BOLYAI SOCIETY
MATHEMATICAL STUDIES, 11

Paul Erdős and his Mathematics. I,
Budapest, 2002, pp. 85–146.

Turbulent Years: Erdős in his Correspondence with Turán from 1934 to 1940

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1. Introduction

The correspondence between Erdős and Turán lasted 42 years, became very frequent as soon as Erdős went to Manchester in 1934 and continued until the death of Pál Turán on September 26, 1976. It is a rich source of documents about their mathematics, about their lives, about history. Here I quote only a very small fraction of their letters and I do not intend to give here a systematic, methodical elaboration even of this part of the correspondence. However, I hope that the present paper gives some indication of their essence and contributes some strokes to the portrait of Paul Erdős.

The Erdős phenomenon — his importance in the development of mathematics — has several components: his mathematical achievements (his theorems, problems and conjectures), his influence on mathematical research, and the influence and stimulation of mathematicians through his personal contact with several hundreds of mathematicians all over the world.

Correspondence was an important part of Erdős's life-style, of his daily routine; in fact it proved to be the most efficient means for his apostle-like, large-scale activities. He formulated thousands of problems and described related results, passing them on in his letters through geographical and political borders, stimulating close friends or casual acquaintances to work with him.

Much of his joint work was done exclusively through correspondence. Another remarkable aspect of Erdős's letters is that one can reconstruct the events of world history and politics by reading his observations and predictions, worries and warnings. After the war he communicated information about destruction and survivors between continents. In more peaceful years news, marriages, family events, trips, dinners, music and history, all found their way into his letters.

There are surveys of various fields of Erdős's mathematics, many personal recollections, and an illustrious list of biographical articles. All these are of great value in obtaining an adequate account of his work and influence. Also, many articles, interviews, films and books have appeared about him, some during his life and many after his death: statistics, extreme data, legends, stories, opinions, presumptions and suppositions, they range from true, authentic, competent to some superficial, misrepresented or even false. As to the many "mosts" of quantitative character about him, which are emphasized so much, let me quote Erdős: "There is an old 'Hungarian' saying — Non numerantur, sed ponderantur".[1]

In his excellent paper "A Taste of Erdős on Interpolation" D. S. Lubinsky [123] writes: "One thing that was clear at the conference associated with these proceedings is that a lot of the very greatest work of Erdős (and especially in collaboration with Turán) was done in the late 1930's and early 1940's, a time of great difficulty for both of them." These few lines gave me the last impulse in my decision to write this article.

Here I restrict myself to the period 1934–1941, and mostly to the years 1938–1939. I feel this period was decisive in many respects in their lives.

In the present context I will focus on Erdős's letters, and I will quote only a few from those of Turán.

The paper is organized as follows. In Section 2 I give some background on Erdős's life in the thirties, including lists of names of those friends, teachers, colleagues and collaborators who are mentioned in the letters. Section 3 contains some preliminary, general comments about Erdős's mathematics and about his joint work with Turán before 1941. In Section 4 there are some remarks about the character of their correspondence. The quotations from the letters are divided into five groups. Section 5 contains some letters in their entirety and some snapshots from the letters 1934–1936 (between Erdős in Britain and Turán in Budapest). In Section 6 there are some quotations from the letters from spring 1938 (a period of special importance

[1] "They are not counted but weighed" [3].

in Erdős's leaving Europe). Section 7 contains quotations from letters in the USA-period (1938–1941), including "the most successful year" 1939. In Section 8 the correspondence with the mediations of Erdős's father in 1940 is illustrated. The closing Section 9 presents Erdős's first letter after the war, after a break of four years.

2. Brief Historical Background, 1930–1940

Erdős's life in the thirties can be divided into three periods: 1930–1934 (university years in Budapest), 1934–1938 (years in Britain and Budapest) and 1938–1940 (years in the USA).[2]

1930–1934: University years

Erdős, at age 17, enrolled in the Péter Pázmány University[3] in Budapest in 1930 and graduated from there with a Ph.D. in 1934. At that time the mathematical atmosphere in Budapest was dominated by analysis (approximation theory, interpolation theory, Fourier series, harmonic analysis), but graph theory (D. König) and number theory were represented too. The mathematicians whom he met as a student and/or who influenced his mathematics in these years were Lipót Fejér (1880–1959), László Kalmár (1905–1976), Dénes König (1884–1944), Simon Sidon (1892–1941), Mihály Bauer (1874–1945), Jenő Egerváry (1891–1958), József Kürschák (1864–1933), Richárd Obláth (1882–1959), Gusztáv Rados (1862–1942), József Suták (1865–1954), Adolf Szücs (1884–1945), Pál Veress (1893–1945).

Erdős said in an interview [149]: "I learned a lot from Lipót Fejér, but very probably, I learned the most from László Kalmár."

In the early thirties the life-long friendship of a legendary group of mathematicians was formed, including László Alpár (1914–1991), Pál Erdős (1913–1996), János Erőd, Ervin Feldheim (1912–1944), Géza Grünwald (1913–1944), Tibor Grünwald (later T. Gallai) (1912–1992), Eszter Klein (1910–), Dezső Lázár (1913–1943), Béla Lengyel (1913–), György Szeke-

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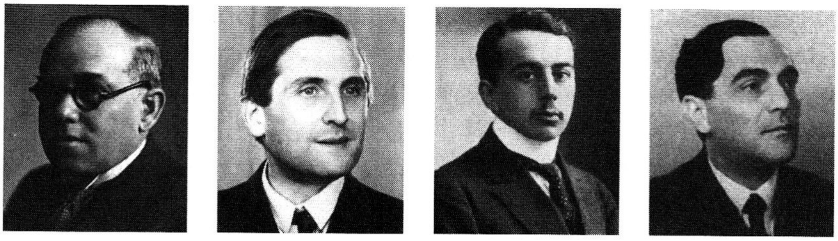
[2] For the most complete biographical account published so far, see Babai [4].

[3] The university was renamed Loránd Eötvös University in 1948.



Picture 1. Lipót Fejér, László Kalmár, Dénes König, Pál Veress

res (1911–), Pál Turán (1910–1976), Márta Wachsberger (later M. Svéd) (1911–) and Endre Weiszfeld (later E. Vázsonyi) (1913–).[4]

According to their reminiscences, their regular meetings at the Statue of Anonymus[5] in the City Park or their day-long excursions around Budapest had a great impact on their future life. It was a kind of open, peripatetic university, brought about by the socio-political discrimination of the era that afflicted most of them.

The intensive collaboration among members of this group of young mathematicians started already in their early twenties. Joint papers of Erdős–Szekeres [85], Szekeres–Turán [152, 153], G. Grünwald–Turán [110], Erdős–G. Grünwald [66, 68], Erdős–T. Grünwald [70], Erdős–Feldheim [63], Erdős–Lengyel [78], Erdős–Turán [86, 40, 88, 89, 90], Erdős–T. Grünwald–Weiszfeld [71], and G. Grünwald–Turán [110] appeared already in the thirties.

It might be that this intensive collaboration, which was quite unusual at that time, have been influential in shaping the future character of the mathematical life in Hungary. It is interesting to observe that — as far as I know — with a single exception (Erdős–Obláth [80]), none of them had a joint paper with any of their "masters", neither had their masters with each other.

1934–1938, Britain–Budapest

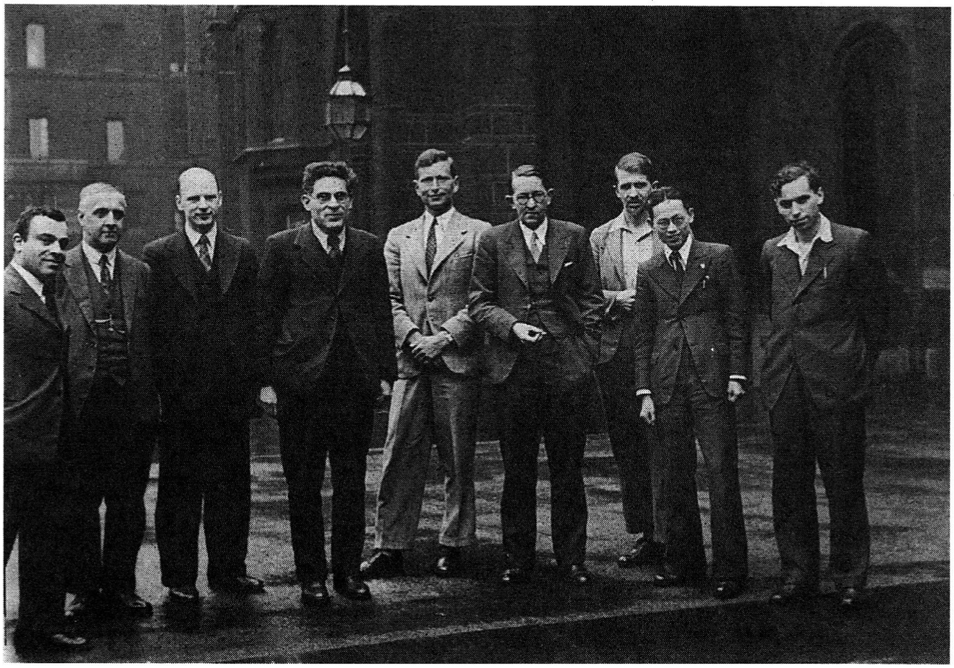
Arranged by Mordell and the Royal Society, Erdős had a fellowship at the University of Manchester. As he says "I went to Manchester in 1934 for three

[4] Alpár, Erdős, Klein, Lengyel, Szekeres, Vázsonyi and Wachsberger survived the war by leaving the country in the late 30's.

[5] Famous but unidentified Hungarian chronicler from the 12th century. He always signed his work as the Notay of King Béla.

Picture 2. Top row: László Alpár, Pál Erdős, Ervin Feldheim, Géza Grünwald
Middle row: Tibor Grünwald, Eszter Klein, Dezső Lázár, Görgy Szekeres
Bottom row: Pál Turán, Márta Wachsberger, Endre Weiszfeld

reasons: I didn't like the Horthy regime, as a Jew I didn't have prospects and I had the invitation [to Manchester]". He spent these four years mostly in Britain, besides Manchester, he visited Cambridge, Oxford, London, Sunderland, Sheffield and Bristol, returning to Budapest three times a year, each time for a few weeks. In Britain, he soon became acquainted with H. Davenport, T. Estermann, G. H. Hardy, H. Heilbronn, J. Gillis, Chao Ko, E. Landau, Levin, J. E. Littlewood, K. Mahler, L. J. Mordell, J. von Neumann, R. Rado, Todd, S. Ulam. No doubt, number theory was outstanding and dominating here. Very soon Erdős had joint papers with Davenport (1936, [23]), Gillis (1936, [65]), Ko (1938, [75, 76]), and Mahler (1938, [79]).



Picture 3. Cambridge, in the thirties: Mahler, Bailey, Zilinskas, Mordell, Heilbronn, Davenport, DuVal, Chao Ko, Erdős

1938–1940, USA

In 1938 Erdős decided to leave Europe. He had a fellowship at the Institute of Advanced Studies in Princeton. Leaving Budapest on September 3, he returned to Britain and sailed on the Queen Mary for New York on September 28. He was not to return to Europe until ten years later. He spent the first year mostly in Princeton, but he also visited a lot of other places during this period.

In a short time, he met S. Bochner, R. Brauer, J. Douglas, K. Gödel, W. Hurewicz, A. E. Ingham, Mark Kac, Sh. Kakutani, T. Nakayama, G. Pólya, H. Rademacher, I. J. Schönberg, C. L. Siegel, O. Szász, G. Szegő, J. D. Tamarkin, A. Tarski, Olga Taussky, E. Wigner, A. Wintner. Very soon he had joint papers with Wintner (1939, [99]), Kac (1939, [73]) and others.

3. Joint work of Erdős and Turán before 1942

As a very simplified comment, I would say that most of the time number theory dominated Erdős's mathematics. Analysis (interpolation theory, polynomials etc.) originated from his Budapest period. For the probabilistic aspect, the year 1938–1939 is decisive.

About the relationship between Erdős and Turán, how it started and how it developed, let me quote Erdős [60].:

"I met Turán in September 1930 at the University of Budapest, though we knew each others existence since we both worked for the mathematical journal for the high school students[6] and our first joint paper appeared there, i.e. a solution of a problem which we obtained independently. At our first meeting I asked him if the sum of reciprocals of the primes diverges or converges. He informed me that it diverges and he told me about the prime number theorem. 7 or 8 years earlier I learnt from my father that the number of primes is infinite."[7]

"We met and discussed mathematics nearly every day, until I went to Manchester, in 1934, after that date until early September 1938 I spent half my time in England, half in Hungary, but when I was in England, we corresponded a great deal."

"We collaborated for 46 years and wrote 24 papers on various branches of mathematics, our personal and scientific contacts never really stopped [until the death of Paul Turán] except during the dark days of world war II 1942–1945, when I was in USA and Turán was in Hungary."

During the years 1934–41 — and also before that — both Erdős and Turán were deeply involved in number theory and also in analysis: interpolation, polynomials.

They informed and asked each other about their own problems and results in number theory, but in most cases they did not work together on those problems. Quoting Erdős [57]: "[In 1930] we immediately noticed our common interest in number theory and in prime numbers in particular. ... Turán was more interested in analytic methods and I in elementary ones."

