Combinatorics, Probability and Computing (2003) 12, 319–344.
DOI: 10.1017/S0963548303005613 Printed in the United Kingdom

Hereditary Extended Properties, Quasi-Random Graphs and Induced Subgraphs

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Received 4 September 2002; revised 6 December 2002

This is a continuation of our work on quasi-random graph properties. The class of quasi-random graphs is defined by certain equivalent graph properties possessed by random graphs. One of the most important of these properties is that, for fixed v, every fixed sample graph L_v has the same frequency in G_n as in the p-random graph. (This holds for both induced and not necessarily induced containment.) In [9] we proved that, if the frequency of just *one* fixed L_v – as a *not necessarily induced subgraph* – in every 'large' induced subgraph $F_h \subseteq G_n$ is the same as for the random graphs, then (G_n) is quasi-random. Here we shall investigate the analogous problem for *induced* subgraphs L_v. In such cases (G_n) is not necessarily quasi-random.

We shall prove, *among other things*, that, for every *regular* sample graph L_v, $v \geqslant 4$, if the number of induced copies of L_v in every induced $F_h \subseteq G_n$ is asymptotically the same as in a p-random graph (up to an error term $o(n^v)$), then (G_n) is the union of (at most) two quasi-random graph sequences, with possibly distinct attached probabilities (assuming that $p \in (0,1)$, $e(L_v) > 0$, and $L_v \neq K_v$).

We conjecture the same conclusion for every L_v with $v \geqslant 4$, *i.e.*, even if we drop the assumption of regularity.

We shall reduce the general problem to solving a system of polynomials. This gives a 'simple' algorithm to decide the problem for every given L_v.

1. Notation

We shall use notation that is mostly standard. For a (simple) graph G, $v(G)$ and $e(G)$ denote the *number* of vertices and edges, and $V(G)$ and $E(G)$ denote the *set* of vertices and edges, respectively. For graphs, the (first) subscript will almost always denote the number of vertices. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X, and $e(X)$ denotes the number of its edges. Given two disjoint subsets $X, Y \subseteq V(G)$, then $G[X, Y]$ denotes the bipartite subgraph of G induced by them, and $e(X, Y) = e(G[X, Y])$.

† Research supported by the Hungarian Research Council, Grants T026069, T038210, and OTKA T032236.

We shall mostly have a *sample graph* $L = L_v$ with v vertices, ($V(L) = \{a_1, a_2, \ldots, a_v\}$), and a graph G with some copies of L. The vertices of a copy $L \subseteq G$ will typically be denoted by $\{b_1, b_2, \ldots, b_v\}$.

- A *not necessarily induced* (or NNI) labelled copy is given by a function $\psi : V(L) \to V(G)$ mapping different a_is into different $b_i := \psi(a_i)$s, where we assume (only) that if $(a_i, a_j) \in E(L)$, then $(\psi(a_i), \psi(a_j)) \in E(G)$. Let $\mathbf{N}(L \subseteq G)$ denote the number of labelled NNI copies of L in G.
- A *labelled induced* copy of $L \subseteq G$ is given by a function $\psi : V(L) \to V(G)$ mapping different a_is into different b_is, where $(\psi(a_i), \psi(a_j)) \in E(G)$ if and only if $(a_i, a_j) \in E(L)$. Denote the number of labelled *induced* copies of $L \subseteq G$ by $\mathbf{N}^*(L \overset{*}{\subseteq} G)$. If we wish to *emphasize* that $L \subseteq G$ is an induced graph, we shall write $L \overset{*}{\subseteq} G$.

We shall use $u_n \sim v_n$ if $u_n/v_n \to 1$ as $n \to \infty$.

The complementary graph of H is denoted by $\overline{H}$.

2. Introduction

This paper is strongly connected to our previous papers [8, 9]: it is a continuation of [9]. Therefore we give here only a shortened introduction. For a longer one see [9].

One of the important questions of modern mathematics and computer science is how random-like objects can be generated in nonrandom ways, when an individual event could be considered random, and in which sense.

Thomason [12, 13], Frankl, Rödl and Wilson [6] and Chung, Graham and Wilson [4] have given some characterization of random-like graph sequences.

Our starting point is a theorem of Chung, Graham and Wilson [4]. There many (actually seven) graph properties $\mathbf{P}_\ell$ are considered,[1] all possessed by (binomially distributed) p-random graphs. They prove that all these properties are equivalent to each other in some well-defined sense. A graph sequence is called *p-quasi-random* if it satisfies one of these properties $\mathbf{P}_\ell$ (and therefore all the others as well).

