPS OF $\mathbb{R}/\mathbb{Z}$

A. BÍRÓ[1,2], J.-M. DESHOUILLERS[2] and V. T. SÓS[1]

Abstract

Let α be a real irrational number and $\mathcal{A}=(x_n)$ be a sequence of positive integers. We call $\mathcal{A}$ a characterizing sequence of α or of the group $\mathbb{Z}\alpha \mod 1$ if

$$\lim_{\substack{n\in\mathcal{A}\\ n\to\infty}} \|n\beta\| = 0$$

if and only if $\beta\in\mathbb{Z}\alpha \mod 1$.

In the present paper we prove the existence of such characterizing sequences, also for more general subgroups of $\mathbb{R}/\mathbb{Z}$. In the special case $\mathbb{Z}\alpha \mod 1$ we give explicit construction of a characterizing sequence in terms of the continued fraction expansion of α. Further, we also prove some results concerning the growth and gap properties of such sequences. Finally, we formulate some open problems.

1. Introduction

The problem we study in the present paper is rooted in ergodic and automata theory and primarily motivated by questions of Dorothy Moharam and Arthur Stone.

We noticed that Kraaikamp and Liardet [2] proved closely related results for $\mathbb{Z}\alpha \mod 1$. Our work is also connected with results of Petersen [5]. We shall make more detailed remarks at the corresponding section.

Our first result states that for any real α, there exists a sequence $\mathcal{A}$ characterizing α in the above sense.

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THEOREM 1. *Let α be a real number. There exists a sequence $\mathcal{A}$ of integers such that, for any real β, we have*

$$\lim_{n\in\mathcal{A},n\to\infty}\|n\beta\|=0$$

if and only if $\beta\in\mathbb{Z}+\mathbb{Z}\alpha$.

We give three different proofs of Theorem 1.

The first one (Section 2) (in the most intererresting case when α is irrational) is based on the consideration of the 2-dimensional sequences $(n\alpha, n\beta)$ modulo 1 and a compactness argument. This ineffective proof is the shortest self-contained one and it can also be extended to countable subgroups of $\mathbb{B}/\mathbb{Z}$.

THEOREM 2. *Let G be a countable subgroup of $\mathbb{R}/\mathbb{Z}$. There exists a sequence $\mathcal{A}$ of integers that characterizes G, that is to say that the sequence $(n\beta)_{n\in\mathcal{A}}$ tends to zero modulo 1 if and only if the class of β modulo 1 belongs to G.*

The second proof (Section 3) relies on Fourier analysis: by essence, it works with rather dense characterizing sequences. It can be extended to multidimensional cases and, by block construction, to countable cases; and it is likely to generalize for compact Abelian groups in place of $\mathbb{R}/\mathbb{Z}$.

The third one (Section 4) provides an explicit construction of a characterizing sequence $\mathcal{A}$ in terms of the continued fraction expansion of α; moreover, we give necessary conditions for a sequence to be a characterizing sequence of α.

In the last section, we add some remarks and formulate some open problems.

2. Proof of Theorem 2 via a compactness argument

LEMMA 2.1. *Let $1,\alpha_1,\alpha_2,\dots,\alpha_t\in\mathbb{R}$ be linearly independent over $\mathbb{Q}$. Let $\varepsilon>0$, $T\geqq 1$, $\varepsilon T\leqq\frac{1}{4}$, and $n_0>0$, $n_0\in\mathbb{Z}$. If for a real number β we have for $n\geqq n_0$*

$$\|n\alpha_1\|,\|n\alpha_2\|,\dots,\|n\alpha_t\|\leqq\varepsilon\ \Rightarrow\ \|n\beta\|\leqq T\varepsilon,$$

then $\beta\equiv K_1\alpha_1+\dots+K_t\alpha_t \pmod 1$ with some integers $K_1,K_2,\dots,K_t$ satisfying $|K_1|+|K_2|+\dots+|K_t|\leqq T$.

PROOF OF LEMMA 2.1. It is clear that the set

$$\{(n\alpha_1,n\alpha_2,\dots,n\alpha_t,n\beta)\colon n\in\mathbb{Z}\}$$

is not dense in $(\mathbb{R}/\mathbb{Z})^{t+1}$, so the numbers $\alpha_1, \alpha_2, \ldots, \alpha_t, \beta$ and 1 cannot be linearly independent over $\mathbb{Q}$. This implies that there are integers $K_1, K_2, \ldots, K_t, K$ such that

$$K_1\alpha_1 + K_2\alpha_2 + \cdots + K_t\alpha_t + K\beta \equiv 0 \pmod 1,$$

where we may assume that $K > 0$. So

$$\beta \equiv -\frac{K_1}{K}\alpha_1 - \frac{K_2}{K}\alpha_2 - \cdots - \frac{K_t}{K}\alpha_t + \frac{K_{t+1}}{K} \pmod 1 \tag{2.1}$$

with some integer K_{t+1}.