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[6] KöMal (Középiskolai Matematikai és Fizikai Lapok), the Mathematical and Physical Journal for Secondary Schools, founded in 1893 and since then is a determining component of mathematical education in Hungary.

[7] see also the comment on p. 112.

Perhaps one can say that in number theory during this period Erdős and Turán were involved in problems of different characters with few exceptions. These exceptions were the problem of $r_k(n)$ and the "Sidon" problem. (See on p. 104) In these years they wrote three short joint papers on number theory ([86], [87], [88]) and also [93], appearing in 1941. However, as we know, these "small" — 3 pages long — papers [88], [93] have a tremendous impact even today, especially the former. These papers became classics in combinatorial number theory and in combinatorics.

The role of combinatorics in Erdős's oeuvre, how combinatorial number theory developed, how combinatorics became more and more an essential part of Erdős's work, how the interaction between number theory and graph theory is manifested in his mathematics, is a long story. I formulate some thoughts about it in [147].

Erdős started working on interpolation theory, on polynomials, before leaving for Britain and continued this quite intensively after that. With changing intensity he has never stopped being interested in these subjects. His last paper on interpolation theory appeared in the year before his death (Erdős–Szabados–Vértesi [83]).

In [58] Erdős wrote: "Turán and I worked jointly for many years on interpolation. Our first big success was in 1934 when we proved that for every general point group the Lagrange interpolation polynomials converge in the mean. It is hard to understand why this was not discovered much earlier. As a whole we were very lucky in our joint work, most of it was new, we had bad luck only twice. We 'discovered' in 1934 that for every point group there is a continuous function $f(x)$ and a point x_0, $-1 < x_0 < 1$ for which the Lagrange interpolation polynomials $L_n\big(f(x)\big)$ diverge at x_0. We soon found out that S. Bernstein anticipated us by a few years. Our second [piece of] bad luck was the rediscovery of some theorem of W. Markov on extremal problems of polynomials".

In 1991 Erdős wrote [61]:

"We [Erdős and Turán] started our work 60 years ago. We proved that if the point group of the interpolation process is the set of roots of the Legendre polynomial, then for every Riemann integrable function

$$\int_{-1}^{1} \big(f(x) - \mathcal{L}_n\big(f(x)\big)\big)^2\, dx \to 0.$$

In fact, we proved a much more general theorem. We also obtained very sharp results about the distribution of the roots of orthogonal polynomials and also about the distribution of the roots of polynomials defined by various interpolatory properties. This work was carried on by many mathematicians and the subject is still very much alive (see the recent book of Szabados and Vértesi [150])".

One can find all these, and also the trace of the "bad luck" in the letters.

They wrote the sequence of papers [89, 90, 92] on interpolation in these years. Their paper on the discrepancy of sequences [91] originated from their work on interpolation. Their second and third papers on the distribution of the roots of polynomials [94], [95] appeared only in 1948 (when they met again, after 10 years, in Princeton).

Comparing with the rather short time of the preparation of [88], [93], it is interesting to observe the dates below:

Turán lectured in the Mathematical and Physical Association (Matematikai és Fizikai Társulat) in Budapest,

— on Interpolation I in May 1934, submitted the paper in March 1936, it appeared in 1937,

— on Interpolation II in Dec. 1935, submitted the paper in Nov. 1937, it appeared in 1938,

— on Interpolation III in April 1937, submitted the paper April 1939, it appeared in 1940, (dedicated to Professor L. Fejér on his sixtieth birthday).

Their last joint paper on the subject appeared in 1961 [97], both of them had papers on these or related topics (some with other coauthors) even later.

In these years Erdős (and Turán, too) worked on interpolation and polynomials also with other young Hungarians in Budapest: their joint papers appeared in 1936 with Feldheim [63], in 1938 with G. Grünwald [66, 67, 68], in 1938 with B. A. Lengyel [78] and in 1939 with T. Grünwald [70].

My impression is (and the correspondence supports this) that beside the dominating impact of Fejér, also Szegő, Kalmár, Pólya, and Fekete influenced their work in analysis; this can also be seen from the numerous remarks and references in their papers.

Though the larger part of the mathematics in the letters of that period is on interpolation theory, polynomials, distribution of zeros of polynomials, the quotations below will not reflect that. The correspondence on analysis was mostly on their joint work, they exchanged problems, results, proofs, thoughts about it, questions, answers went back and forth. Only the presentation of both sides of the correspondence on this subject could give a realistic picture. However this would be against my intention here.

For an authentic survey on these topics see Borwein [14], Erdélyi [28], Lubinsky [123] and Vértesi [164] in this volume, [5], Erdélyi–Vértesi [29], for a detailed source see, e.g., the book of Szabados and Vértesi [150].

4. Correspondence

The 42 years of correspondence between Erdős and Turán illustrate their mathematics and reflect the similarities and dissimilarities of their interest and personalities. It demonstrates the different atmosphere and circumstances in which they lived.

In the letters they not only discuss their joint work, but they also inform each other about their own results in other areas, and about results and problems of others (news of mathematics in Budapest, Britain, USA and all over the world). Usually Erdős only sketched the proofs while Turán gave quite detailed ones, occasionally several pages long. In 1938–1940 also Erdős, when he became more and more pessimistic about the future, started writing more detailed proofs. It is quite remarkable that they gave lectures not only about their joint works but also about each others' results in seminars (Erdős to the Seminar of Hardy and Mordell in Britain, Turán to the Seminar of Fejér in Budapest).

Their personal problems, caused by the outside world and their private life, have their place in the letters. The letters also testify their deep concern for their friends and colleagues, and the actual support they provided.[8]

As far as I know, Erdős wrote mostly to Turán, but several of these letters were addressed also to E. Klein, or occasionally to somebody else, too. He often asked Turán to inform others about certain parts of the letters.

Some letters of Erdős had no dates. In 1976–77, after the death of Pál Turán, I showed these to Erdős and he added here and there some comments and also indicated the assumed dates. His memory was legendary. He remembered even after several decades when and where he had met somebody, had posed or worked on or had published some problems. However these dates are not certain, therefore I added them in square brackets.

[8] One should not forget that in 1934 Erdős was 21, Turán was 24.

No letter from Turán to Erdős has a date on it. Although Erdős keeps asking Turán to date the letters, unfortunately Turán had never done so in these letters. Though the contents often yields some information on the date, here I do not even give an indication of it.

Erdős's character is well reflected in his letters: he is jumping from topics to topics. I had to make here several compromises. I kept the chronological order of the letters, however sometimes this makes it more difficult to follow the history of some mathematical results.

I provide supplementary explanations to the quotations, where necessary, in [ ]. If there is just a part of the letter, I indicate it by [..] before and after the quotation. All the letters were written in Hungarian. Both Erdős and Turán had a very special style and vocabulary. Therefore, alongside the English translation, a few hard-to-translate words and expressions will also be given in Hungarian.

Let me point out in advance that many of the Erdős-topics have roots in that period and are mentioned in the letters. The letters also testify that sometimes (like in the case of the Erdős–Ko–Rado theorem) the paper appeared several years after obtaining the result. In the commentaries after the letters I give some additional information.

Let me emphasize once more, it is my intention to present mostly the letters from Erdős to Turán and to give just a very few quotations from the letters of Turán to Erdős.

5. Letters and Snapshots, 1934–1936

Unless explicitely indicated, the letters below are from Erdős to Turán.

The almost random selection from the possible quotations reflects, besides the enormous richness of Erdős's mathematics, his strong attachment to his friends, to the life in Budapest, his concern about others, how attentive and helpful he was, how interested he was in other matters beside mathematics. Let me start with one of the first letters from Erdős to Turán.

1934

Dear Turán P.,[9]

I succeeded to prove in an elementary way that the number of b-abundant integers less than n is at most $n/e^{e^{b/10}}$. I do not know whether or not your results are better than that. I fought a lot but without success with

$$\sum_{d|n} \frac{1}{\log d} \quad \text{and} \quad \sum_{p|n} \frac{1}{\log p}.$$

Presumably I arrive Friday, we can play tennis already on Saturday. Continuum greetings

C with you
$\mathcal{E}. \mathcal{P}.$

Commentary. As Szekeres writes [151]: "In those early years George Cantor was his [Erdős's] hero and his letters to his friends always ended with the phrase, 'let the spirit of Cantor be with you'. This was later shortened to 'C with you'."

b-abundant numbers are those integers n for which the sum of the divisors is $> bn$. Already Euler considered perfect numbers, for which the sum of divisors is $2n$. Davenport [21] and Chowla [19] proved that abundant numbers have an asymptotic density. In [31] Erdős gave an elementary proof. This was one of the first favorite subjects of Erdős. See also [32, 33]. The papers [34], [38], [39] emerged from these and are fore-runners of probabilistic number theory, see Erdős–Kac [73, 74], and Erdős–Wintner [99]. This also led Erdős to the characterization of additive number theoretic functions, considering the behavior of $\big(f(n+1) - f(n)\big)$.

For the importance of that subject in Erdős's work, and how it led to the investigation of the value distribution of additive arithmetical functions and finally also to probabilistic number theory see Turán [161], Baker–Bollobás [7], Ruzsa [139], the book of Elliott [26], [27] and Sárközy [142].

Autumn 1934

[..] I proved the following: let $\varepsilon_n \to 0$, and let c_n denote the density of integers having a divisor between n and $n^{1+\varepsilon_n}$. Then $c_n \to 0$ as $n \to \infty$. [..]

[9] In Hungarian: Kedves Turán P.; in Hungarian we put the family name first.

Commentary. Erdős published this result in [37] as a best possible strengthening of a result of Besicovitch [11]. For further results see [157]. For more about the importance of Erdős's work on the distribution of divisors see the books Halberstam-Roth [115] and Hall–Tennenbaum [116].

07. XII. 1934. (Manchester)

TP,[10] I learned that you were ill. I hope by the time you receive my letter you will be better. I will arrive on Thursday. We have continuum amount of things to discuss. A few days ago I proved in an elementary way the following: Every integer large enough and not of the form $4k+1$ is the sum of the square of a prime and of a "quadrat frei" number (I can determine the bound, how large is enough). I lectured on your method at Hardy [at Hardy's seminar] quite in details. At Mordell's seminar I only mentioned your results. Estermann proved the following: The number of solutions of the equation $ax^2+by^2=n$ is $< 2d(n)$ (where $d(n)$ is the number of divisors of n).

C with you, see you
E. P.

Picture 4. Erdős, in Mordell's garden

Commentary. The above result of Erdős appeared in [35]. In 1954 [51] he proved that there is a sequence of integers $a_1 < a_2 < \ldots <$ for which $\sum_{a_n < x} 1 < c(\log x)^2$ and every integer can be written in

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[10] very often they used only initials of each other.

the form $p + a_i$ where p is a prime. For more results and history see [115].

1934 or 35

Dear Turán P. and Klein E.

Gyuri [Szekeres] should send his results about Abelian groups to Kalmár immediately. I am glad that Márta [Wachsberger] cheered up. I recommend the problem about the decomposition of the square [into distinct squares]. (Tell it to GT [Tibor Grünwald]), does he show some sign of life?. Do not let our company die, TP should buy a pullover as soon as possible. Nobody has written to me but you. Try to exert your influence to change this. I wrote to everybody because I write my letters to the whole company.

[..]

Send our paper on interpolation theory to Hardy as soon as possible, very probably he will answer then everything, he is a very bad correspondent. [..]

Commentary. The problem above is discussed in the paper Brooks–Smith–Stone–Tutte [17], Tutte [163].

1934 or 1935

Klein E, TP

I received your letter Monday morning. Now I am answering it.

I understood Borel the following way: In the decimal expansion of every real number with the exception of a set of measure 0 the frequency of three consecutive 1's is asymptotically $n/1000$. From this one can see the general theorem, too. Did you try to struggle with the determination of the frequency of the digits of squares? Do you have any results concerning the decomposition of the square? In a few days I will write to ϱ [$\varrho \leftrightarrow$Ro; Rózsa Politzer, later Rózsa Péter]. Solve the following: Let $E(x)$ denote the integer part of x, then

$$E(x) + E\left(x + \frac{1}{m}\right) + E\left(x + \frac{2}{m}\right) + \ldots + E\left(x + \frac{m-1}{m}\right) = E(mx).$$

Just now, during the lecture of Mordell I produced a neat solution for the problem of GT and Kemény–Spitz on the decomposition of the plane. I was happy to learn about the earning of 10P, Mordell expressed his highest appreciation of the solution of M. Wachsberger and E. Weiszfeld. I also told him the solution of P. Turán and he liked it, too. I told him just yesterday your Hardy–Ramanujan method, today LMS [J. of London Math. Soc.] arrived, he also read it from there and he liked it very much. I will arrange what you have to pay for the reprints.

There is a paper of Chowla in the July issue of the LMS about quadratic forms. Try to steam roll [lehengerelni] the problem, which is not completely solved there. Todd likes very much Epszi's [Epszi=Eszter Klein] solution about the convex four-gon. I would like to see more sign of life in your next letter. I was happy to learn that Márta [Wachsberger] is showing some sign of life, but she should write. It is interesting that the first part of P. Turán's letter (about the movie) is almost unreadable, later it improves remarkably. I am glad that Markov is crushed.

Now you will have no excuse to add Schur's theorem [to our paper], the best would be if we wrote it down at Christmas and sent it then after [we send the paper on] Markov. I think that in our Interpol[ation] paper only the square-integral should be written down in details. I will write again in the next few days.