Here we need only the following one of the quasi-random graph properties of [4]. Let $p \in (0,1)$, and let $v = v(L)$.[2]

We consider the following property of a graph sequence (G_n):

$\mathbf{P}_1^*(v)$: for fixed $v \geqslant 4$, and *for each* graph L_v,

$$\mathbf{N}^*(L_v \overset{*}{\subseteq} G_n) = (1 + o(1)) p^{e(L_v)} (1-p)^{\binom{v}{2} - e(L_v)} \cdot n^v \quad \text{as} \quad n \to \infty. \tag{2.1}$$

$\mathbf{P}_1^*(v)$ says that the graph G_n contains each graph L_v of order v with the same frequency as the p-random graph. $\mathbf{P}_1^*(v)$ refers to the induced copies. The analogous property for NNI copies is defined similarly:

$\mathbf{P}_1(v)$: for fixed $v \geqslant 4$, and *for each* graph L_v,

$$\mathbf{N}(L_v \subseteq G_n) = (1 + o(1)) p^{e(L_v)} \cdot n^v \quad \text{as} \quad n \to \infty. \tag{2.2}$$

[1] More precisely, properties of graph sequences!
[2] Sometimes we shall use $\eta = e(L_v)$; in other cases we shall write $e(L_v)$.

Trivially, $\mathbf{P}_1^*(v)$ and $\mathbf{P}_1(v)$ are equivalent for fixed v and p, because knowing the distribution of the NNI copies, the inclusion–exclusion method yields the distribution of the induced copies.

According to the Chung–Graham–Wilson theorem, both $\mathbf{P}_1(v)$ and $\mathbf{P}_1^*(v)$ are quasi-random properties. This implies the following result.

Corollary 2.1. *If* (2.1) – *or* (2.2) – *holds for a given $v \geqslant 4$ for every graph L_v (of v vertices), then it holds for arbitrary other graphs L_μ (for arbitrary $\mu \geqslant 3$), e.g.,*

$$\mathbf{N}^*(L_\mu \overset{*}{\subseteq} G_n) = (1 + o(1))p^{e(L_\mu)}(1-p)^{\binom{\mu}{2}-e(L_\mu)}n^\mu. \tag{2.3}$$

In [8] we proved that the Szemerédi partition of graphs is crucial to the theory of quasi-random graphs.

Given a graph G with two disjoint subsets of vertices, X and Y, the edge density between X and Y is defined by

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

Definition 1. (ε-regularity) Given a graph G and two disjoint vertex sets $X, Y \subseteq V(G)$, we shall call the pair (X, Y) ε-regular, if, for every $X' \subset X$ and $Y' \subset Y$ satisfying $|X'| > \varepsilon|X|$ and $|Y'| > \varepsilon|Y|$, we have

$$|d(X', Y') - d(X, Y)| < \varepsilon.$$

Our main result in [8] was as follows.

Theorem A. (Simonovits and T. Sós) *(G_n) is p-quasi-random if and only if:*

$\mathbf{P}_S(p)$: *for every $\varepsilon > 0$ and κ there exist two integers, $\Omega(\varepsilon, \kappa)$ and $n_0(\varepsilon, \kappa)$, such that, for $n > n_0$, $V(G_n)$ has a partition into k classes $U_1, \dots, U_k$, with $\kappa < k < \Omega(\varepsilon, \kappa)$, $||U_i| - n/k| < \varepsilon n/k$ such that, for all but at most εk^2 pairs (i, j), $1 \leqslant i < j \leqslant k$,*

$$(U_i, U_j) \text{ is } \varepsilon\text{-regular, and } |d(U_i, U_j) - p| < \varepsilon.$$

In our previous paper [9], we investigated those properties $\mathbf{P}$ which *do not imply* quasi-randomness of graph sequences (G_n) on their own, but do imply quasi-randomness if they are assumed not only for the whole graph G_n but also for every sufficiently large *induced* subgraph $F_h \overset{*}{\subseteq} G_n$. We called such properties *hereditarily extended properties*. The consideration of such extensions is motivated by the fact that sufficiently large induced subgraphs of random-like graphs must also be random-like: being a random graph is a 'hereditary property'. Similarly, being a quasi-random graph is also a hereditary property.