We now prove that $\frac{K_i}{K}$ is an integer for $1 \leqq i \leqq t$. Let $n = rK$, then

$$n\beta \equiv -K_1 r\alpha_1 - K_2 r\alpha_2 - \cdots - K_t r\alpha_t \pmod 1. \tag{2.2}$$

Assume that $\frac{K_1}{K}$ is not an integer. Then there is an integer $1 \leqq R < K$ such that $\left\| \frac{RK_1}{K} \right\| \geqq \frac{1}{3}$. For this R and for any $\delta > 0$ we can choose an r which is large enough and satisfies

$$\left\| \frac{R}{K} - r\alpha_1 \right\| < \delta, \text{ and } \|r\alpha_2\|, \ldots, \|r\alpha_t\| < \delta.$$

Then, by (2.2), $\left\| n\beta + \frac{RK_1}{K} \right\| < (K_1 + \cdots + K_t)\delta$, while

$$\|n\alpha_1\|, \ldots, \|n\alpha_t\| < K\delta.$$

If δ is small enough, this gives us $\|n\alpha_1\|, \ldots, \|n\alpha_t\| < \varepsilon$, but $\|n\beta\| > \frac{1}{4}$. This contradiction shows that K divides K_1, and similarly K divides $K_2, \ldots, K_t$.

So, by (2.1),

$$\beta \equiv K_1\alpha_1 + \cdots + K_t\alpha_t + \frac{K_{t+1}}{K} \pmod 1$$

with some integers $K_1, \ldots, K_t$ (changing a bit our notations). We now prove that $\frac{K_{t+1}}{K}$ is also an integer. If this is not the case, then we can choose $0 < R < K$ such that $\left\| \frac{RK_{t+1}}{K} \right\| \geqq \frac{1}{3}$. Let $n = R + rK$. For any $\delta > 0$ we can choose r such that $\|n\alpha_1\|, \ldots, \|n\alpha_t\| < \delta$ (and r is large enough), and then similarly as above, we will have

$$\left\| n\beta - R\frac{K_{t+1}}{K} \right\| < (K_1 + \cdots + K_t)\delta.$$

If δ is small enough, this is a contradiction, so $\dfrac{K_{t+1}}{K}$ is also an integer.

This means that

$$\beta \equiv K_1\alpha_1 + \cdots + K_t\alpha_t \pmod 1$$

with integers $K_1, \dots, K_t$, and it is easy to see that our condition can be satisfied only in the case $|K_1| + |K_2| + \cdots + |K_t| \leqq T$.

LEMMA 2.2. *Let $\alpha_1, \alpha_2, \dots, \alpha_t$ and ε, T, n_0 be as in Lemma 1. If $\delta > 0$ is given, we can choose N such that if*

$$\|n\alpha_1\| \leqq \varepsilon,\ \|n\alpha_2\| \leqq \varepsilon, \dots,$$

$$\dots, \|n\alpha_t\| \leqq \varepsilon \Rightarrow \|n\beta\| \leqq T\varepsilon$$

for $n_0 \leqq n \leqq N$, then

$$\|\beta - K_1\alpha_1 - \cdots - K_t\alpha_t\| < \delta,$$

with some integers $K_1, K_2, \dots, K_t$ satisfying $|K_1| + \cdots + |K_t| \leqq T$.

PROOF. This is an easy consequence of Lemma 2.1 and the compactness of $\mathbb{R}/\mathbb{Z}$.

PROOF OF THEOREM 2. It is clear that there are finitely generated subgroups G_k such that $G = \bigcup_{k=1}^{\infty} G_k$, and $G_k \leqq G_{k+1}$ for each k. Since G_k is finitely generated and $G_k \leqq \mathbb{R}/\mathbb{Z}$, the torsion subgroup of G_k is finite cyclic, let its order be n_k. Then n_kG_k is a finitely generated torsion free Abelian group, hence it is free. Let $\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{t(k),k}$ be free generators of n_kG_k. Then, for $l < k$ we have $\dfrac{n_k}{n_l}\alpha_{i,l} \in n_kG_k$ $(1 \leqq i \leqq t(l))$, where $\dfrac{n_k}{n_l}$ is obviously an integer. Hence

$$\frac{n_k}{n_l}\alpha_{i,l} = \sum_{j=1}^{t(k)} c_{j,i,k,l}\ \alpha_{j,k}, \tag{2.3}$$

and let

$$M_k = \max_{l<k, 1\leqq i\leqq t(l)} \sum_{j=1}^{t(k)} |c_{j,i,k,l}|\,. \tag{2.4}$$

Let $0<N_1<N_2<\dots<N_k<\dots$ be integers such that if for $N_k<n\leqq N_{k+1}$, $n_k\,|\,n$ one has

$$\left\|\frac{n}{n_k}\alpha_{1,k}\right\|\leqq\frac{1}{4k(M_k+1)},\dots,\left\|\frac{n}{n_k}\alpha_{t(k),k}\right\|\leqq$$

$$\leqq\frac{1}{4k(M_k+1)}\Rightarrow\left\|\frac{n}{n_k}\beta\right\|\leqq\frac{1}{4},$$

then

$$\left\|\beta-K_{1,k}\alpha_{1,k}-\dots-K_{t(t),k}\ \alpha_{t(k),k}\right\|<\delta_k$$

with some integers satisfying $|K_{1,k}|+\dots+|K_{t(k),k}|\leqq(M_k+1)k$, where δ_k will be chosen later. In view of Lemma 2.2 we can define such a sequence recursively.