Write! C with you
$\mathcal{E}.\ \mathcal{P}.$

1935 — perhaps 1934

[..] Today I proved the following: Let $f(n) > n$, $n \to \infty$, and c_n denote the density of those integers which have one and only one divisor between n and $f(n)$. Then $c_n \to 0$. [..]

Commentary. See the Commentary to the letter of Autumn 1934.

10. II. 1935. (Cambridge)

TP, Klein E,

Now I am in Cambridge, infinitely many mathematicians are here, Landau, Heilbronn, Linfoot and the others from Cambridge. It is a pity that you are not here. Now the following problems are studied here: Take the unit square and a measurable set in it. Given that for every point of the square there is a rectangle with sides parallel to the sides of the square so that the density of the point set in it is $> c > 0$ (density means the following: we take the measure of the set in the rectangle and divide it with the area of the rectangle). Prove that the measure of the point set cannot be arbitrary small, i.e. it is $> f(c)$. Besicovitch proved that. However if we delete from the condition that the sides of the rectangle are parallel to the sides of the square, only the spirit of Cantor knows whether the theorem remains true or not. We do not know either what happens if instead of rectangles we consider convex curves.

Tomorrow, on Sunday, I will have lunch at the Rados' place, also Heilbronn and Landau will be there. I was introduced to Landau, according to his habit he addressed me in Hebrew, yesterday he (trivial being) was in the synagogue, even more, he wanted Heilbronn to go with him, but he [Heilbronn] ran away. Siegel proved that the Klassenzahl $> d^{(1/2-\varepsilon)}$. TP, I have received your postcard. Markov! [Write it down.]

Write, C with you.
$\mathcal{E}. \mathcal{P}.$

Commentary. The problem above about measurable sets in a rectangle is discussed in the monograph of Saks [141]. For rectangles of arbitrary direction the answer is no, this was proved by Nikodym [130].

11. III. 1935. (Manchester)

[..] As to dying, even Cantor and Galois died, and it is very probable that even Hilbert and Einstein will die. Moreover, it is most likely that not even the one who solves the Continuum problem will live forever. I thought I would arrive on the 21st, but Titchmarsh and Hasse will be here and therefore it is possible that I would only arrive Saturday morning at 6:20. In any case, call me Saturday noon and we meet on the 24th, Sunday on the excursion. I think Gy. Szekeres will be there. [..]

[..] What is with GT? Does Epszi show any sign of life?

C with you, see you in 12 days
$\mathcal{E}. \mathcal{P}.$

1. V. 1935. (Manchester)

TP, today Mordell asked me to speak in his seminar about your theorem on the prime factors of polynomials. [..]

6. V. 1935 (Manchester)

TP, in our correspondence I lead by 3 over 0. Now it is Sunday afternoon and I make an excursion alone, only van der Waerden's Höhere Algebra is with me, just now I am reading the second proof of the Wohlordnungssatz.

This morning I proved that the number of integers $< n$ with the number of prime-factors $> \log\log n$ is asymptotically $n/2$. This was an old problem. However, I cannot prove that the number of integers $m \leq n$ for which the largest prime-factor of m is greater than that of $m+1$ is asymptotically $n/2$. This must be true, it is very easy to prove if we consider the smallest prime factor instead of the largest one. [..]

[..] Let $a_i > 0$, $\sum_{i=1}^{\infty} a_i = 1$. Prove that

$$a_1 + (a_1 \cdot a_2)^{1/2} + (a_1 \cdot a_2 \cdot a_3)^{1/3} + \ldots + (a_1 \cdot a_2 \ldots a_n)^{1/n} + \ldots$$
$$< e(a_1 + a_2 + \ldots).$$

I cannot prove the convergence of the left side either.

We meet in 4 weeks on the excursion.

Write, C with you.
E. P.

P.S.: I proved the convergence just in this minute.

Commentary. About the problem above on the largest prime factor Erdős wrote in [61] (almost 60 years later!): "This problem seems un-attackable. Pomerance and I have some much weaker results [81]."

For the proof of the inequality see the letter of 7. XII. 1938.

4. XI. 1935 (Manchester)

[..] Do you meet on *Wednesdays* [at the Statue of Anonymus]?

I tried to read topology, but I was not interested in it. In my despair I read from Reye Geometrie der Lage 120 pages, it is a beautiful book. Even more *sensation:* The ζ function interests me and I proved that $\zeta(1+it) = o(\log t)$ and that $\limsup \frac{\zeta(1+it)}{\log t} > e^c$, unfortunately my proofs are the same as Weyl's.[11] [..]

---

[11] It should be $\limsup \frac{\zeta(1+it)}{\log\log t} > e^c$,

Commentary. Eszter Klein — at my request on 7 June 2001 in Sydney — kindly put down her recollections about how the "Anonymus group" started:

"In the autumn of 1928 we were first year students at the university. Lipót Fejér was lecturing on analysis, the seminar to that lecture was given by Pál Veress. In one of these seminars Veress mentioned the book of Pólya–Szegő and suggested that a few of us join together and work through the problems (exercises) in that book — he could not find any better introduction to analysis. This suggestion started our small group. The Statue of Anonymus [in the City Park] was a very fitting place for the meetings. As far as I remember, we used to meet on *Wednesday* afternoons. We borrowed a copy of the book from the library and every week copied 2-3 exercises from the book to work on it for the next week. Very probably this started in the Spring of 1929, the first members of the group were Pál Turán, Márta Wachsberger, Eszter Klein, but the group was growing fast. György Szekeres from the Technical University, occasionally Miklós Ság also from the Technical University joined us. Soon we started working on a problem book in physics, too. Even the summer break did not interrupt the regularity of our meetings, in case of bad weather we met in the room of the Mathematical Seminar.[12] As far as I remember, Pál Erdős joined us in the next year, bringing Tibor Grünwald with him. Occasionally Géza Grünwald, who studied at the University of Szeged, and László Molnár, who studied in Zürich, also showed up."

15. XI. 1935 (Manchester)

[..] Here is a beautiful conjecture: Let $f(z)$ be a rational function, if $f(z), f\big(f(z)\big), f\Big(f\big(f(z)\big)\Big), \ldots$ converges in a region, then [excluding some trivial cases] it must converge to a constant. This is an old conjecture, even Julia who investigated a lot of things of this kind, was unable to settle it. [..]

I talked to Hardy, he asked about you, he gave us continuum many reprints. [..]

See you in 4 weeks
C with you. Write
E. P.

---

[12] at the University, at Múzeum krt. 6-8.

Picture 5. Gallai, Erdős, Svéd at the Statue of Anonymus, around 1990

Commentary. Most probably Erdős himself has never worked on Julia-sets. On the subject see, e.g., the book of Milnor [127].

29. XI. 1935 (Manchester)

[..] I hope I can prove the following old conjecture: almost every integer has two divisors such that $d_1 < d_2 < 2d_1$. [..]

[Erdős's addendum on the letter 44 years later] "Even today (5. IV. 1977) it is not proved".

Commentary. Erdős proved in [47] that the density of integers which have two divisors such that $d_1 < d_2 < 2d_1$ exists, and it was proved by Maier and Tenenbaum only 50 years after Erdős wrote about this in his letter, that the density is 1 [124]. For more results and history see Halberstam–Roth [115] and Hall–Tenenbaum [116].

Let $r_k(n)$ denote the maximum number of elements of a sequence of integers in $[1, n]$ not containing a k-term arithmetic progression.

In December 1935–January 1936 Erdős spent a few weeks in Budapest. Very probably Erdős and Turán worked on the problem of $r_k(n)$ during his stay in Budapest, but possibly they started working on that much earlier. Erdős, in a handwritten recommendation letter for Szemerédi (in 1974) writes that "we conjectured in 1932 that $r_k(n) = o(n)$".

Here are some quotations from the letters:

[From Turán to Erdős] (Budapest)

Dear EP

[..] I wrote down the three-term arithmetic progression; if $r(N)$ denotes the number of elements of the sequence of maximal density below N [not containing a 3-term arithmetic progression], then, as you said, $r(17) = 8$ is true, with a few checking $r(18) = 8$ is also true. From $r(17) = 8$ not only $r(N) < \frac{8}{17}N$ follows, but iterating your idea also $\left(\frac{8}{18} + \varepsilon\right)N$ follows for $N > N_0(\varepsilon)$. Namely it is trivial that $r(35) \le 16$, $r(71) \le 32$ etc. From $r(18) = 8$, $r(N) < \left(\frac{8}{19} + \varepsilon\right)N$. [..]

[..] $r(N) < \frac{1}{3}N$ starts very late, because $r(41) \ge 16$, namely 1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41 is such a sequence. [..]

C with you Write
T.P.[13]

10. II. 1936 (Manchester)

TP! $r(21) = 9$ 1, 3, 4, 8, 9, 16, 18, 19, 21. $r(22) = r(23) = 9$.

I also proved that for n large enough $r(n) < (1/3 + \varepsilon)n$.*

What is with E. Weiszfeld? Why doesn't he write? On Thursday I go to Cambridge, I will stay there until the 19th. An older theorem of mine is the following: Let $a_1, a_2, \dots, a_x, \dots$ be an infinite sequence of integers. If the number of a_i's up to m is $> m^{1-c_1/\log\log m}$ (c_1 is a not

[13] TöPö = TP phonetically in Hungarian.

too large constant) then for infinitely many n's the equation $n = a_i^2 \ldots a_r^2$ has more than $n^{c_2/\log\log n}$ solutions. I wrote in an earlier letter $m^{c/\log\log m}$ instead of $m^{1-c/\log\log m}$, in that form the theorem is obviously not true. The American paper of Lipi [Lipót Fejér] about multiple monotone power series has appeared.

C with you Write
$\mathcal{E}.\ \mathcal{P}.$

[At the end of the letter Erdős added:]

*The proof is wrong!

21. II. 1936 (Manchester)

[..] The Szekeres conjecture is the following: The maximal number of integers up to $\dfrac{3^k+1}{2}$ such that no k of them form an arithmetic progression is at most 2^k. According to that $r(41) = 16$. [..]

[..] Problems: (1) (Littlewood) on a triangle-shaped billiard table a moving ball either describes a closed curve or gets arbitrarily close to every point. (It is not solved even in the special case when we have a right angled triangle). We suppose that if the ball arrives into a corner, it will be reflected along the same curve.

(2) Let $f(x)$ be a polynomial of degree n, the coefficient of x^n be 1. The sum of the length of the intervals where $|f(x)| \leq 1$ is < 4. It doesn't seem to be too difficult, but I could not prove it yet. [Added:] I proved it in this moment.

(3) Do you remember my conjecture:

$(z-a)^n$, $|a| = 1$ is the polynomial for which the area [of the set of points] in the unit disc where it [i.e., the polynomial's absolute value] is > 1 is the largest one (the zeros are in the unit disc). This conjecture is false! I am interested very much in the question, whether it is possible to find a polynomial, which is almost everywhere > 1. Now I am reading Hilbert's Grundlagen der Geometrie, I have already read 100 pages, it is very interesting.

See you in 4 weeks.
C with you. Write!
$\mathcal{E}.\ \mathcal{P}.$

[From Turán to Erdős] (Budapest)

[..] So now $r(N) < \left(\frac{8}{21} + \varepsilon\right) N$, this is already quite close to 1/3. It is possible that also for 24 we can only take 9, if so, we would have $r(N) < \frac{1}{3}N + 8$.

Why do you say that $r(41) = 16$ is the triumph of the Szekeres' conjecture? [..]

Commentary. They submitted quite soon, on 6 June 1936, their less than 3 pages long paper "On some sequences of integers" to J. London Math. Soc. In this paper [88] they prove that $r_3(N) < \left(\frac{4}{9} + \varepsilon\right) N$ for N large enough, and they formulate the following:

"It is probable that $r(n) = o(n)$".[14]

They write about Szekeres' conjecture:

"An immediate and very interesting consequence of this conjecture would be that for every k there is an infinity of k combinations of primes forming an arithmetic progression.

Another consequence of it would be a new proof of a theorem of van der Waerden that would give much better limits than any of the previous proofs."

Behrend [10] constructed a sequence that disproves Szekeres' conjecture.

The problem about primes forming k-term arithmetic progressions is still open.

Erdős writes in [62]: "we did not at first realize the difficulty of our conjecture" ...

For almost twenty years nothing happened, the first significant result was obtained by K. F. Roth [134, 135]. The next breakthrough is due to Szemerédi who proved the conjecture for $k = 4$ [154], and then for every k [155]. In the last three decades this "originally innocent-looking problem" became the core of several theories, like a branch of ergodic theory, see, e.g., Furstenberg [103], Furstenberg and Katznelson [105] and Furstenberg [104] in this volume. For a recent very important development see Gowers [106, 107, 108]. The Szemerédi Regularity Lemma [156] (the basic lemma in Szemerédi's proof), became a general powerful method in combinatorics, for

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[14] Erdős writes somewhere that originally they formulated their conjecture only for $r_3(n)$ but they meant the general case.

its importance see the survey papers by Komlós–Simonovits [119] and Komlós–Shokoufandeh–Simonovits–Szemerédi [120]. Not even a survey paper but a monograph could give a worthy overview of the subject, of the generalizations, of different density theorems, of it's influence on and relation to ergodic theory and the Freiman, Freiman–Ruzsa theory etc.

—

13. III. 1936 (Manchester)

[..] I proved the following in a quite elementary way. Let $0 < a_1 < a_2 < \dots a_x \leq n$ be integers, $a_i \nmid a_k a_\ell$, then $x < \pi(n) + 2n^{2/3}$, on the other hand x can be greater than $\pi(n) + n^{2/3}/(\log n)^2$. [..]