Let $\beta_L(p)$ and $\gamma_L(p)$ denote the 'densities' of *labelled induced* and *labelled not necessarily induced* copies of L in a p-random graph, respectively:

$$\beta_L(p) = p^{e(L_v)}(1-p)^{\binom{v}{2}-e(L_v)} \quad \text{and} \quad \gamma_L(p) = p^{e(L_v)}. \tag{2.4}$$

In [9] we have considered graph sequences for which, for a fixed L_v,

$$\begin{aligned} &\text{for every induced subgraph } F_h \overset{*}{\subseteq} G_n \\ &\mathbf{N}(L_v \subseteq F_h) = \gamma_L(p)h^v + o(n^v). \end{aligned} \tag{2.5}$$

Of course, (2.5) a.s. holds for any sequence of p-random graphs, or, more generally, for any sequence of p-quasi-random graphs (G_n). The question is whether (2.5) implies p-quasi-randomness.

Observe that in (2.5) we used $o(n^v)$ instead of $o(h^v)$: that is, for small values of h we allow a relatively much larger error-term. When $h = o(n)$, condition (2.5) is automatically fulfilled. (See also Lemma 3.1.)

One of our main results in [9] was as follows.

Theorem B. *Let L_v be a fixed sample graph, with $e(L_v) > 0$, and let $p \in (0,1)$ be fixed. Let (G_n) be a sequence of graphs for which (2.5) holds. Then (G_n) is p-quasi-random.*

Consequently, (2.5) holds for every other graph L_μ as well.

Theorem B means that, instead of assuming that, for a fixed $v \geqslant 4$, (2.2) holds *for every graph on v vertices* (as in $\mathbf{P}_1(v)$), it is enough to assume it just for one specific L_v, but in the stronger, *hereditarily extended sense* of (2.5). Moreover, Theorem B holds even for $v = 3$.

The structure of the paper. The paper is organized as follows. In the next section we explain why we cannot directly generalize Theorem B to the case of the induced subgraphs. We also explain that, since the number of induced copies of L_v in a p-random graph *is not a monotone function* of p, we should try to prove only that, under some conditions, a graph sequence (G_n) is the union of *two* quasi-random sequences.

In Section 3 we formulate our main result, Theorem 3.2, for 2-class counterexamples, and a general conjecture (Conjecture 3.5). Then we formulate two theorems showing that for P_3 the situation is completely different from what we conjecture for other sample graphs. Finally we formulate some positive results, *e.g.*, that for regular graphs our general conjecture is true.

In Section 4 we establish our main tool, the 'copy polynomial', which is used to reduce the graph-theoretical problem to solving some polynomial equations.

In Section 5 we prove Theorem 3.2, and then we argue that our theorems enable us to decide algorithmically whether, for some (L_v, p), our conjecture holds.

In Section 7 we prove that for regular graphs our conjecture holds. Section 8 contains the proofs of our assertions on the 'strange' case of P_3.

3. New results

The aim of this paper is to investigate phenomena analogous to the one described in Theorem B for the *induced case*, *i.e.*, when (2.5) is replaced by the following

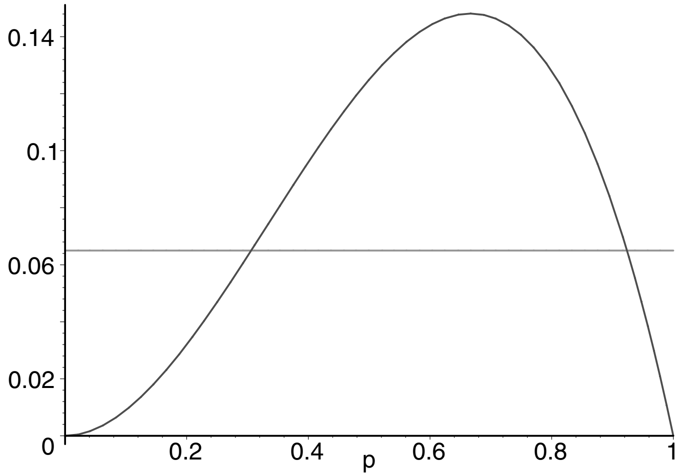

Figure 1.

condition:

$$\begin{gathered}\text{for a fixed } (L_v, p), \text{ for every induced subgraph } F_h \stackrel{*}{\subseteq} G_n, \\ \mathbf{N}^*(L_v \stackrel{*}{\subseteq} F_h) = \beta_L(p)h^v + o(n^v).\end{gathered} \tag{3.1}$$

We shall (implicitly) use the following.