Now define

$$\mathcal{A}_G=\bigcup_{k=1}^{\infty}\left\{n>0\colon N_k<n\leqq N_{k+1},\ n_k\,|\,n,\ \text{and}\right.$$

$$\left.\left\|\frac{n}{n_k}\alpha_{1,k}\right\|,\dots,\left\|\frac{n}{n_k}\alpha_{t(k),k}\right\|\leqq\frac{1}{4k(M_k+1)}\right\}.$$

Then $\lim\limits_{n\to\infty,n\in\mathcal{A}_G}\|n\beta\|\to 0$, if $\beta\in G$.

Indeed, let $\beta\in G_l$ for some l. Then, for $k>l$ if $n\in\mathcal{A}_G$ and $N_k<n\leqq N_{k+1}$, then $n\beta=\dfrac{n}{n_k}\left(\dfrac{n_k}{n_l}n_l\beta\right)$. Then, since $n_l\beta=\sum\limits_{i=1}^{t(l)}c_i\ \alpha_{i,l}$, one has by (2.3) and (2.4) $\dfrac{n_k}{n_l}\,n_l\,\beta=\sum\limits_{j=1}^{t(k)}d_j\,\alpha_{j,k}$ with $\sum\limits_{j=1}^{t(k)}|d_j|\leqq CM_k$, where $C=\sum\limits_{i=1}^{t(l)}|c_i|$, and this shows

$$\|n\beta\|\leqq\frac{C}{4k}.$$

Assume conversely that $\lim\limits_{n\to\infty,n\in\mathcal{A}_G}\|n\beta\|=0$. Then, for large enough $n\in\mathcal{A}_G$, we have $\|n\beta\|\leqq\dfrac{1}{4}$, and, by the definition of $\mathcal{A}_G$, this shows that for k large enough

$$\|n_k\beta-K_{1,k}\alpha_{1,k}-\dots-K_{t(k),k}\alpha_{t(k),k}\|<\delta_k$$

with some integers $|K_{1,k}|+\dots+|K_{t(k),k}| \leqq (M_k+1)k$.

Let $H_k=\{\alpha\in\mathbb{R}/\mathbb{Z}\colon n_k\alpha=K_{1,k}\alpha_{1,k}+\dots+K_{t(k),k}\alpha_{t(k),k}$ with integers $|K_{1,k}|+\dots+|K_{t(k),k}| \leqq (M_k+1)k\}$.

Then we have

$$\|\beta-\alpha_k\|<\frac{\delta_k}{n_k} \tag{2.5}$$

with some $\alpha_k\in H_k$. Of course, H_k is a finite set. We now choose δ_k. Let

$$\varepsilon_k=\frac{1}{2}\min_{\alpha'\in H_{k+1},\alpha\in H_k,\alpha\neq\alpha'}\|\alpha'-\alpha\|$$

and choose the sequence δ_k in such a way that $\delta_k>\delta_{k+1}$ and $\delta_k<\varepsilon_k$ for all k furthermore $\delta_k\to 0$.

Since (using (2.5) for k and $k+1$)

$$\|\alpha_{k+1}-\alpha_k\|<\frac{\delta_k}{n_k}+\frac{\delta_{k+1}}{n_{k+1}}<2\varepsilon_k,$$

and $\alpha_{k+1}\in H_{k+1}$, $\alpha_k\in H_k$, so $\alpha_{k+1}=\alpha_k$ by the definition of ε_k.

This shows that the sequence α_k is quasistationary, so (by $\delta_k\to 0$) $\beta=\alpha_k\in H_k$ for some k. So it is enough to show that $H_k\subseteqq G_k$ ($\leqq G$). But it is clear, because $n_kH_k\subseteqq n_kG_k$ by definition, and G_k contains the unique cyclic subgroup of order n_k of $\mathbb{R}/\mathbb{Z}$. □

3. Proof of Theorem 1 via Fourier technique

The strategy here is to build, by blocks, a characterizing sequence $\mathcal{A}$ such that, for any β not in $\mathbb{Z}+\mathbb{Z}\alpha$, there are infinitely many blocks over which the mean value of $\cos^2(\pi n\beta)$ is less than 0.95. We restrict ourselves to the case when α is irrational.

The following lemma is an easy consequence of Vaaler's lemma ([4], pp. 6–8).