Commentary. This was published in 1938 [40] where an analogous problem for the condition $a_i a_j \neq a_k a_\ell$ is also considered. (See also p. 124.)

The paper has a great importance in the history of combinatorics. It contains the first germs of extremal problems both for integers and for graphs: graph theory is applied to number theory the first time, finite geometry is applied to graph theory the first time, and the first extremal graph theorem is formulated here: if a graph on n vertices contains no cycle of length four, than the number of edges is at most $cn^{3/2}$ where c is an absolute constant. For more details see [13, 145] in this volume. See also the letters of 20. XI. 1936, 7. XII. 1938, and the commentary after the second one.

The general extremal problem — to formulate it in a very simplified form — is as follows: how large can a certain structure be if it does not contain a certain given substructure? It is remarkable that, unlike extremal graph theory, extremal number theory — as a systematic theory — began to develop only recently, though the problem above, or the $r_k(n)$ problem, the additive and multiplicative Sidon-problems etc. were present already in the thirties (before extremal graph theory existed). The interesting history of the interaction between combinatorics-graph theory and number theory is demonstrated nowhere else better than in the oeuvre of Paul Erdős. This was the subject of [147].

14. VIII. 1936 (Blackpool)

[..] Now I have been away already for a week. There is not much news. Yesterday I flew 5 minutes, it was very pleasant, the landing was not very unpleasant either. [..]

13. X. 1936 (Manchester)

TP, I have received your letter. Mordell sends the enclosed 10 shillings to Sidon, I think one can change it better in Pest [Budapest], perhaps Heilbronn and Davenport will also send some money. [..]

[From Turán to Erdős] (Budapest)

[..] Sidon usually visits me twice a week and he is very talkative. First he wanted to send the money back to Mordell. Already there are 12 [mathematicians] who are willing to give 1 Pengő each month.[15]

[From Turán to Erdős] (Budapest)

[..] Sidon visited me twice, I gave him 35P, he asked the amount for 2 weeks, I still have 48P. He showed two of his papers in preparation, one starts as: "Herr Erdős believes ...", it is related to your Fourier-paper. It is time to finish it. His paper in Studia has appeared, I had to force him to give me a copy of it.

Now I continue, Sidon was here, I gave him 10P, I wanted to give him all, he didn't want to accept it, he gave up his apartment. I am afraid he will do what he used to say.

Lengyel came to say 'good by'. The dissertation of Götö [Tibor Grünwald] has been corrected, it is a very nice one. I thank you very much for your help with the Proceedings-paper. [..]

Commentary. Legends and truth

Everybody knows the anecdote: Erdős and Turán visited Sidon and he greeted them with saying: "visit some other time and some other person". This story may suggest that Erdős and Turán had no real connection with Sidon. The letters prove just the opposite, in the given period they mention Sidon very frequently in their letters. The letters testify that both their mathematical and human contacts were very profound. About their human relation to Sidon,

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[15] Pengő was the largest unit of Hungarian currency. At that time there were 25 Pengő to the pound.

(who was about twenty years older) it had a threefold feature: since Sidon "the tragic fated Hungarian mathematician" ([161]) had some psychological difficulties, they followed with care the process of the publication of his papers, they supported Sidon and also collected money from others (Erdős abroad and Turán in Budapest) for him, often saving him from starvation. Meeting him regularly also helped him to keep in touch with the outside world.

The next letter is the first "problem letter" as the fore-runner of the many "problem papers" that formed a very important and characteristic part of Erdős's publications.

It is interesting how precisely Erdős describes in 1961 [54] the start of his problem lectures and problem papers. "I gave several talks on unsolved problems at various places (Moscow, Leningrad, Peking, Singapore, Adelaide). In the autumn of 1959 I gave a series of talks on unsolved problems at the Mathematical Institute of the Hungarian Academy of Sciences[16] and most of the problems discussed here were discussed in my lectures.

My first talk on unsolved problems was given on November 16, 1957 at Assumption University Windsor, Ontario, Canada, a paper on this talk appeared in the Michigan Mathematical Journal 4 (1957), 291–300, [53] and there is a considerable overlap between this paper and the present one."

I think that the paper [52] is already a sort of problem paper.

20. XI. 1936 (Manchester)

TP! trivial being! I have received your letter, you should have written already a week ago.

The spirit of Cantor was with me for some length of time during the last few days, the results of our encounters are the following:

(1) The quadrature-convergence is also possible at point-groups where the distance between two consecutive zeros on the unit circle is $< c/n^3$. Very probably this cannot be sharpened.

(2) One can find a continuous function $f(x)$ such that the arithmetic means of the Lagrange parabolas belonging to the $T_n(x)$ abscissas diverge everywhere. I use the method of G.G.

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[16] [54] appeared in the periodical of the Institute, when the editors started a new section of unsolved problems.

(3) For every $x = x_0$ there is a continuous $f(x)$ for which $f(x_0) = 0$ and $\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{m=1}^{n} \mathcal{L}_m\big(f(x_0)\big) = +\infty$. This is the truth for $U_n(x)$ as well as for $T_n(x)$ abscissas. I use a method similar to the one I showed that for $T_n(x)$ abscissas and a proper $f(x)$, for $x_0 = \frac{1}{2}$, $\lim_{n\to\infty} \mathcal{L}_n\big(f\big(\frac{1}{2}\big)\big) = \infty$.

(4) For every Riemann integrable $f(x)$ and $T_n(x) =$ abscissas $\lim_{n\to\infty} \int_{-1}^{1} \big|f - \mathcal{L}_n(f)\big|^k = 0$. The same method [is used] as in our joint paper.

(5) I can solve the following problem: find the polynomial of degree n, $\big|f(x)\big| \le 1$ for which $\big|f'(x_0)\big|$ takes the maximal value, under side condition $f''(x_0) = 0$. This is a generalization of Schur's problem. The extremal polynomials are our polynomials.

(6) Let $\big|f(x)\big| \le 1$ in $[-1, 1]$, if $\big|f(x)\big| \ge T_n(x)$ for $x = x_0$, then $\big|f'(x)\big| \le \big|T_n'(x)\big|$ [for $x = x_0$].

(7) Let $a_1 < a_2 < \ldots a_x \le n$ be a sequence of positive integers such that none of them divides the product of k others, then $x < \pi(n) + O\left[\frac{n^{2/(k+1)}}{(\log n)^2}\right]$, the remainder term is sharp.

(8) Let $a_1 < a_2 < \ldots < a_x \le n$ be a sequence of positive integers such that all the products $a_i a_j a_\ell$ are distinct, then $x < \pi(n) + O(n^{2/3+\varepsilon})$, here one can improve the remainder term, just I am not able to do that.

(9) Let $m = p_1 p_2 \ldots p_x$, $p_1^{c_1} = p_2$, $p_2^{c_2} = p_3, \ldots p_{x-1}^{c_{x-1}} = p_x$, $c_i \ge 1$, then for almost all m the number of c_i's larger than k is $x/k + o(x)$.

Problems:

(I) Determine the maximum of $f'(x)$ if $-1 \le x \le 1$, $\big|f(x)\big| \le \big|\sqrt{1 - x^2}\big|$ ($f(x)$ has degree n). Very probably the maximum is $2n - 2$. $\sqrt{}$**

(II) Determine the minimum of $\int_{-1}^{1} \big|f(x)\big|\, dx$ if $\max\big|f(x)\big| = 1$ for $-1 \le x \le 1$ ($f(x)$ is of degree n). This would solve the following problem: at a given total variation what is the maximum of $\big|f'(x)\big|$. For an $f(x)$ of degree 2 this maximum is $\frac{1+\sqrt{2}}{2}$ if the total variation is 1.

(III) Let $a_1 < a_2 < \ldots \le a_x \le n$ be a sequence of integers such that all the products $a_{i_1} \ldots a_{i_r}$ are distinct, then $x < \pi(n) + O(n^{1/2})$.

I will write about the other problems later. I have forwarded Sidon's letters. I hope, Feldheim has already sent the paper of Marcinkiewicz, I have sent Marcinkiewicz our first Annals paper. I have never written such a long letter.

Greetings to the whole company.

C with you, write and solve the problems.
$\mathcal{E}.\ \mathcal{P}.$

Will your Bloch paper appear in Acta?

$\sqrt{}$** [Erdős's comment] The condition means that it is inside the circle. I state that the extremal polynomial is the following: Let $x_1, x_2, \ldots, x_{n-1}$ be the zeros of $T_{n-1}(x) = 0$ and let $f(x)$ be the extremal polynomial, then $f(-1) = f(1) = 0$ $f(x_i) = (-1)^i\sqrt{1 - x_i^2}$, i.e. $f(x)$ is tangent to the circle in the zeros of $T_{n-1}(x)$ and $|f'(1)| = |f'(-1)| = 2n-2$. This must be true. It is beautiful. Isn't it?

Commentary.

The assertion formulated in (2) was already stated in [68] but Erdős noted in [49] that the proof gave only a weaker result, and he stated that he is able to get the result for almost all x. In fact, the proof of this appeared in a paper of Erdős and Halász [72].

Problem I was solved by Q. I. Rahman [133].

For more information on interpolation, polynomials see Borwein–Erdélyi [15], Erdélyi [28], Borwein [14], [123], the monographs Borwein–Erdélyi [16], Szabados–Vértesi [150] and the papers Erdős–Halász [72], [63], Névai [129], Vértesi [164].

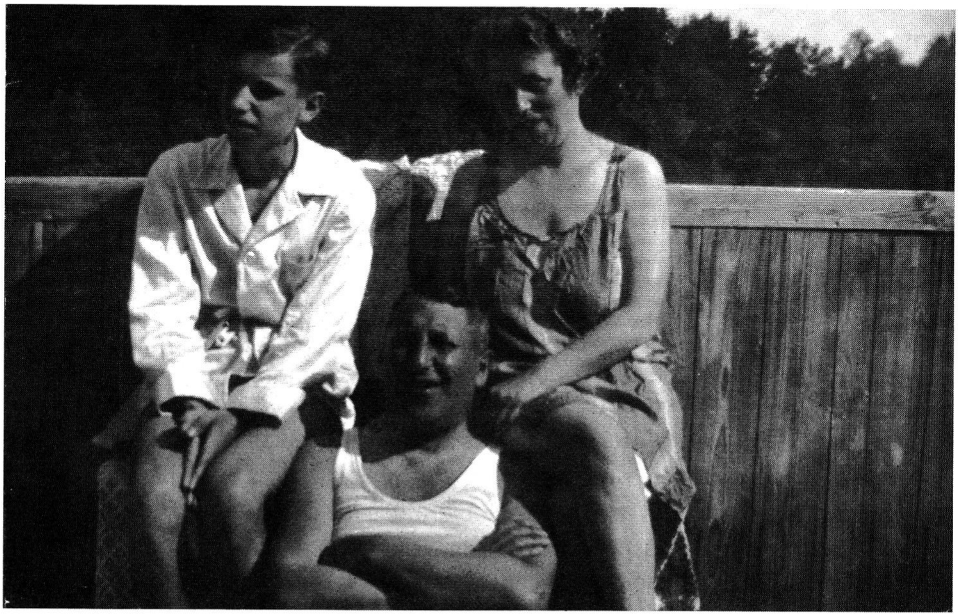
[From Turán to Erdős] (Budapest)

[..] Gy. Szekeres was here on Saturday, we were together at the Casals-concert, in the break I asked him to prove the Minkowski theorem about determinants. After the break we had the Schumann cello-concerto, he proved it during the first movement. I had needed half an hour. Obviously the proof was the same. [..]

[..] I received the letter of Hardy, he is extremely kind, I will write to Rogosinski too. [..]

[From Turán to Erdős] (Budapest)

[..] Your father mentioned the prime number theorem [in the school, in his class] in grade VIII and spoke about twin primes. Perhaps we shall live to see in the future that these problems will become common topics of discussion and people will queue up to get tickets for lectures of famous mathematicians. [..]



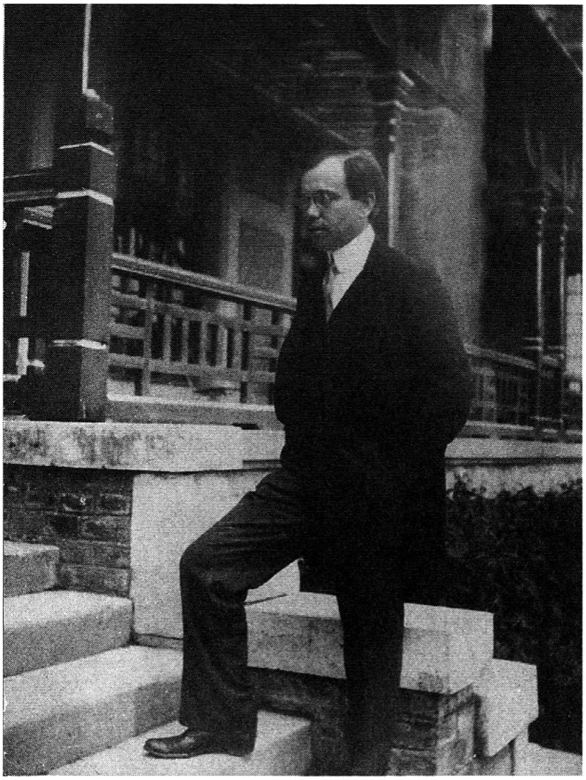
Picture 6. Erdős and his parents, Lajos Erdős and Mrs Erdős (Anna Wilhelm), August 18, 1930, Reifnitz

Commentary.