Lemma 3.1. *A graph sequence* (G_n) *satisfies* (3.1) *if and only if there exists a sequence of positive numbers* $\vartheta_n \to 0$ *for which, if* $h > \vartheta_n n$, *then, for every induced subgraph* $F_h \stackrel{*}{\subseteq} G_n$,

$$|\mathbf{N}^*(L_v \stackrel{*}{\subseteq} F_h) - \beta_L(p)h^v| \leqslant \vartheta_n h^v. \tag{3.2}$$

To simplify the formulation of our results, we exclude the cases $L_v = K_v$ and $e(L_v) = 0$. We shall see that the situation for the induced case is much more involved, because, if $G_{n,p}$ is a p-random graph, then the expected number of $\mathbf{N}^*(L_v \stackrel{*}{\subseteq} G_{n,p})$ is not monotone for fixed n while p increases.[3,4]

Clearly, $\beta_L(p)$ (in (2.4)) is a function of p which is monotone increasing in $[0, e(L_v)/\binom{v}{2}]$, monotone decreasing in $[e(L_v)/\binom{v}{2}, 1]$ and vanishes in $p = 0$ and in $p = 1$: see, for example, Figure 1 For every $p \in (0, e(L_v)/\binom{v}{2})$ there is a *unique* probability $\overline{p} \in (e(L_v)/\binom{v}{2}, 1)$, yielding the same expected value. Therefore, hereditarily assuming the number of induced copies does not determine the probability uniquely, unless $p = e(L_v)/\binom{v}{2}$.

[3] We shall speak loosely about random and generalized random graphs, in the following sense: they are distributions on the sets of n-vertex graphs. Still, we shall say that G_n is a $G_{n,p}$-random graph, or later, that $G_n = G(V_1, V_2, u, v, s)$, where the latter is a generalized random graph: see Definition 4.

[4] Here we use the $\stackrel{*}{\subseteq}$ in two places: $F_h \stackrel{*}{\subseteq} G_n$ and $L_v \stackrel{*}{\subseteq} F_h$. They are completely different: the question does not make sense if we replace $F_h \stackrel{*}{\subseteq} G_n$ by $F_h \subseteq G_n$.

Definition 2. Given a graph L_v, the probabilities p and $\overline{p}$ are called *conjugate* if $\beta_L(p) = \beta_L(\overline{p})$, that is,

$$p^{e(L_v)}(1-p)^{\binom{v}{2}-e(L_v)} = \overline{p}^{e(L_v)}(1-\overline{p})^{\binom{v}{2}-e(L_v)}, \tag{3.3}$$

and $p \neq \overline{p}$[5].

Remark 1. A random graph sequence (G_n) with edge probability u satisfies (3.1) if and only if $u \in \{p, \overline{p}\}$.

Example 1. If $e(L_v) = e(\overline{L}_v)$, then for every p the conjugate probability is $\overline{p} = 1 - p$. This is the case, for instance, if L_v is self-complementary.

Obviously, not only p-quasi-random or $\overline{p}$-quasi-random sequences satisfy (3.1), but any sequence obtained by merging two such sequences. These are the 'typical' or 'good' sequences satisfying (3.1). The most we can expect is that if (G_n) satisfies (3.1) then (G_n) is a merged sequence with two conjugate probabilities.

Definition 3. (Strong counterexamples) Given a graph L_v, and a $p \in (0,1)$, we call a graph sequence $\mathcal{G} = (G_n)$ a *strong counterexample* sequence for (L_v, p) if it satisfies (3.1) but it is *not* a quasi-random graph sequence, nor *a merged sequence with conjugate probabilities.*

This notion is not 'empty': we shall see in Theorem 3.3 that there are strong counterexamples for P_3, but we think that basically there are no other cases with strong counterexamples.

We will show that there are two reasons for the existence of strong counterexamples:

- there may occur 'strange' *algebraic coincidences*,
- there are some *degenerate counterexamples*.

Remark 2. If (G_n) is a strong counterexample sequence for (L_v, p), then the same sequence is also a strong counterexample sequence for $\overline{p}$. Further, the complementary graphs $(\overline{G}_n)$ form a sequence of strong counterexamples for $\overline{L}_v$ and $1-p$ (and $1-\overline{p}$).

To formulate our main results, we generalize the notion of random graphs as follows (see [8] for a more general notion of the r-class generalized random graph).

Definition 4. (2-class generalized random graph) Define the random graph $G_n \in \mathcal{G}(V_1, V_2, u, v, s)$ as follows: $V(G_n) = V_1 \cup V_2$ (where $V_1 \cap V_2 = \emptyset$). We join independently the pairs in V_1 with probability u, in V_2 with probability v, and the pairs (x, y) for $x \in V_1$ and $y \in V_2$ with probability s. We shall call this graph *trivial* if $u = v = s$ and *nontrivial* otherwise.

Remark 3. (a) Often, instead of saying 'take a 2-class generalized random graph' from $\mathcal{G}(V_1, V_2, u, v, s)$ and then 'something' almost surely holds, we shall simply write 'let $G_n = G(V_1, V_2, u, v, s)$', then ...