LEMMA 3.1. *Let $k\geqq 2$. There exist two 1-periodic real valued functions $\varphi_k^{\pm}$ such that*

$$\varphi_k^-\leqq \mathbf{1}_{[-1/k,1/k]}\leqq\varphi_k^+,$$

$$\varphi_k^{\pm}(x)=\sum_{|q|<5k}c_q^{\pm}e(qx),$$

$$\left|c_0^{\pm}-\frac{2}{k}\right|\leqq\frac{1}{5k}\quad\text{and}\quad\left|c_q^{\pm}\right|\leqq\frac{11}{5k}.$$

Our second tool is Koksma's inequality ([3], Theorem 5.1).

LEMMA 3.2. *Let $x_1,\dots,x_N$ be a sequence of real numbers and φ a 1-periodic function. We have*

$$\left|\frac{1}{N}\sum_{n=1}^{N}\varphi(x_n)-\int_0^1\varphi(x)dx\right|\leqq D_N^*V(\varphi),$$

where $V(\varphi)$ is the total variation of φ on $[0,1]$ and

$$D_N^*=D^*(x_1,\dots,x_N)=\sup_{0<t\leqq 1}\frac{1}{N}\left|\mathrm{Card}\{n\leqq N;x_n\in[0,t[\}-Nt\right|.$$

We first select a sequence of integers (H_k) such that H_k/k is increasing and tends to infinity and which satisfies, for $k\geqq 2$:

$$\text{for}\quad m=1,\dots,15\,k,\quad\text{we have}\quad \|m\alpha\|>\frac{100\,k}{H_k},$$

and

$$D^*(\{n\alpha\};\,1\leqq n\leqq H_k)\ \max(V(\varphi_k^+),V(\varphi_k^-))\leqq\frac{1}{5k}. \tag{3.1}$$

This is possible, since α is irrational (cf. Corollary 1.1 in [3]).

Now we define $N_1:=1$, and, by induction, $N_k:=N_{k-1}+H_k$; then we define

$$\mathcal{A}_k:=\{n\in\,]N_{k-1},N_k];\ \|n\alpha\|<1/k\},$$

and finally

$$\mathcal{A}=\bigcup_{k=2}^{\infty}\mathcal{A}_k.$$

It is clear that the sequence $(\|n\alpha\|)_{n\in\mathcal{A}}$ tends to 0, which easily implies that for any β in $\mathbb{Z}+\mathbb{Z}\alpha$, we also have $(\|n\beta\|)_{n\in\mathcal{A}}$ tends to 0.

Let now β be a real number which does not belong to $\mathbb{Z}+\mathbb{Z}\alpha$. We first show that the set $\mathcal{K}$ of integers k satisfying

$$\forall\,q\in\mathbb{Z}\text{ with }|q|<5k\colon\ \|\beta+q\alpha\|\geqq\frac{50k}{H_k} \tag{3.2}$$

is not finite. Let us assume on the contrary that $\mathcal{K}$ is finite, which means

$$\exists K,\ \forall k\geqq K,\ \exists q\in\mathbb{Z}\colon |q|<5k\text{ and }\|\beta+\alpha q\|<\frac{50\,k}{H_k}.$$

Let k and q be such integers; since $\beta \notin \mathbb{Z} + \mathbb{Z}\alpha$, we have $\|\beta + q\alpha\| \neq 0$, and, since H_l/l tends to infinity, we can find $l \geqq k$ $(\geqq K)$ such that $\frac{50(l+1)}{H_{l+1}} \leqq \|\beta + q\alpha\| < \frac{50l}{H_l}$. By our assumption, we can find q' with $|q'| < 5(l+1)$ such that $\|\beta + \alpha q'\| < \frac{50(l+1)}{H_{l+1}}$. This implies

$$\||q - q'|\alpha\| < \frac{50(l+1)}{H_{l+1}} + \frac{50l}{H_l} \leqq \frac{100l}{H_l};$$

thus, there exists a positive integer $m\colon = |q - q'| < 5k + 5(l+1) \leqq 15l$ such that $\|m\alpha\| \leqq \frac{100l}{H_l}$, which contradicts the definition of H_l.

Let us now consider k in $\mathcal{K}$ and let

$$S_k\colon = \sum_{n \in \mathcal{A}_k} \cos^2(\pi n\beta) = \sum_{n \in \mathcal{A}_k} \left(\frac{1}{2} + \frac{1}{4} e(n\beta) + \frac{1}{4} e(-n\beta) \right).$$

We have

$$\begin{aligned} S_k &\leqq \sum_{N_{k-1} < n \leqq N_k} \varphi_k^+(n\alpha) \cos^2(\pi n\beta) \\ &\leqq \frac{1}{2} \sum_{N_{k-1} < n \leqq N_k} \varphi_k^+(n\alpha) + \frac{1}{2} \sum_{|q| < 5k} |c_q^+| \left(\left| \sum_{N_{k-1} < n \leqq N_k} e\left(n(q\alpha + \beta)\right) \right| \right), \end{aligned} \tag{3.3}$$

where we used the fact that $|c_q^+| = |c_{-q}^+|$. For the first term, we use Koksma's lemma and relation (3.1): we write $\psi_k^+(x) = \varphi_k^+(x + N_{k-1}\alpha)$ and get

$$\sum_{N_{k-1} < n \leqq N_k} \varphi_k^+(n\alpha) = \sum_{h=1}^{H_k} \psi_k^+(h\alpha) \leqq H_k \int_0^1 \psi_k^+(x)dx + \frac{H_k}{5k},$$

and, by Vaaler's lemma, this leads to

$$\sum_{N_{k-1} < n \leqq N_k} \varphi_k^+(n\alpha) \leqq \frac{12H_k}{5k}. \tag{3.4}$$