Both Erdős's mother and father were mathematics teachers, his father wrote expository articles in the high school journal and about teaching of Mathematics. When I asked György Szekeres not so long ago, how and when Erdős became interested so much in number theory, his answer was: in the crib. The above quotation may support that.

[From Turán to Erdős] (Budapest)

[..] The first seminar of Fejér was yesterday. Now Lipi [Lipót Fejér] wants to introduce a new system, he wants to discuss more Fourier series, he wants to turn the interest of the students towards that. Of course he realizes that now he is overwhelmed with problems [milyen bajok szakadtak



Picture 7. Lipót Fejér, Tátralomnic 1912

árva fejére] because he has neglected to influence the students, now all his PhD students investigate problems about which he does not know anything or just a bit. He also complained saying that now he must work hard to cope with his students, although in 1905 he had already given lectures on analytic number theory, before Landau's book appeared. [..]

> **Commentary.** Both Erdős and Turán were PhD students of Fejér and they both wrote their PhD thesis on number theory [30], [158].

6. The spring of 1938

Both the gradually deteriorating political situation, Erdős's decision — taking a few months — to leave Europe can be traced from the letters February-September 1938.

In 1934–1938 Erdős returned to Hungary quite regularly, three times a year. Each time he gave the exact date much in advance, when he would arrive and what he was planning to do already on the first day. It was still so in his letter:

22. II. 1938 (Manchester)

TP, sad news, the spirit of Cantor took Landau.[17]

Trivial being, why don't you write. The spirit of Cantor avoids me, yesterday I was thinking of number theory a lot, besides a few conjectures I had no success.

Write! See you in 4 weeks.

C with you
$\mathcal{E}. \mathcal{P}.$

III. 1938 (Manchester)

[..] I get back around 22. [..]

In the letter below there is still no indication to change his plan to return to Budapest.

12. III. 1938 [the day of the Anschluss] (Manchester)

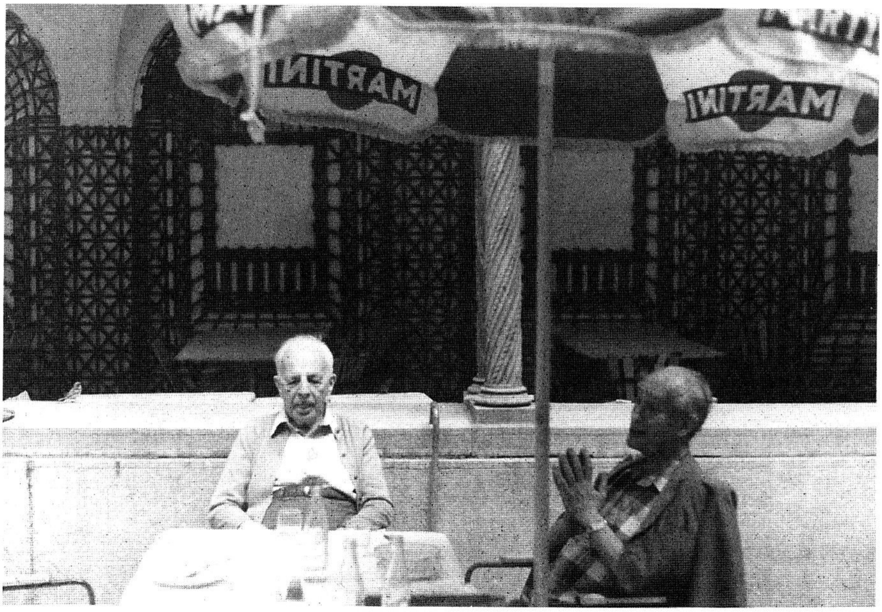
TP I received your card. Davenport has a friend in Australia, but Davenport does not think that Australia is a good place, we will speak about it, (if we live), now that Szegő became a full professor, I expect more from him. Kemény wrote about the Generali [Insurance Company], "Erdős and Turán were active in that", I am glad that our action helped him.

Let $r < k$, $2k \leq n$, let A_i be a set of k-element combinations [k-tuples] of $a_1 < \ldots < a_n$.

Let $A_1, A_2, \ldots, A_x$ be a family of k-element combinations with the property that every two of them has at least r common elements, than $x \leq \binom{n-r}{k-r}$. Yesterday I proved the conjecture for $r = 1$ in the special case $k \mid n$. The conjecture could have beautiful applications in number theory.

C with you
$\mathcal{E}. \mathcal{P}.$

[17] In Erdős's language, Landau died.



Picture 8. Gábor Szegő and Pál Turán, in the sixties

This is the moment when the deteriorating political situation (most probably the Anschluss on March 12) made Erdős change his plans, he cancelled his trip back to Budapest. He was hesitating for about three months to come back, finally he decided to return at the beginning of July to take farewell and to leave Hungary.

27. III. 1938. (Manchester)

TP, inscrutable fate wanted us not to meet at spring, I hope that during the summer (at the end of May) we can meet again. Don't forget Interpol[ation] III. My parents wrote, that you did something concerning the ζ function, write about it as soon as possible. I hope that Szegő will come to Budapest during the summer, however I am not sure about it. It would be very good if you could write to him before it is too late, perhaps now he can do something for you in America. I have much more confidence in it than in Australia.

With the help of an idea of Ko I settled my combinatorial problem for $r = 1$. I am not sure whether I wrote that problem to you, to be on the safe side I write it now: Let us consider a family of k-combinations of an n-element set with the property that every two combinations have at

Picture 9. Ko and Erdős, early thirties

least r common elements, then the number of k-combinations is $\le \binom{n-r}{k-r}$. I forgot: $n \ge 2k$.

The theorem would have applications in number theory.

The events of the last few weeks cause great excitement, one can divide the people into three groups, there are people who expect the world war by 38, by 39 or by 40, we did not meet anybody who expected it for 41, anyhow, hurry up with the writing of Interpol III. Write! Regards to everybody

C with you
$\mathcal{E}. \mathcal{P}.$

Commentary. The result above and the generalization of it for arbitrary r was published only more than twenty years later, in

1961 [77]. This Erdős–Ko–Rado paper became the classical, seminal paper of extremal set theory. As Erdős writes about it in [61]: "Perhaps my most quoted theorem is our theorem with Ko and Rado". For a survey see, e.g., Frankl [100] or Füredi [102].

20. IV. 1938 (Cambridge)

[..] Try to convince Sidon to write his papers more clearly, by the way Sidon is quite well known here. [..]

[..] Perhaps I can talk to Hardy [to help you to leave Hungary], however I expect more from Szegő because I think Hardy has no more capacity. [..]

28. IV. 1938 (Manchester)

[..] Now I cannot send [money] for Sidon, I think at the moment his life is not endangered by starvation, I must give a lot to Ko. [..]

[..] You write that we will speak about it when we meet. You are too optimistic, you had better write it down as soon as possible. [..]

30. IV. 1938 (Longdale)

[..] Write immediately to Hardy, perhaps he can do something. I have talked to Mordell today, he suggests the same, however his opinion is that Hardy first helps those who do not even have a home anymore. [..]

[..] The situation of Sidon is more serious than I thought, I will try to do something. [..]

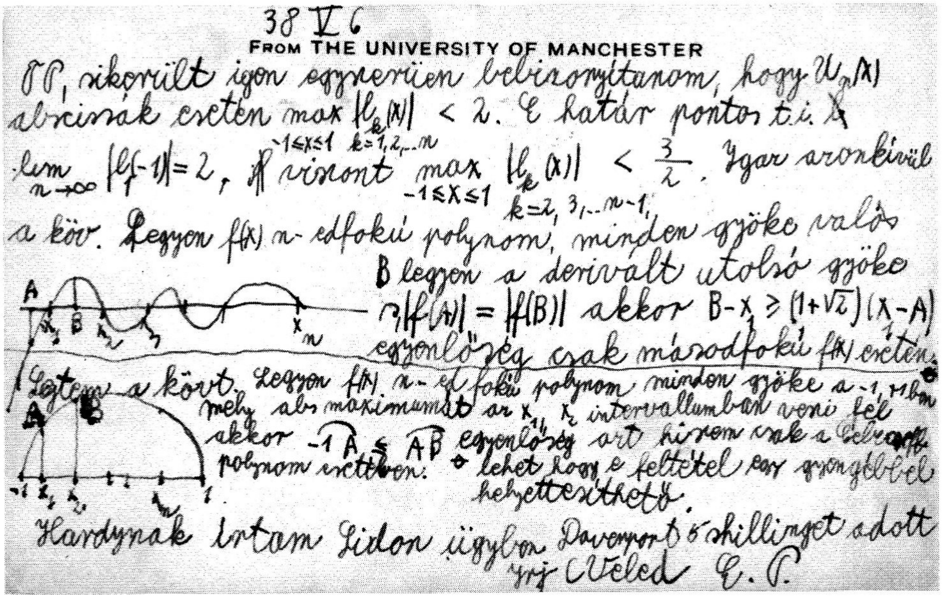
20. V. 1938 (Manchester)

[..] It is almost certain that I will not be coming home in the summer, so really only the spirit of Cantor knows when we can meet again. Write! I hope you have already written to Hardy. [..]

[From Turán to Erdős] (Budapest)

[..] .. one cannot do anything but to flee from this source of loathing (az utálat eme forrásától) into mathematics. Perhaps I have never been engaged with it so much as I am now. [..]

[..] Since I do not recommend that you come back home — there are a lot of new difficulties with the passport, even with the existing ones — sooner or later you have to rescore the character of your letters, you should write some sketch of the proofs of your theorems, because it may happen that we shall never have the opportunity to discuss them in person.



38 VI 6

FROM THE UNIVERSITY OF MANCHESTER

P.P., sikerült igen egyszerűen bebizonyítanom, hogy $U_n(x)$ abcissák esetén $\max_{-1\le x\le 1,\ k=1,2,\dots n} |l_k(x)| < 2$. E határ pontos t.i. $\lim_{n\to\infty} |l_1(-1)| = 2$, viszont $\max_{-1\le x\le 1,\ k=2,3,\dots n-1} |l_k(x)| < \frac{3}{2}$. Igaz azonkívül a köv. Legyen $f(x)$ n-edfokú polynom, minden gyöke valós B legyen a derivált utolsó gyöke $|f(A)| = |f(B)|$ akkor $B - x_1 \geq (1+\sqrt{2})(x_1 - A)$ egyenlőség csak másodfokú $f(x)$ esetén.

Legyen $f(x)$ n-edfokú polynom minden gyöke a -1, $+1$-ben mely abs maximumát az x_1, x_2 intervallumban veszi fel akkor $\widehat{-1A} \leq \widehat{AB}$ egyenlőség azt hiszem csak a Tchebyscheff polynom esetében. Lehet hogy e feltétel egy gyengébbel helyettesíthető.

Hardynak írtam Sidon ügyében Davenport 6 shillinget adott

Veled E. P.

Picture 10. A letter from Manchester, on Interpolation

I am interested in your idea how you can simplify the proof of the everywhere unboundedness of $\sum \left| \ell_k(x) \right|$? You should write about this. [..]

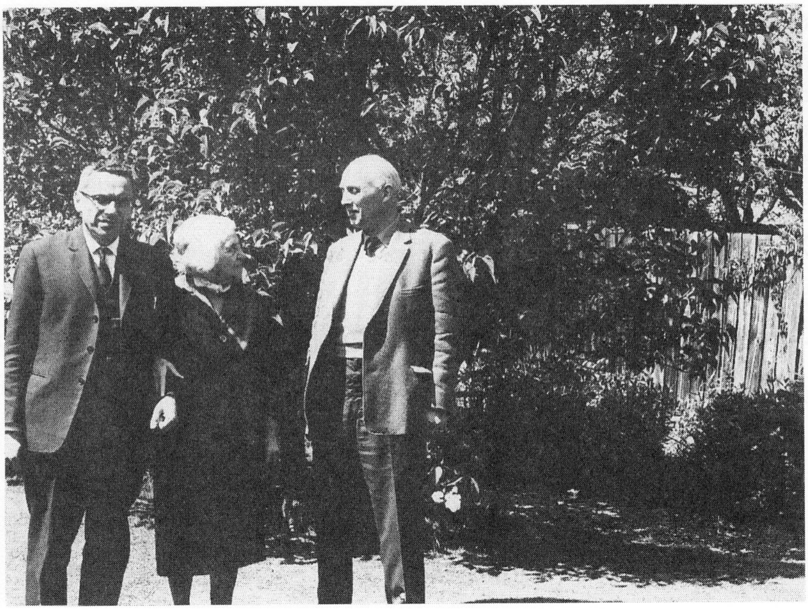
8. VI. 1938 (Manchester)

[..] If your theorem about $n, n+n^{1/2+\varepsilon}$ is OK, write it down immediately !!!!!! and send it to Hardy, you know that the days are running short. [..]

[..] Show this letter also to G.T. [..]

12. VI. 1938 (Sheffield)

[..] Now I am in Sheffield at the Rados'. Rado is playing now the beautiful part of Beethoven op 96 that you trivi[al being] never whistled to me, who knows when you can whistle it to me, you can imagine how many times he had to play it. [..]