[5] For the 'peak' $p = e(L_v)/\binom{v}{2}$, $p = \overline{p}$: p is 'self-conjugate'.

(b) In many cases we have a sequence (G_n), where each G_n is in a $\mathcal{G}(V_1^{(n)}, V_2^{(n)}, u, v, s)$ and we are not interested in the actual sets $V_1^{(n)}$ and $V_2^{(n)}$, only in that, for some constant $c > 0$, $cn < |V_1^{(n)}|, |V_2^{(n)}| < (1-c)n$. In these cases we shall simply write that $G_n \in \mathcal{G}_c(u, v, s)$, or $G_n \in \mathcal{G}(u, v, s)$.
(c) If $u \in (0,1)$ is fixed and $|V_1| > cn$, then almost surely

$$\mathbf{N}^*(L_v \overset{*}{\subseteq} G(V_1, V_2, u, v, s)) - \mathbb{E}(\mathbf{N}^*(L_v \overset{*}{\subseteq} G(V_1, V_2, u, v, s))) = o(n^v),$$

because the standard deviation is small. So we do not have to distinguish between the expected value or the almost sure value.

Remark 4. Assume that $G_n \in \mathcal{G}(V_1, V_2, u, v, s)$ for $cn < |V_i| < (1-c)n$. If (G_n) satisfies (3.1) then the two parts $G[V_i]$ form random graphs satisfying (3.1) and therefore, by Remark 1,

$$\{u, v\} \subseteq \{p, \overline{p}\}. \tag{3.4}$$

Our main result is as follows.

Theorem 3.2. (Two-class counterexample) *If there is a strong counterexample sequence* (G_n) *for a fixed sample graph* L *and for a probability* $p \in (0,1)$, *then there is also a strong counterexample sequence of form* $G_n \in \mathcal{G}(V_1, V_2, u, v, s)$ $(s \neq u)$ *with* $|V_1| \sim n/2$, *and satisfying* $\{u, v\} \subseteq \{p, \overline{p}\}$.

Remark 5. If, for some fixed $c > 0$, for (L_v, p) there is a strong counterexample sequence of form $G_n = G(V_1, V_2, u, v, s)$ $(s \neq u)$ with $c \leqslant |V_1| \leqslant (1-c)n$, then any $\tilde{G}_n = G(W_1, W_2, u, v, s)$ is also a strong counterexample sequence, assuming that $c \leqslant |W_1| \leqslant (1-c)n$.

So we will ignore the actual classes V_1, V_2: we shall say that the *structure* $\mathcal{G}(u, v, s)$ is a strong counterexample for (L_v, p).

The following theorem shows that, for (P_3, p) and $(\overline{P}_3, p)$, for some $p \in (0,1)$, there exist strong counterexample sequences.

Theorem 3.3. *Let* $L_v = P_3$. *Then we obtain the following properties.*
(a) *For every* $p \geqslant \frac{1}{\sqrt{3}}$, $(p \neq \frac{2}{3})$ *there exists an* $s \in [0,1]$, *namely,*

$$s = s(p) := 3p\frac{1-p}{3p-1} \tag{3.5}$$

such that the sequence $G_n \in \mathcal{G}(p, p, s)$ *is a strong counterexample sequence for* (P_3, p).
(b) *For* P_3, *and* $p_c = \frac{1}{\sqrt{3}} \approx 0.577$, *let the conjugate probability be* $\overline{p}_c$.[6] *For every* $p \leqslant \overline{p}_c$, $(p \neq \frac{2}{3})$, *taking*

$$s^* := 3\overline{p}\frac{1-\overline{p}}{3\overline{p}-1} \in [0,1], \tag{3.6}$$

the sequence $\mathcal{G}(\overline{p}, \overline{p}, s^*)$ *gives a strong counterexample sequence for* (P_3, p).