Combined with (3.3), this leads to

$$S_k \leqq \frac{6H_k}{5k} + \frac{10k}{2} \times \frac{11}{5k} \times \max_{|q|<5k} \|q\alpha + \beta\|^{-1},$$

and, by (3.2), we get

$$S_k \leqq \frac{6H_k}{5k} + \frac{11H_k}{50k} \leqq \frac{71}{50}\frac{H_k}{k}.$$

In the same way we proved (3.4), we get

$$|\mathcal{A}_k| \geqq \sum_{N_{k-1}<n\leqq N_k} \varphi_k^-(n\alpha) \geqq \frac{8H_k}{5k};$$

we thus have

$$S_k \ \leqq 0.9\ |\mathcal{A}_k|,$$

and since $\mathcal{K}$ is infinite, we get

$$\liminf_{k\to\infty} \frac{S_k}{|\mathcal{A}_k|} \leqq 0.9,$$

which implies that we do not have

$$\lim_{\substack{n\in\mathcal{A}\\ n\to\infty}} \cos^2(\pi n\beta) = 1,$$

and this proves Theorem 1. □

4. Proof of Theorem 1 via continued fraction expansion

Let us first consider the case when α is rational. Let $\alpha = p/q$, with coprime p and q; then $\mathbb{Z}+\mathbb{Z}\alpha$ is the group of integral multiples of $1/q$ and we simply take $\mathcal{A} = \{q\mathbb{Z}\}$. It is clear that $\|n\alpha\| = 0$ whenever n belongs to $\mathcal{A}$. In the other direction, if β is irrational, $(n\beta)_{n\in\mathcal{A}}$ is dense modulo 1 and hence $\lim_{n\in\mathcal{A},\, n\to\infty} \|n\beta\| \neq 0$ modulo 1; if $\beta = r/s$, with coprime r and s and s is not a divisor of q, the rational $rq(ks+1)/s$ is never an integer and again $\lim_{n\in\mathcal{A},\, n\to\infty} \|n\beta\| \neq 0$.

We now consider the case when α is irrational. We prove a constructive form of Theorem 1.

THEOREM 1*. *Let α be an irrational number, $\alpha=[a_0;a_1,a_2,\dots]$ be its continued fraction expansion and (p_n/q_n) be the sequence of its convergents. Let $\mathcal{A}=(x_n)$ be the monotone sequence formed by the integers*

$$\{rq_m;\ 1\leqq r\leqq a_{m+1},\ m=1,2,\dots\}.$$

$\mathcal{A}$ is a characterizing sequence of $\mathbb{Z}\alpha$ mod 1:

$$\lim_{n\to\infty}\|x_n\beta\|=0$$

if and only if $\beta\in\mathbb{Z}+\mathbb{Z}\alpha$.

REMARK. We may further notice, as it will be clear from the proof, and as it follows from [2], that the sequence $\mathcal{A}=(q_m)$ itself is a characterizing sequence when the coefficients a_n are bounded.

The proof relies on a characterization of the elements of $\mathbb{Z}+\mathbb{Z}\alpha$ in terms of the convergents of α.

Here we use the following theorem of Kraaikamp–Liardet:

LEMMA 4.1. (Kraaikamp–Liardet, [2]) *A real β does not belong to $\mathbb{Z}+\mathbb{Z}\alpha$ if and only if we have*

$$\|q_n\beta\|>\frac{1}{4}q_n|q_n\alpha-p_n|$$

for infinitely many n's.

PROOF OF THEOREM 1*. (a) If $\beta\equiv k\alpha$ mod 1 then for $1\leqq r\leqq a_{m+1}$

$$\|\,rq_m\alpha\,\|<\frac{r}{a_{m+1}q_m}\leqq\frac{1}{q_m},$$

hence

$$\lim_{n\to\infty}\|\,x_nk\alpha\,\|=0.$$

(b) If $\beta\not\equiv k\alpha$ mod 1, by Lemma 4.1, we have a sequence $n_i\to\infty$ such that

$$\|\,q_{n_i}\beta\,\|>\frac{1}{4}\,q_{n_i}|q_{n_i}\alpha-p_{n_i}|>\frac{1}{4}\,\frac{1}{a_{n_i+1}+1}. \tag{4.1}$$

Since

$$\|\,rq\beta\,\|=r\,\|\,q\beta\,\|\qquad\text{if}\qquad\|\,q\beta\,\|<\frac{1}{2r},$$

by (4.1) we have

$$\max_{1\leqq r\leqq a_{n_i+1}}\|\,rq_{n_i}\beta\,\|>\frac{1}{10}.$$

□

In the next Theorem concerning the "characterization of characterizing sequences" we use the expansions of integers and real numbers $\beta \in (-\alpha, 1-\alpha)$ in terms of the continued fraction expansion and convergents of α.