Picture 11. Erdős, Erdős's mother, Rado, early seventies

23. VI. 1938 (Manchester)

[..] Still I might be coming home. Who knows what will happen in the future. [..]

[end of June, 1938] (London)

TP, if everything is all right, in 10 days I will be in Pest [Budapest], we must finish Interpol[ation] III, because we cannot tell whether we can meet at all or when, also this meeting will be the gift of Cantor's spirit. Now I am in London, it is quite cold. Greetings to everybody

C with you
E. P.

Erdős spent the summer of 1938 in Budapest. On September 3 he arrived at the final decision to immediately leave Hungary, his parents, friends and masters without knowing when he would be able to return, if at all. He returns to Britain through Italy–Switzerland–France.

He writes in [59]:

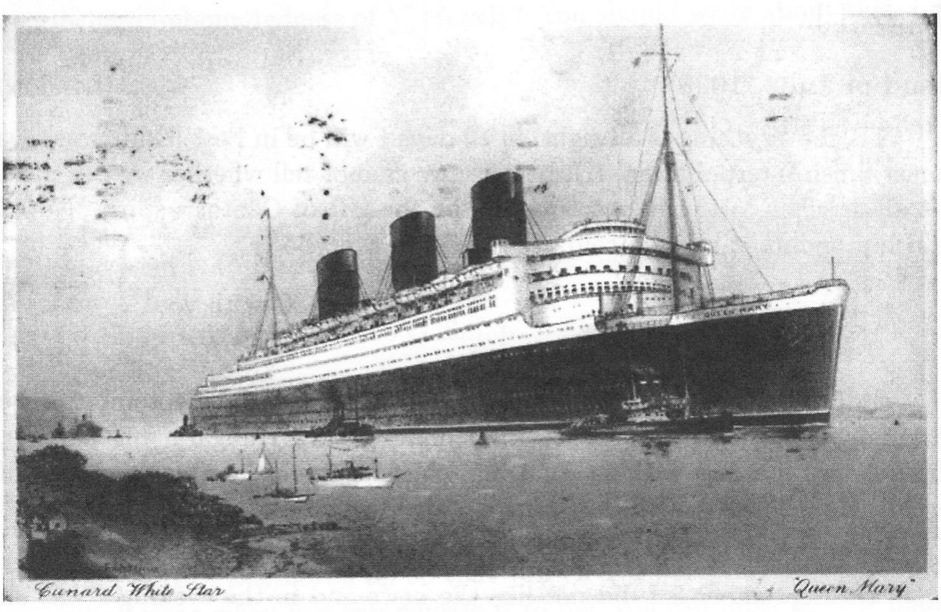
"On September 3 I did not like the news and in the evening I was on my way to England and three and a half weeks later to the USA — we corresponded until 1941 [with Turán] then there was an enforced gap of four years and we started [to correspond] as soon as possible. We wrote our first postwar joint paper on the difference of consecutive primes [96] by correspondence."

18. IX. 1938 (Manchester)

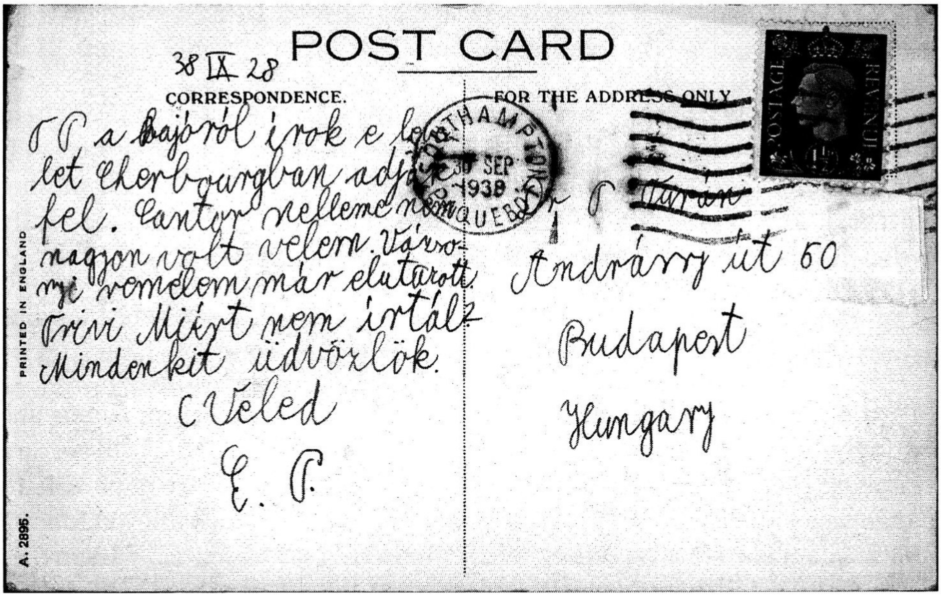
[..] It was just in time to finish Interpol[ation] III. If nothing happens, I will sail on the 28th.

Greetings to everybody
C with you
$\mathcal{E}. \mathcal{P}.$

Erdős leaves Britain on September 28 for Princeton, USA. He sends the postcard below already from the boat.



Picture 12. Queen Mary, Postcard


POST CARD

38 IX 28

CORRESPONDENCE.

T P a hajóról írok e levelet Cherbourgban adják fel. Cantor szelleme nem nagyon volt velem. Vázsonyi remélem már elutazott. Trivi Miért nem írtál? Mindenkit üdvözlök.

C Veled

E. P.

FOR THE ADDRESS ONLY

T. P. Turán
Andrássy út 60
Budapest
Hungary

PRINTED IN ENGLAND

A. 2895.

Picture 13. Queen Mary, Postcard, the letter

28. IX. 1938

T.P., I am writing from the boat, it will be mailed in Cherbourg. Cantor's spirit was not really with me. I hope Vázsonyi has already left. You trivial [being]. Why didn't you write?

Greetings to everybody
C with you
E. P.

7. Letters from USA, September 1938–1940

The letters from 1938–1940 show a very strong double feature: an extremely high intensity of mathematics and a permanent worrying about the political situation, about the frightening facts in Hungary and all over the world.

19. XI. 1938 [Princeton]

T.P.

[..] I sent lots of gossip to my parents, read it when you go to see them in connection with Interpol[ation] III. Today I talk to Chevalley until 11, unfortunately there is not much understanding among the French mathematicians; lots of gossip, triviality, intrigues, Lebesgue and Borel especially love each other. The Interpol[ation] II is all right. Tomorrow, with Wigner, I will go to visit Einstein. [..]

26. XI. 1938 (Princeton)

[..] I got a letter from Epszi, she shows some sign of life, of course she has [only] ε time. At the moment Brauer, Hamburger and Hellinger are occupying themselves with Jordan's theorem.[18] I hope, Weyl will be able to change this situation. I still cannot prove that uniform distribution follows from a quadrature-convergence. Write (more!) The trivial E. Vázsonyi is silent. Call D. Lázár and beat him to death. I do not have any information about him, how is his Epszi? [here Epszi=ε=child]. [..]

7. XII. 1938 (Princeton)

[..] I have not heard anything about you for weeks. The other day I read Knopp's proof of the following theorem of Carleman:

Let $a_i > 0$, $\sum\limits_{i=1}^{\infty} a_i = 1$, then

$$a_1+(a_1 \cdot a_2)^{1/2}+(a_1 \cdot a_2 \cdot a_3)^{1/3}+\ldots (a_1 \cdot a_2 \ldots a_n)^{1/n}+\ldots < e(a_1+a_2+\ldots).$$

Obviously

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{a_1 + 2a_2 + \ldots + i \cdot a_i}{i(i+1)},$$

now

$$\frac{a_1 + 2a_2 + \ldots + i \cdot a_i}{i(i+1)} \geq \frac{\sqrt[i]{i! a_1\, a_2 \ldots a_i}}{i+1} > \frac{1}{e}\sqrt[i]{a_1\, a_2 \ldots a_i}$$

i.e.

$$e(a_1 + a_2 + \ldots) > a_1 + (a_1\, a_2)^{1/2} + \ldots$$

This is beautiful, isn't it? Show it also to the others.

---

[18] By this term Erdős regularly refers to people detained in a camp or prison.

Davenport and I proved that every number large enough is the sum of 16 fourth powers. I proved that the number of integers of the form $x_1^4 + x_2^4 + x_3^4 + x_4^4$ up to n is $> n^{83/110} > n^{3/4}$. 16 is the best possible, because $31 \cdot 16^n$ is not the sum of 15 fourth powers.

Let $0 < a_1 < a_2 < \ldots < a_x \le n$ be a sequence of integers such that all the products $a_i\ a_j$ are distinct, then $x < \pi(n) + O\big(n^{3/4}/(\log n)^{3/2}\big)$. The remainder term is sharp, earlier I was only able to prove $O(n^{3/4})$. With Bochner we are working on the following problem: what is the necessary and sufficient condition for an entire function with only real roots to be approximable by polynomials with only real roots. For example e^{x^2} cannot be approximated.

I think the next German Math. congress will be held in Dachau. Recently also Remak left number theory and prefers to deal with the Jordan theorem.[19] I hope Brauer will be here soon. Shohat will try to do something for you. I received Erőd's letter. What will happen to him? I talked to Bochner, Pólya solved our problem already 25 years ago.

One can prove quite easily the following theorem. Let the highest coefficient of $f(x)$ be 1, all the roots real, $x_1 < x_2 < \ldots < x_n$ and $x_n - x_1 \le 1$, then the sum of the length of the intervals where $|f(x)| \le 1$ is at least 2, the theorem is not true if $x_n - x_1 > 1$, one can ask the following question: Let all the roots of $f(x)$ belong to $[-1, 1]$. For which $f(x)$ is the sum of lengths of the intervals where $|f(x)| \le 1$ minimal and what is the value of that minimum? Greetings to everybody, write, trivial being.

C with you
$\mathcal{E}.\ \mathcal{P}.$

Commentary. For the result with Davenport see [24]. Later Davenport proved a sharper result in [22].

About sequences with distinct pairwise products Erdős proved the weaker upper bound in 1936 [40]. It is interesting that Erdős published the above improvement — where the order of magnitude is best possible — only in 1969 [55]. The same lower bound has already been proved in [40].

For the above and related results on polynomials see the papers of Erdős [44, 45] and Borwein [14], Erdélyi [28].

[19] See the footnote to the letter of 26. XI. 1938.

11. XII. 1938

TP! trivial being, the following problem is due to Sidon: Let $0 < a_1 < a_2 < \ldots < a_x \leq n$ be integers, the sums $a_i + a_j$ all distinct, how large can x be? He proved $x > n^{1/4}$ is possible, it is easy to improve it to $n^{1/3}$, but I was unable to give an explicit example, yesterday I found the following: Let $a_i = i^2 n + i$, $i = 1, 2, \ldots, n/2$, obviously $a_i < n^3$ and if $a_i + a_j = a_k + a_\ell$, then $i^2 + j^2 = k^2 + \ell^2$ and $i + j = k + \ell$, which is impossible. I was not able to improve $n^{1/3}$ and I was not able to prove that $x = o(\sqrt{n})$. The following problem came up: if $a_1 < a_2 < \ldots < a_x < \ldots$ is an infinite sequence, all the $a_i + a_j$ sums are distinct, then $\sum\limits_{i=1}^{\infty} \frac{1}{a_i^{1/2}}$ is convergent.

Schur proved in the first volume of Math. Zeitschrift (p. 377) among others the following theorem: Let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ be a polynomial of integer coefficients, with all the roots in $[-1, 1]$ and let n be large enough, then $a_0 > \left(\sqrt{2} - \varepsilon\right)^n$. Schur gets the proof by investigating the discriminant (i.e., the discriminant is maximal for $\int P_n(x)$), you know that paper. I give a simpler proof in the following way:

We may suppose that $f(-1), f(0), f(1) \neq 0$ (if not, we divide by $x(x-1)(x+1)$). Let $x_1, x_2, \ldots, x_n$ be the roots, then

$$|a_0|\,|x_1 x_2 \ldots x_n| \geq 1, \qquad \left|a_0 \prod_{i=1}^{n} (1 + x_i)\right| \geq 1, \qquad \left|a_0 \prod (-1 + x_i)\right| \geq 1,$$

that is

$$a_0^2 x_1^2 x_2^2 \ldots x_n^2 \geq 1, \quad a_0^2 \prod (1 - x_i^2) \geq 1.$$

Now obviously $\min\left[(x_1^2 x_2^2 \ldots x_n^2), \prod (1 - x_i)\right] \leq 1/2^n$ that is $a_0 \geq \left(\sqrt{2}\right)^n$ q.e.d. $\left(\sqrt{2}\right)^n$ is not sharp, very probably one can improve it if one also uses that $a_0^{2n-2} \prod\limits_{j>i} (x_i - x_j)^2$ is an integer, so what is

$$\max\min\left[(x_1 x_2 \ldots x_{n-1})^{n-1}, \left(\prod (1 + x_i)\right)^{n-1}, \right.$$
$$\left. \left(\prod (1 - x_i)\right)^{n-1}, \prod_{i<j} (x_i^2 - x_j^2)\right].$$

Obviously $a_0 > \frac{1}{A^{1/(n-1)}}$, if $A = \max\min\left[(x_1 x_2 \ldots x_n)^{n-1} \ldots\right]$.

Perhaps Erőd can solve that, if he is interested in the problem.

Schur proves, by giving an example, that $a_0 < \left(\sqrt{1+\sqrt{2}}\right)^n$.