[6] $\overline{p}_c = -\frac{1}{2\sqrt{3}} + \frac{1}{2} + \frac{4\sqrt{3}}{\sqrt{6}} = 0.7486098314.$

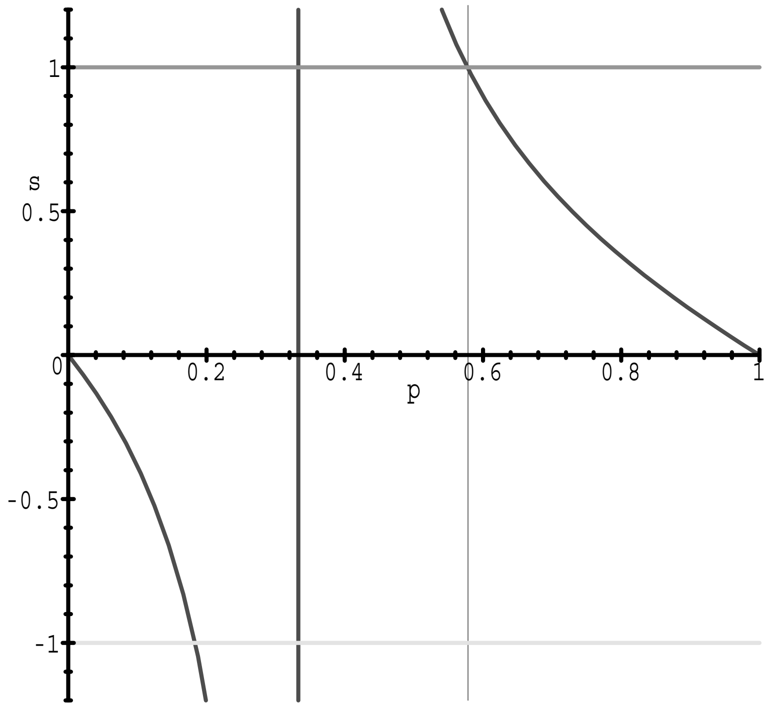

Figure 2. $s(p) := 3p\frac{1-p}{3p-1}$

(c) $\mathscr{G}(u, v, s)$ *is a strong counterexample for* (P_3, p) only if $\mathscr{G}$ *is one of the sequences given in* (a) *or* (b)*:* $u = v \in \{p, \overline{p}\}$ *and* s *and* s^* *are given by* (3.5) *and* (3.6)*, respectively.*

This means that for $p \in (p_c, \overline{p}_c)$ we have two different strong counterexample sequences. (The next theorem will assert that there are no further strong counterexample sequences for P_3.)

To understand the situation for P_3, consider Figure 2, where one can see that $s = s(p)$ in (3.5) is negative in $(0, \frac{1}{3})$, then it becomes positive but larger than 1, and becomes a probability (*i.e.*, $s \in [0, 1]$) only for $p \geqslant \frac{1}{\sqrt{3}}$.

So, for example, we get strong counterexample sequences for $p = \frac{4}{5}$, with $s = \frac{12}{35}$, or for $p = \frac{1}{\sqrt{3}}$, with $s = 1$.

The sharpness of this theorem is expressed by Theorem 3.4 below: it asserts that these are essentially the only strong counterexample sequences for P_3. The proof of this theorem will be given elsewhere.

Theorem 3.4. (Structure of P_3-counterexamples) *If, for* $\mathscr{G} = (G_n)$,

$$\mathbf{N}^*(P_3 \overset{*}{\subseteq} F_h) = p^2(1-p)h^3 + o(n^3) \tag{3.7}$$

holds for every $F_h \overset{*}{\subseteq} G_n$*, then* $\mathscr{G} = (G_n)$ *can be split into four subsequences* $\mathscr{G}_i$*, where*

(a) $\mathscr{G}_1$ *is p-quasi-random;*

(b) *$\mathscr{G}_2$ is $\overline{p}$-quasi-random;*
(c) *for each $G_n \in \mathscr{G}_3$, $V(G_n)$ can be partitioned into two parts: $V(G_n) = V_1^n \cup V_2^n$ so that both $G_n[V_1^n]$ and $G_n[V_2^n]$ are p-quasi-random,*[7] *$d(V_1^n, V_2^n) = s + o(1)$, $s \neq p$, and V_1^n and V_2^n are joined $o(1)$-regularly;*
(d) *$\mathscr{G}_4$ is like $\mathscr{G}_3$, but p and s are replaced by $\overline{p}$ and s^*, respectively.*

We think that P_3 and $\overline{P}_3$ are exceptional sample graphs.

Conjecture 3.5. *Let L_v be fixed, $v \geqslant 4$ and $p \in (0,1)$. Let (G_n) be a graph sequence satisfying* (3.1). *Then (G_n) is the union of two sequences, one being p-quasi-random and the other $\overline{p}$-quasi-random (where one of these two sequences may be finite, or even empty).*

A possible weakening of Conjecture 3.5 could be that, for given L_v, there are only finitely many values of p for which there exist strong counterexample sequences.

We can prove the conjecture only for some special cases.

Theorem 3.6. (Regular graphs) *Let L_v be a regular sample graph, and let $p \in (0,1)$. If, for a graph sequence (G_n),* (3.1) *holds, then (G_n) is the union of p-quasi-random and $\overline{p}$-quasi-random graph sequences.*

We can also prove the conjecture for some small graphs.