Every positive integer n has a unique expansion in the form

$$n = \sum_{k=0}^{K} b_k q_k,$$

where $0 \leqq b_0 < a_1$, $0 \leqq b_k \leqq a_{k+1}$, $b_k = a_{k+1} \Rightarrow b_{k-1} = 0$ for $k \geqq 1$.

Further, every $\beta \in (-\alpha, 1-\alpha)$ has a unique expansion in the form

$$\beta = \sum_{k=0}^{\infty} d_k \theta_k,$$

where $\theta_k = q_k\alpha - p_k$ and we have the restriction $0 \leqq b_0 < a_1$, $0 \leqq d_k \leqq a_{k+1}$ and $d_k = a_{k+1} \Rightarrow d_{k-1} = 0$ for $k \geqq 1$, furthermore $d_{2i} \neq a_{2i+1}$ for infinitely many i. (See, e.g., [7].)

We prove the following necessary conditions for a sequence to be a characterizing sequence:

Proposition 1. *For the sequence $n_i \to \infty$ we have*

$$\lim_{i\to\infty} \|n_i\alpha\| = 0$$

if and only if for the expansions

$$n_i = \sum_{k=k_0(i)}^{K(i)} b_k(i) q_k$$

$k_0(i) \to \infty$ holds, where $b_{k_0(i)}$ is the first nonvanishing coefficient: $b_{k_0(i)}(i) > 0$.

Proof of Proposition 1. We use the following lemma.

Lemma 4.1. *Let*

$$n = \sum_{j=r}^{s} b_j q_j$$

be the expansion of n where $b_r > 0$. Then $\|n\alpha\| > |\theta_{r+1}|$.

Proof. We have

$$n\alpha = \sum_{j=r}^{s} b_j q_j \alpha \equiv \sum_{j=r}^{s} b_j \theta_j.$$

Using that $b_r > 0$ implies $b_{r+1} \leqq a_{r+2} - 1$ and that $\operatorname{sign} \theta_j$ is an alternating sequence, we get

$$\left| b_r\theta_r + \sum_{j \geqq 1} b_{r+j}\theta_{r+j} \right| \geqq b_r|\theta_r| - (a_{r+2}-1)|\theta_{r+1}| - |\theta_{r+2}| \geqq |\theta_{r+1}|,$$

where we used also

$$\sum_{j \geqq 1} a_{r+2j+2}|\theta_{r+2j+1}| = |\theta_{r+2}|,$$

which follows from the recursive formulas.

Definition. Let $n_i \to \infty$ be a sequence of integers,

$$K = \{k\colon \text{ there exist } i \text{ such that } b_k(i) \neq 0\}$$

(i.e. $k \in K$ if and only if q_k occurs in the expansion of at least one n_i).

For a characterizing sequence (n_i) K cannot contain arbitrary long gaps:

Proposition 2. *If K contains arbitrary long gaps, then there exist β such that $\beta \not\equiv k\alpha \bmod 1$ but $\lim_{i\to\infty} \| n_i\beta \| = 0$. Moreover, there are uncountably many β with the property $\lim_{i\to\infty} \| n_i\beta \| = 0$.*

Proof of Proposition 2. Suppose $\mathcal{A} = (n_i)$ is a characterizing sequence, $K(\mathcal{A})$ defined as above. Suppose $K(\mathcal{A})$ has arbitrary large gaps.

Let the sequences $k_j, l_j, m_j,\ j = 1, \dots$ be such that

$$[k_j, l_j] \cap K(\mathcal{A}) = \emptyset,$$
$$l_j - k_j > 10j, \qquad m_j = \left[\frac{k_j + l_j}{2}\right],$$

and the intervals $[k_j, l_j]$ are disjoint.

Let $\varepsilon_j = 0$ or $\varepsilon_j = 1$ for each $j \geqq 1$, and let $\beta = \sum_{j=1}^{\infty} \varepsilon_j\theta_{m_j}$. The cardinality of the set of such numbers β is that of the continuum. We prove that

$$\lim_{i\to\infty} \|n_i\beta\| = 0$$

holds for any such β.

By Proposition 1 we may suppose that $b_k(i) = 0$ for $k < m_{j_0}$ if $i > t_0(j_0)$. Observe that $b_k(i) \neq 0$ implies $|k - m_j| > 2j$ for $j = 1, \dots$.