Your letter arrived now. Everything is in order with Interpol III.

Let $b - a = k$ (in units $1/n$) and suppose that the number of roots in (a,b) is $= k - k^{1/2+\varepsilon}$. To the left of α up to 0 and to the right from β to π intervals of length $\frac{\pi}{n}\left(1+\frac{1}{10\,k^{1/2-\varepsilon}}\right)$!! (not $\frac{\pi}{n}$) and I continue in that way then also this polynomial according to Riesz's theorem takes its maximum in $[a,b]$ etc. and the degree of it is $n - c_n/k^{1/2-\varepsilon}$, and now it is trivial how one can continue it. (I will think on your theorem on Legendre polynomials.) So the main thing is that one can work with units larger than π/n. One more small thing: Let $f(x)$ be a polynomial of degree n, highest coefficient 1 and all the roots in $-1, 1$, than there exists an interval of length 1, a, $a+1$, $-1 < a < a+1 < 1$ such that if $a < x < a+1$ then $|f(x)| < 1$, even more generally it is true: if there is an $0 \leq x_0 \leq 1$ with $|f(x_0)| > 1$, then $|f(x)| < 1$ if $-1 < x < 0$. Davenport proved that almost every number is the sum of 4 cubes. Write

C with you, greetings to everybody
E. P.

Commentary.

Erdős writes in [61]: "Denote by $f(n)$ the number of solutions of $n = a_i + a_j$.

I first met Sidon in 1932. He posed two problems: Does there exist an infinite sequence A for which $f(n) > 0$ for all $n > n_0$, but so that for every $\varepsilon > 0$ $f(n)/n^{\varepsilon} \to 0$ as $n \to \infty$. ...

Sidon's second question. Find a sequence A for which $A(x)$ [the number of elements $< x$ in the sequence A] is as large as possible and for which $f(n) = 0$ or 1, i.e. the sums $a_i + a_j$ are all distinct. Such sequences are called now Sidon sequences. Sidon was led to these problems by his study of lacunary trigonometric series."

For the construction of a long Sidon sequence, the modification of the above idea, taking $i^2 \bmod p$ instead of $i^2 \bmod n$ gave the best possible result according to the order of magnitude. Together with the upper bound this was published in the Erdős–Turán paper [93]. The improved lower bound which gave the asymptotically best possible result, was given independently by Erdős [46] and Chowla [20] using a construction of Singer. There is still the

Conjecture: for the longest Sidon sequence in $[1, n]$ it holds that $x = n^{1/2} + O(1)$.

In their paper [93] Erdős and Turán proved that $f(n) = C$ for $n > n_0$ is impossible and they formulated the famous

Conjecture: If $f(n) > 0$ for $n > n_0$ then it cannot be bounded, "which have resisted so far all attacks".

In 1969 Erdős proved the multiplicative analog of this conjecture [56].

The infinite version of Sidon's problem, how large the counting function $A(x)$ of an infinite Sidon sequence can be, is still unsolved. Erdős and Turán would be pleased to learn about the result of Ruzsa [140], which superseded the first breakthrough of Ajtai–Komlós–Szemerédi [1]. See more in [140].

These problems of Sidon (or the problem of $r_k(n)$) can be considered as prototypes of much more general problems in various additive or multiplicative structures for one equation or even for systems of equations, see, e.g., T. Sós [148], Babai–Sós [6], Ruzsa [137, 138].

"The most successful year"

Now returning to Lubinsky's observation (on p2), as another documentation of his opinion let me quote from Babai [4], p. 61:

"Erdős concludes what even in 1995, after 1400 articles, he considers his most successful academic year [1939]. The crop includes: the Erdős–Kac and Erdős–Wintner theorems on additive functions, his third paper with Turán on interpolation [92]; (the preceding papers in the series also appeared in the respected Princeton periodical) and the result that the product of consecutive integers is never a square."

The Erdős–Kac and Erdős–Wintner papers were written up in an extremely short period. Erdős met Kac and Wintner for the first time in February 1939, both the Erdős–Kac and Erdős–Wintner papers were received by the Journals in March 1939. (See also Commentary to the first letter from 1934).

The Erdős–Turán Interpolation III paper took a much longer — several years long — period to be published. (See p. 93.)

Erdős never stopped being interested in the product of consecutive integers and had a great satisfaction when 35 years later, in 1975 with Selfridge they proved the general result [82], see also Hildebrand [117] in this volume.

The letters from this period help us follow the emergence and development of these four results. Erdős immediately wrote about all of these to Turán.

10. I. 1939 (Princeton)

[..] In the last few days I investigated the product of consecutive integers. I wrote to Epszi about it. Gy. Szekeres suggested [investigating] this.

[..] Yesterday I was in Baltimore. I met Wintner and Kac, the latter came not so very long ago from Poland, he sends his compliments to Sidon. (this is not irony). [..]

[..] Wintner and I proved that the sufficient conditions in "On the density of some sequences III" [39] (I am sure you have it) are also sufficient.[20] Jessen and Wintner proved the following theorem: Let $f(m)$ be an additive number theoretical function, let $\psi(c)$ be the density of the numbers for which $f(m) > c$, then $\psi(c)$ is either abs continuous or singular (i.e., it changes on a set of measure 0), if $f(p) = 1/p$ then $\psi(c)$ is singular, very probably this is always the case but I cannot prove it.

Don't let G. T. sleep, try to get him to work, this is the only thing which can save him. [..]

1. II. 1939 (Princeton)

TP I received your letter of 16 Jan. I still do not understand your infinite Sidon-sequence example. I will think it over. It would be good to submit Interpol[ation] III as soon as possible.

I hope your flu is over and also the others are well. [..]

[..] Your lemma seems to be very difficult. On the 9th I will be at Harvard, on the 10th at Providence. I lost the address of Erőd, I send his letter to you.

That

$$\binom{n}{k} = y^2 \quad \text{for} \quad k > 3$$

is impossible, I prove it the following way (we may suppose that $n \geq 2k$).

Let $n - i = a_i x_i^2 \quad * \qquad (i = 0, 1, 2, \ldots, k-1)$.

---

[20] it should be: necessary.

In a letter to Epszi that you have very probably seen I prove that all the a_i's are distinct. If $\binom{n}{k} = y^2$ then

$$\frac{a_0 a_1 \ldots a_{k-1}}{k!} = \frac{u^2}{v^2};$$

now

$$a_0 a_1 \ldots a_{k-1} \leq k!$$

Let $rp \leq k < (r+1)p$, then in $k!$, p has power at least r, but in $a_0 a_1 \ldots a_{k-1}$ at most $r+1$ (the a's are quadratfrei [square free]) so if in $a_0 a_1 \ldots a_{k-1}$ it would occur to a power greater than in $k!$ then in $(a_1 a_1 \ldots a_{k-1})/k!$ it would only have power one, and this is impossible**, however for $k \geq 4$

$$a_0 a_1 \ldots a_{k-1} \geq 1.2.3.5. \ldots (k+1) > k!$$

contradiction.

---

* The a_i's are quadratfrei and all their prime factors are less than k.

** you see that $a_0 a_1 \ldots a_{k-1} \mid k!$

Send Interpol III as soon as possible. Unfortunately I have to write proofs, who knows, when, where — if at all — we shall meet.

C with you
$\mathcal{E}.\ \mathcal{P}.$

Commentary. Erdős proved the theorem above about binomial coefficients in [43]. Further, he conjectured that

$$\binom{n}{k} = y^m$$

impossible if $k \geq 2$, $n \geq 2k$, $m > 2$. Later he proved this in [50] for $k > 3$. For $k = 2, 3$, the conjecture was proved only recently by Győry [112]. For the newest results and history see the paper of Győry [113]. Obviously, the question is closely related to the product of consecutive integers, see also the Commentary after the letter of 8.III. 1939 and [113].

4. III. 1939 (Cranbury)

TP, trivi being, I received your letter. Now I am having lunch here. Yesterday I proved that the product of consecutive odd integers is never a power.

C with you
$\mathcal{E}.\ \mathcal{P}.$

8. III. 1939 (Princeton)

I proved recently that the product of consecutive integers is never a power.

Kac is here now. We proved some very general theorems about additive functions, for example: Let $f(n)$ be additive, $|f(p)| \leq 1$, $\sum \frac{f(p)}{p}$ divergent. Then the density of integers for which $f(m) > \sum_{p \leq m} \frac{f(p)}{p}$ is 1/2, for $f(m) = \nu(m)$ I proved it earlier (the density of integers, for which $\nu(m) > \log\log m$ is 1/2).

Ko has found a solution of $x^x y^y = z^z$. Write. Live. Greetings to everybody.

C with you. See you
$\mathcal{E}.\ \mathcal{P}.$

Best regards to you and to Sidon

M. Kac

Commentary. Observe that the Erdős–Kac paper [73] was communicated five days later, on 13 March, 1939!

Erdős writes in [55]:

"I was for a long time looking for a theorem like the conjecture of Kac but due to my lack of knowledge of Probability theory ....
...and probabilistic Number Theory was born."

The 150 years old problem about the product of consecutive integers was one of the favorite problems of Erdős. In fact Erdős proved in his papers [41], [43] that the product of consecutive integers is never a square, and for arbitrary l that for k large enough (depending on l) $(n+1)\dots(n+k)$ is never an l'th power. About the general case Erdős wrote in [61] the following: "On diophantine equations, no doubt, my most important result is my joint work with Selfridge [82]". They proved the general result — without any assumption — in that paper.

3. IV. 1939 (Princeton)

[..] Wintner and I proved that if $f(m)$ is an additive function, then the distribution function exists if and only if $\sum \frac{\|f(p)\|}{p}$ and $\sum \frac{\|f(p)\|^2}{p}$ are convergent. ($\|f(p)\| = f(p)$ if $|f(p)| < 1$ and 1 if $|f(p)| > 1$.)

I have proved that if $\left|f(p)\right| < \frac{1}{p^c}$ then the distribution function is singular, and on the other hand from a criterion of Wintner I have proved that if $f(p) = \frac{1}{\log p}$ then the distribution function is absolutely continuous, and if for example $f(p) = (-1)^{(p-1)/2}/(\log\log p)^{3/4}$ then the distribution function is transcendental entire.

I conjecture the following: Let $f(x)$ be a polynomial of degree n without zero in the unit disc, $f(-1) = f(+1) = 0$, $\max\limits_{1\le x\le 1} \left|f(x)\right| = 1$ then $\int\limits_{-1}^{1} \left|f(x)\right| \le 2 - \frac{2}{n+1}$ with equality in case $f(x) = 1 - x^n$. [..]

Commentary. Again, observe that the Erdős–Wintner paper [99] was communicated two weeks later, on 26 March, 1939.

On the enormous impact of the papers of Erdős and Kac [73], [74] and of Erdős and Wintner [99] see, e.g., Elliott [26] and [27] in this volume.

11. V. 1939

TP, trivi being! I have not heard about you for a continuum amount of time.

My parents wrote that you succeeded in bowling over the Lemma. Yesterday morning I succeeded to prove that for continuous functions the step-parabolas converge in every inner point. This follows almost trivially from $\sum\limits_{k=1}^{n} v_k(x)\ell_k^2(x) = 1$ and from the π/n distances of the roots. Namely, it is obviously enough to prove that

$$\sum_{\substack{|x_k-x_0|>\frac{f(n)}{n}\\ f(n)\to\infty}} \ell_k^2(x_0) \to 0,$$

even it is enough [to show] that

$$\sum_{|x_k-x_0|>\varepsilon} \ell_k^2(x_0) \to 0.$$

However, from the π/n distance it trivially follows that $\ell_k(x) - L_k(x) \to 0$ where $L_k(x)$ is the Legendre fundamental function. From this it is trivial that

$$\sum_{|x_k-x_0|<\varepsilon} \ell_k^2(x_0) \to 1 \quad \text{i.e.} \quad \sum_{|x_k-x_0|<\varepsilon} v_k(x_0)\ell_k^2(x_0) \to 1$$

and since $v_k(x_0) > c > 0$ (in an inner point), everything is trivial. No doubt you have heard about my other interpol papers. [..]

[..] Interpol[ation] III is already at Annals. Sadly enough we shall not meet in the summer and so I do not know when I can tell you the proofs, I hope we shall meet in 1940 if we are still alive. [..]

[..] Two unsolved problems:

(1) Let $f(n)$ be an arithmetic function which is $+1$ or -1. Prove that for arbitrary c one can find d and n such that $\left|\sum_{k=1}^{n} f(kd)\right| > c$.

(2) Does there exist a sequence $x_1, x_2 \ldots$ of complex numbers for which $|x_i| = 1$, such that for an arbitrary interval of length α the number of elements in the sequence with index at most n is $\frac{\alpha}{2\pi}n + O(1)$? $O(1)$ is independent from α and n, i.e. the sequence is Gleichverteilt [uniformly distributed] in the highest degree. I think that such a sequence does not exist.

Hardy was here last year and gave a lecture about things related to Ramanujan's results; of course he discussed your method. I enclose a few stamps. One more problem: Let $0 < \theta < 1$ be an arbitrary number. Denote $(n\theta) = x_n$. It follows from our theorem that $\max \prod(x - x_i) < (1+\varepsilon)^n/2^n$. For every θ one cannot say more. The question is, how much can one prove for almost all θ. (Obviously $(n\theta)$ is on the unit circle and x_n is its projection.) Greetings to everybody.