Theorem 3.7. *Let L_v be a sample graph, $v = 4$ or $L_v = K(2,3)$, and $p \in (0,1)$. If, for a graph sequence (G_n),* (3.1) *holds, then (G_n) is the union of p-quasi-random and $\overline{p}$-quasi-random graph sequences.*

Theorem 3.7 will be proved in a continuation of this paper.

As we have mentioned, there is a singular, trivial case of counterexamples.

Construction 3.8. (Degenerate counterexamples) *If L_v is connected, and $L_v \neq K_v$, and if G_n is the vertex-disjoint union of $\ell_n \geqslant 2$ complete graphs, then $\mathbf{N}^*(L_v \overset{*}{\subseteq} G_n) = 0$: (G_n) is a sequence of strong counterexamples for L_v and $p = 0$.*

To avoid this and similar counterexamples, we shall always assume that $p \in (0,1)$. We have already excluded $e(L_v) = 0$ and $e(\overline{L}_v) = 0$.

By Theorem 3.2, to prove that Conjecture 3.5 holds for some specific (L_v, p), it is enough to prove that there are no two-class generalized random graph counterexamples. As we shall see, *this reduces to proving that some algebraic equations on (u, v, s) have only the trivial solutions $u = v = s$.* So, Theorem 3.2 can often be used to prove that Conjecture 3.5 holds for certain sample graphs.

[7] Here $G_n[V_1^n]$ is not a graph of n vertices!

If, for some $p \in (0,1)$, there exists a counterexample sequence, then, by Theorem 3.2, we may restrict ourselves to the 2-class generalized random graph counterexample sequences $G_n(V_1, V_2, u, v, s)$ and these may be of three different types:

'counterexamples of the first kind'	$G_n(V_1, V_2, p, p, s)$,
'counterexamples of the second kind'	$G_n(V_1, V_2, \overline{p}, \overline{p}, s)$,
'mixed case'	$G_n(V_1, V_2, p, \overline{p}, s)$, $(p \neq \overline{p})$.

We shall see that for P_3 there are no 'mixed' counterexamples. So Conjecture 3.5 would imply that there are no 'mixed' counterexamples at all.

A corollary of Theorem 3.2 is as follows.

Algorithm 3.9. *There is a finite algorithm such that, if there is no strong counterexample for (L_v, p), then the algorithm will 'prove' this.*

Indeed, in Lemma 4.1 we shall prove that we can reduce the problem to deciding whether a given system of polynomials has roots in a 3-dimensional cube: see Section 5.4. We do not claim that this algorithm is 'efficient'.

Remark 6. All the theorems of this paper are formulated for labelled graphs (induced or not necessarily induced); however, all our results easily extend to unlabelled graphs.

4. The copy polynomials

We shall introduce some polynomials counting the induced copies of L_v in $F_h \overset{*}{\subseteq} G_n = G(V_1, V_2, u, v, s)$. The simplest way to define them is as follows.

Definition 5. (Copy polynomials) Let $L = L_v$ be a fixed 'sample graph' and $k = 0, \dots, v$. For a fixed k we partition the vertices of L into two classes A and B with $|A| = k$, $|B| = v - k$. Let $\eta = e(L_v)$. Then we define $\mathbb{P}^k_{u,v}(s)$ by

$$\mathbb{P}^k_{u,v}(s) := \binom{v}{k} u^\eta (1-u)^{\binom{v}{2}-\eta} - \sum_{\substack{A \subseteq V(L_v) \\ |A|=k}} u^{e(A)}(1-u)^{\binom{k}{2}-e(A)} v^{e(B)} (1-v)^{\binom{v-k}{2}-e(B)} s^{e(A,B)} (1-s)^{k(v-k)-e(A,B)}. \quad (4.1)$$

Here the terms of the $\sum$ are the probabilities that, if we choose k (labelled) points in V_1 and $v - k$ points in V_2, then we get an induced (labelled) L_v. The first term counts these L_v if $u = v = s$. The meaning of these polynomials is expressed in the following lemma.