In the proof below we use

(a) $$\sum_{k>m} b_k|\theta_k| \leqq \sum_{k>m} a_{k+1}|\theta_k| = \sum_{k>m} (|\theta_{k-1}| - |\theta_{k+1}|) < 2|\theta_m|;$$

(b) $$\sum_{k<m} b_k q_k < \sum_{k<m} a_{k+1} q_k = \sum_{k<m} (q_{k+1} - q_{k-1}) < 2q_m;$$

(c) with some fixed $0<\lambda<1$,

$$|\theta_{k+j}| < \lambda^j |\theta_k| \quad \text{and} \quad q_{k-j} < \lambda^j q_k \quad \text{for} \quad j \geqq 2;$$

(d) $q_k\theta_m \equiv q_m\theta_k \pmod 1$.

Now, by

$$n_i\beta = \sum_{j=1}^{\infty} \varepsilon_j n_i \theta_{m_j} = \sum_{j=1}^{\infty} \sum_{k>l_{j_0}} \varepsilon_j b_k(i) q_k \theta_{m_j}$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3,$$

where

$$\sum\nolimits_1 = \sum_{j=1}^{j_0} \sum_{k>l_{j_0}},$$
$$\sum\nolimits_2 = \sum_{j>j_0} \sum_{l_{j_0}<k<m_j-j},$$
$$\sum\nolimits_3 = \sum_{j>j_0} \sum_{l_j<k}.$$

Using (a), (c) and (d) we get

$$\sum\nolimits_1 \equiv \sum_{j=1}^{j_0} \varepsilon_j \left(\sum_{k>m_{j_0}+j_0} b_k(i)\theta_k \right) q_{m_j} \leqq$$
$$\leqq 2\sum_{j=1}^{j_0} q_{m_j} \left|\theta_{m_{j_0}+j_0}\right| \leqq 4q_{m_{j_0}}\theta_{m_{j_0}}\lambda^{j_0} \leqq 4\lambda^{j_0}.$$

Using (b) and (c) we have

$$\left|\sum\nolimits_2\right| \leqq 2\sum_{j>j_0} \left(\sum_{k<m_j-j} b_k(i) q_k \right) |\theta_{m_j}| \leqq 2\sum_{j>j_0} q_{m_j-j} \left|\theta_{m_j}\right|$$
$$\leqq 2\sum_{j>j_0} \lambda^j < c_0\lambda^{j_0}.$$

Finally,

$$\sum\nolimits_3 \equiv \sum_{j>j_0} \varepsilon_j \Bigg(\sum_{k>m_j+j} b_k(i)\theta_k \Bigg) q_{m_j}$$

and

$$\Big| \sum\nolimits_3 \Big| \leqq 4 \sum_{j>j_0} q_{m_j} \big|\theta_{m_j+j}\big| \leqq 4 \sum_{j>j_0} \lambda^j < c_0 \lambda^{j_0}.$$

5. Some complements and open problems

5.1. On the growth of a characterizing sequence for $\mathbb{Z}+\mathbb{Z}\alpha$.

The proof in Section 2 usually provides rather dense sequences, but that of Section 4 provides rather sparse sequences. In the case when α has bounded quotients, we can even get a sequence $\mathcal{A}=(x_m)$ such that

$$0 < \underline{\lim}\ x_{m+1}/x_m \leqq \overline{\lim}\ x_{m+1}/x_m < +\infty.$$

This is close to the best possible result we can expect: if the sequence $\mathcal{A}=(x_m)$ is such that $\lim\limits_{m\to\infty} x_{m+1}/x_m \to \infty$, then Eggleston ([1]) proved that the set of α's such that $(\|x_m\alpha\|)$ tends to 0 is not countable; its Hausdorff dimension is 1, hence $\mathcal{A}$ cannot be a characterizing sequence.

There is, however, a way to build characterizing sequences $\mathcal{A}=(x_m)$ such that their k-th finite differences are growing arbitrarily fast. Even we can have $\lim\limits_{m\to\infty} x_{m+2}/x_m \to \infty$.

PROPOSITION 3. *Let $\mathcal{B}=(b_n)$ be a characterizing sequence of α and let $\mathcal{B}'=(b_{k_n})$ be any subsequence of $\mathcal{B}$. Then the sequence $\mathcal{A}=(x_n)$, where*

$$x_{2n} = b_{k_n}$$
$$x_{2n+1} = b_{k_n} + b_n,$$

is also a characterizing sequence.

We clearly have $\|x_n\alpha\|$ tends to zero, and if $\beta \notin \mathbb{Z}+\mathbb{Z}\alpha$, either $\|x_{2n}\beta\|$ does not tend to zero or $\|x_{2n}\beta\|$ tends to zero but then $\|x_{2n+1}\beta\| \geqq |\|x_{2n}\beta\| - \|b_n\beta\||$ does not tend to zero, since $\|b_n\beta\|$ does not tend.