C with you! Write! See you!
$\mathcal{E}.\ \mathcal{P}.$

Commentary. The Erdős–Turán inequality on the discrepancy of sequences appeared in 1940 [95].

Very probably the importance of the distribution of the zeros of the interpolation polynomials led Erdős to problem (2). This was conjectured in 1935 by van der Corput and the first result was proved only ten years later by Aardene–Ehrenfest [2]. The letter above shows that Erdős raised the above question on uniformly distributed sequences independently from van der Corput. As we know, the starting point of discrepancy theory is just this conjecture, the discovery of the negative phenomenon that there is no sequence which is too well distributed in the above sense of "highest degree". Discrepancy theory — which has been much studied in the last fifty years and is now an extended theory — is related to many fields, e.g., number theory, combinatorics, ergodic theory etc. See Kuipers–Niederreiter [121], Beck–Chen [8], Drmota–Tichy [25], Matoušek [125], Beck–Sós [9].

Problem (1) is even now one of the famous open questions in discrepancy theory. The discrepancy of the family of all arithmetic progressions was investigated first by Roth [136].

26. VI. 1941.

[..] In the autumn of 1940 I thought I can prove your theorem in an elementary way, that there is no sequence $a_1 < \dots$ such that for $n > n_0$ the number of solutions of the equation $n = a_i + a_j$ is constant, unfortunately my proof was wrong. [..]

Commentary. The proof of this theorem is in [93]. The much stronger result that even on the average the representation function cannot be close to a constant, was proved in 1954 by Erdős and Fuchs for arbitrary sequences of positive integers [64]: Let the function $f(n)$ be defined as the number of solutions of $n = a_i + a_j$, then

$$\sum_{x \le n} f(x) = cn + o\left(\frac{n^{1/4}}{(\log n)^{1/4}}\right)$$

is impossible. Erdős asked the question much earlier, see the letter below of XII. 1945. For the continuation of this beautiful theorem, which also gives the deep explanation of the result of Hardy and Littlewood on the classical circle problem, see Halberstam–Roth [115], Montgomery–Vaughan [128], Ruzsa [139], Sárközy [142].

Erdős himself liked this theorem very much. Let me quote him [61]:

"I should not forget our result with W. Fuchs [64] which certainly will survive the authors by centuries."

The investigation of how regular or smooth the representation function $f(n)$ defined above can be, became a quite intensively investigated topic. See, e.g., the survey Sárközy–Sós [143].

8. Correspondence in 1940 with the mediations of Lajos Erdős

During the period 1940 October–1945 Pál Erdős was staying in the USA. Turán was in forced labor camp (more exactly in and out) in Hungary. Their contact through correspondence did not stop immediately, but the manner of it changed and was quite peculiar. From the USA, Erdős wrote to his parents in Budapest. His father, Lajos Erdős — who was a highly educated mathematics teacher — copied these letters and sent them to Turán to the — changing — address of the forced labor camp where he was. In the other direction: Turán wrote to Erdős's parents who copied and forwarded it to P. Erdős — when this was feasible. It is almost unbelievable that even some of *these* letters survived all these years. Here I give the translation of two letters.

Budapest, 23. IX. 1940 [From Lajos Erdős to Pál Turán]

Dear TP, we had a great week of mail [postahetünk]. On Saturday we received two airmail-letters, from IX. 9, and IX. 14, and today we received a letter by surface mail from VIII. 25. I have a lot of things to write to you and as I am a nice guy, I really will write it down. He [Erdős] had written his letter on IX. 9, before he went to a meeting. He travelled together with a Pole called Tarski, whose wife and children stayed still in Warsaw. Palkó [Erdős] received the algorithm-paper of Rédei to referee. He sends the message to Rédei that a mathematician called Schuster proved in the 1938–39 Monatshefte that for $m = 3p$ there is no Euclidean algorithm, with the exception of $m = 21, 33, 57$. This means that together with his [Rédei's] results the case $m = pq$ is completely settled. He met Szegő at the meeting, they discussed a lot of things. [..]

A few month ago Ottó Szász had an accident, his hip broke, he still uses a stick. Szász mentioned that an American told him he had seen a paper of Fejér whose name was misspelled as: Fejes.[21] Of course Szász corrected him. Szegő went back to California, very probably Pali [Erdős] will get an invitation to give a lecture there, for which he will get $40. However, the travel expenses alone amount to $80. A student of Szegő is grappling with the following theorem:

Prove that $\mathcal{T}_n(x)$ takes its maximum in the interval $[1+it, 1-it]$ at its endpoints. This seems to be not quite trivial and there is no simple proof yet. He [EP] writes on 25 Aug that Schönberg arrived at Princeton, he will stay there for 2 weeks. Siegel and Gödel are in Princeton, but they haven't got permanent jobs. Rademacher is a full professor, but Wintner is only an associate professor. On the 25 Aug he [Erdős] proved the following theorem. Let M be an arbitrary set of cardinality m, $n < m$ and let f be a function

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[21] He mixed up the names L. Fejes [= László Fejes Tóth] and L. Fejér.

which associates to each element of M a subset of M of cardinality $< n$, further let $f(a) \not\ni a$. The elements a and b are independent if $f(a) \not\ni b$ and $f(b) \not\ni a$, a subset of M is called independent if any pair of elements of it are independent. The theorem states that there exists an independent subset of cardinality m. D. Lázár will be interested in it. He proved the theorem for the case when M is not the union of less then m sets of cardinality less then m. The proof uses the generalized Cantor conjecture. It won Gödel's highest appreciation.

That will do for now. Please inform D. Lázár about the set theoretical theorem if you are in a position to do so. What do you know about GT? That swine[22] does not give any sign. We are as well as one can be in these circumstances. Today Veress sent a reprint of his paper about non-planar graphs. Even more, he sent two copies, one to Pali [Erdős].

We hope you are well. Write as soon as possible.

Greetings with love
Erdős Lajos

Commentary. Erdős published the above result on set mappings only in 1950 [48]. In 1935–1936 Turán — in connection with interpolation theory — asked: If a finite set $f(x)$ is associated to every point x of the real line, does there exist an infinite free set, i.e. when $x \notin f(y)$ holds for any two distinct elements? (In graph setting: In a loop-free infinite digraph of finite out-degree, does there exist an infinite independent set?)

In 1949, on March 19, Turán gave a commemorative lecture at the Bolyai Math. Soc., published in [159]. Here one can find the following: "After this has been proved [in the affirmative] by G. Grünwald, D. Lázár succeeded to prove that there exists such a set of continuum many points. Erdős immediately told his [Lázár's] proof to János Neumann [John von Neumann] who was in Budapest at that time on a visit and he at once asked it for the Comp. Math. [122]. Joining his work, this theorem and method were generalized by Sierpinski, Sophie Piccard and Ruziewicz."

On later developments and Erdős's important contribution to it see the survey papers of Hajnal [114] and Komjáth [118]. Quoting Komjáth:

[22] In Hungarian this is applicable to your friend upsetting you.

"Clearly, the underlying structure has nothing to do with the question and eventually a nice theory emerged which culminated in the results of Erdős, G. Fodor and A. Hajnal."

A postcard of Turán from Szászlekence[23] to Lajos Erdős in Budapest:

1. XI. 1940 (Szászlekence)

Dear Mr. and Mrs. Erdős,[24]

I do not remember when I wrote last time, perhaps from Abafája[25]. Many things happened since that, our eventful life did not facilitate writing letters. Now I can turn to mathematics a bit more. I proved that if $p(n)$ denotes the largest prime factor of n, then $\sum \frac{1}{n\,p(n)}$ is divergent.

This is an interesting strengthening of the fact that $\sum \frac{1}{n}$ is divergent, and is quite curious because for "most" integers, $p(n)$ is $n^{c/\log\log n}$ and $\sum \frac{1}{n^{1+c/\log\log n}}$ is convergent. I have also been reading Titchmarsh's function theory, it is really one of the most artistically written books. I have been missing your letters for a very long time, I received the last one at Pomáz[26]: if you do not find it difficult, please write to me those parts of the letters of Pál, of which you think that I am interested in. A young man called Szirmai is with us, according to him he knows you very well and he sends his regards.

[Néni kezét csókolja, Tanár urat sokszor üdvözli]

Sincerely yours
Turán Pál

---

23 one of the forced labor-camps where Turán was

24 In Hungarian: "Kedves Erdős néni és Tanár úr!"

25 One of the forced labor-camps where Turán was.

26 One of the forced labor-camps where Turán was.

9. After World War II

After the summer of 1938 the next time Erdős and Turán could meet was ten years later, in 1948, in Princeton. Erdős returned to Hungary on December 2, 1948. This was the first time I met Erdős.

There is no record about the correspondence of Erdős and Turán between June 1941 and spring 1945. (See the quotation on p. 91.)

After four years break, the first letter after the war, from Erdős to Turán and Tibor Gallai (Grünwald) in the winter of 1945:

XII. 1945.

TP, GT

I do not know your address, therefore I send my letter to my parents.

News in mathematics: Henry Mann (who immigrated from Vienna) proved the $\alpha + \beta$ conjecture. Linnik (student of Vinogradov) proved that the smallest prime number in the arithmetic progression $kx + \ell$ is $< k^c$. He also proved that every large number is the sum of 7 cubes.

Saks, Schauder, Wolf, Remak were killed by the Nazis. Lindenbaum, Ruziewicz met the same fate. Mazurkiewicz, Banach died, Marcinkiewicz disappeared. Steinhaus, Knaster, Kuratowski, all the French mathematicians and the well-known Dutch mathematicians are alive, Hardy is very ill, Helly died.

I am rather tired, I was on a long excursion in the mountains (we always go, never eat)[27]. Vázsonyi is well, he has married, a few weeks ago I visited him.

What happens at the university? Will you get jobs? What are your plans? Do you want to stay at Budapest or do you want to leave?

There are many jobs here (soon there will be more), if the situation calms down a bit, very probably it will be possible to immigrate. For the time being I have no idea what I want to do, I have too little information from Budapest, so I have not made up my mind about returning. Béla Lengyel is well, he has married; has a child, he is an assistant professor in Rochester.

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[27] in Hungarian: "mindig megyünk, sohse eszünk", this is a phrase they used during their day-long excursions around Budapest in the thirties.

Some theorems in number theory, which I proved since 1942:

Let $f(n)$ be additive. If there exist c_1 and c_2 such that for infinitely many n's there exist sequences

$$a_1^{(n)} < a_2^{(n)} < \ldots < a_x^{(n)} < n, \qquad x > c_1 n$$

and

$$\left| f\left(a_i^{(n)}\right) - f\left(a_j^{(n)}\right) \right| < c_2,$$

then

$$f(n) = c \log n + \varphi(n) \tag{1}$$

where

$$\sum \frac{\left(\varphi(p)'\right)^2}{p} < \infty, \qquad \varphi(p)' = \begin{cases} \varphi(p) & \text{if } \varphi(p) < 1 \\ 1 & \text{otherwise.} \end{cases}$$

If (1) holds, such c_1 and c_2 always exist, this means that the condition is necessary and sufficient.

Is the following true: Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers. Let $f(n)$ denote the number of solutions of $n = a_i + a_j$. Prove that

$$\sum_{k=1}^{n} f(k) = cn + O(1)$$

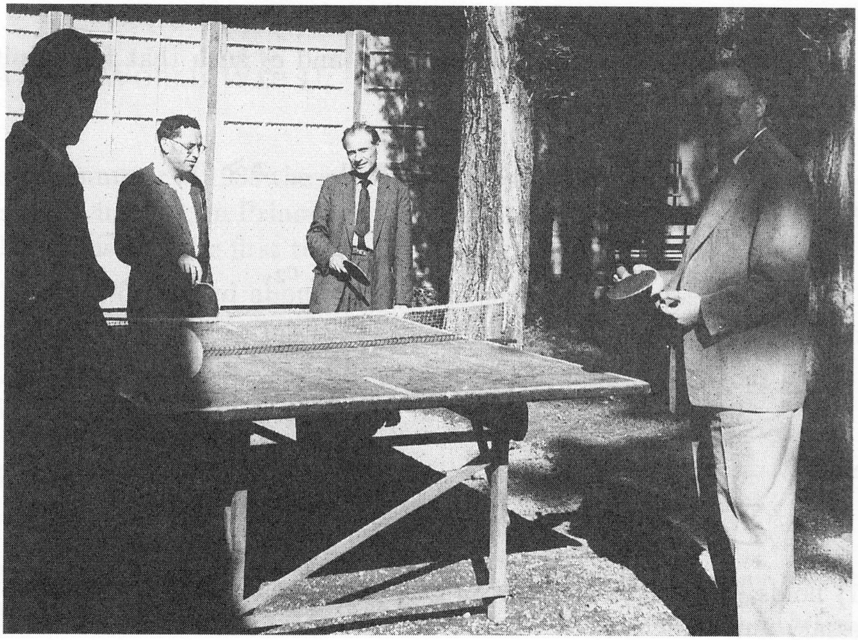
is impossible. I cannot prove this.

I will send reprints as soon as possible. So far packages cannot be sent, most probably it will become possible in the near future, so let me know what you need.

The Svéd's have two very sweet ε's [children], how are your ε's, I wonder when I get a chance to see them? Write!

$\mathcal{E}. \mathcal{P}.$

...LIFE — AND CORRESPONDENCE — CONTINUED ...



Picture 14. Knapowski, Erdős, Szekeres and Turán, at the Number Theory Conference, Balatonvilágos, September 1958, playing table tennis

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