5.2. Connections with a theorem of Petersen

In a theorem of Petersen ([5]) the equivalence of a number of conditions related to the discrepancy of $(\{n\alpha\})$ sequences is proved. In particular, it is proved that if α and β are irrational numbers, then the series

$$\sum_{k\neq 0}\frac{1}{k^2}\frac{\|k\beta\|^2}{\|k\alpha\|^2}$$

is convergent if and only if

$$\beta\in\mathbb{Z}\alpha+\mathbb{Z}.$$

Using the theorem of Kraaikamp and Liardet (see our Lemma 4.1) it is easy to see the following strengthening of this statement:

If $\beta\notin\mathbb{Z}+\alpha\mathbb{Z}$, then the sequence

$$\frac{\|k\beta\|}{k\cdot\|k\alpha\|}$$

even cannot tend to 0. In fact, it follows by considering the numbers $k=q_n$.

5.3. The distribution of $\{n\beta\}$ $(n\in\mathcal{A})$ for $\beta\notin\mathbb{Z}+\mathbb{Z}\alpha$.

PROPOSITION 4. *If $\mathcal{A}$ is a characterizing sequence (i.e. $\lim_{n\in\mathcal{A},n\to\infty}\|n\beta\|=0$ if and only if $\beta\in\mathbb{Z}+\alpha\mathbb{Z}$), then the set $\mathcal{A}'=\{rn\colon n\in\mathcal{A}, 1\leqq r\leqq\|n\alpha\|^{-1/2}\}$ is also a characterizing sequence. If $\beta\notin\mathbb{Q}+\mathbb{Q}\alpha$, then $\{\{n\beta\}\colon n\in\mathcal{A}'\}$ is everywhere dense in $[0,1)$.*

PROOF. (a) It is obvious that $\mathcal{A}'$ is also a characterizing sequence.

(b) Let $\beta\notin\mathbb{Q}+\mathbb{Q}\alpha$ and $H(\mathcal{A}')$ be the set of limit points of $\{n\beta\}$, $n\in\mathcal{A}'$.

$H(\mathcal{A}')$ cannot be reduced to a finite set of rationals, otherwise, these rationals have a common denominator Q and then $\|Qn\beta\|$ $(n\in\mathcal{A})$ tends to zero, whence $Q\beta\in\mathbb{Z}+\mathbb{Z}\alpha$, a contradiction. So, $H(\mathcal{A}')$ either contains an irrational number or a set of rationals with unbounded denominators.

1st Let $\gamma\in H(\mathcal{A}')\cap(\mathbb{R}\setminus\mathbb{Q})$. Let $\varkappa\in\mathbb{R}/\mathbb{Z}$ and $\varepsilon>0$. We can find N such that $\|N\gamma-\varkappa\|<\varepsilon/2$. We can also find $n\in\mathcal{A}$ such that $\|n\beta-\gamma\|<\varepsilon/2N$ and $\|n\alpha\|^{-1/2}\geqq N$. Then $Nn\in\mathcal{A}'$ and $\|Nn\beta-\varkappa\|<N\|n\beta-\gamma\|+\|N\gamma-\varkappa\|<\varepsilon$.

2nd Let $p_n/q_k\in H(\mathcal{A}')$ with $(p_k,q_k)=1$ and $q_k\to\infty$. Let $\varkappa\in\mathbb{R}/\mathbb{Z}$ and $\varepsilon>0$. We first select $\gamma=p/q$ in $H(\mathcal{A}')$ with $q>1/\varepsilon$. We can find N such that $\|N\gamma-\varkappa\|<\varepsilon/2$ and we end the proof as above. □

Now the question is: is it possible to strengthen that?

PROBLEM 1. Let α be irrational. Does there exist a sequence $\mathcal{A}=\{x_n\}$ of positive integers such that

$$\lim_{n\to\infty}\|x_n\alpha\|=0$$

and $(\{x_n\beta\})$ is *uniformly distributed* for every $\beta\notin\mathbb{Q}+\mathbb{Q}\alpha$?

The next problem concerns Theorem 2.

Problem 2. Characterize those subgroups of $\mathbb{R}/\mathbb{Z}$ for which the statement of Theorem 2 is true. What we know from Theorem 2 is that countable groups belong to this family.

Remark that without the condition $\|x_n\alpha\| \to 0$ the problem is solved by the general theorem of Rauzy [6].

Finally, a problem of Dorothy Maharam and Arthur Stone is in the direction of Proposition 1.

Problem 3. Let $\mathcal{A}_\alpha$ be the family of all infinite subsets A of the positive integers with the property that

$$\lim_{n\to\infty, n\in A} \|n\alpha\| = 0.$$

Characterize the families $\mathcal{A}$ of infinite subsets of the positive integers for which $\mathcal{A} = \mathcal{A}_\alpha$ for some α.

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MTA RÉNYI ALFRÉD MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

biroand@renyi.hu

STATISTIQUE MATHÉMATIQUE
UNIVERSITÉ VICTOR SEGALEN BORDEAUX 2 ET A2X
UMR 5465 CNRS–UNIVERSITÉ BORDEAUX 1
F–33076 BORDEAUX Cedex
FRANCE

jean–marc.deshouillers@math.u-bordeaux.fr

MTA RÉNYI ALFRÉD MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

sos@renyi.hu