Lemma 4.1. (Copy polynomials) *Fix an L_v and a $p \in (0,1)$. Assume that $|V_1|, |V_2| > c^* n$ for some fixed $c^* > 0$. Then a graph sequence (G_n) of 2-class generalized random graphs*

$G(V_1, V_2, u, v, s)$ *satisfies* (3.1) *almost surely if and only if* $u, v \in \{p, \overline{p}\}$*; further, s is a common zero of the corresponding system of polynomials of* (4.1).[8,9]

Motivation of Conjecture 3.5. Lemma 4.1 provides some motivation for the conjecture. If for some fixed sample graph L_v and $p \in (0,1)$ Conjecture 3.5 does not hold, then Theorem 3.2 guarantees that there is a generalized random graph counterexample $(G(V_1, V_2, u, v, s))$ where $|V_1| = |V_2| > cn$, and we know that there are only 3 possibilities for $\{u, v\}$, but earlier we did not know the value of s. By Lemma 4.1, we know that $d(V_1, V_2) = s$ is one of the roots of the 'corresponding' copy polynomials.

If we count the number of equations for the induced case, then we mostly find that the system of polynomials is over-determined. Indeed, we generally have a fixed p which determines $\overline{p}$ and therefore we have to solve the three systems of equations.

Equation (6.1) below asserts that all the copy polynomials in (4.1) vanish. Obviously, $u = v = s$ is a solution of (6.1). We wish to motivate the conjecture that there are no other solutions.

If p is fixed, u and v may have only two values, and then the unknown (variable) s must satisfy the system (6.1) of $v-2$ copy polynomial equations, or in the symmetric case, $\lfloor \frac{v}{2} \rfloor$ equations. So for $v \geqslant 4$ we have at least 2 equations for s, more equations than unknowns. And this gets 'worse' as v increases. (On the other hand, as $v \to \infty$, the possibilities for L_v grow exponentially. This could work against the conjecture.)

Proof of Lemma 4.1. By Remark 4, and the conditions of the lemma, $G[V_1]$ and $G[V_2]$ are random graphs satisfying (3.1) and therefore $u, v \in \{p, \overline{p}\}$. So, from now on, although using u, v, we know that they are p or $\overline{p}$.

The basic idea of the proof below is simple: we take a 2-class generalized random graph $G(V_1, V_2, u, v, s)$ and two sets $X \subseteq V_1$ and $Y \subseteq V_2$. That gives a subgraph $G(X, Y, u, v, s)$ and we count the expected value[10] of induced L_vs in it. For each $A \subseteq V(L_v)$ with $|A| = k$ we can easily count those L_vs whose A-vertices are in X and the remaining vertices in Y. Using (3.1) – applied to $G(X, Y, u, v, s)$ – we get an algebraic identity, which reduces to a system of polynomial equations.

Take a $G_n := G(V_1, V_2, u, v, s)$. Let $X \subseteq V_1$, $Y \subseteq V_2$, $|X| = x$, $|Y| = y$. We think of L_v as a graph with $V(L_v) = \{a_1, \dots, a_v\}$ and for any of the 2^v possible 0–1 sequences we have a partition of $V(L_v)$ into A and B.

Let us count the expected value S_k of $L_v \overset{*}{\subseteq} G(X, Y, u, v, s) \subseteq G$, having k vertices in X and $v-k$ vertices in Y.

Put the corresponding k vertices $b_i = \psi(a_i)$ of A into $X \subseteq V_1$, and the others into $Y \subseteq V_2$. The vertices $a_i \in A$ can be put into $X \subseteq V_1$ in $\sim x^k$ ways.[11] The vertices $a_i \in B =$

[8] For a given p we have three choices for $\{u, v\}$: (p, p), $(p, \overline{p})$ and $(\overline{p}, \overline{p})$. They are considered as parameters: we have to solve systems of equations consisting of polynomials of one unknown s.

[9] The expression 'almost surely' could mean here two different assertions: almost surely for each fixed n; or, generating such a generalized random graph for each n, the assertion then holds almost surely for the obtained sequence of graphs. However, here both assertions hold.

[10] which is the typical approximate value, by Remark 3.

[11] The error comes from using $x(x-1)\cdots(x-k+1) \sim x^k$.

$V(L_v) - A$ can be chosen in $\sim y^{v-k}$ ways. The 'expected number' is

$$S_k = x^k y^{v-k} \left(\sum_{\substack{A \subseteq V(L_v) \\ |A|=k}} u^{e(A)}(1-u)^{\binom{k}{2}-e(A)} v^{e(B)}(1-v)^{\binom{v-k}{2}-e(B)} s^{e(A,B)}(1-s)^{k(v-k)-e(A,B)} \right) + o(n^v). \tag{4.2}$$

In each term, the first two factors correspond to the probability that, for the vertices $a_i \in A$, the images, $b_i = \psi(a_i) \in X \subseteq V_1$ are joined according to the given subgraph $L_v[A] \subseteq L_v$, while the next two factors reflect the probability that $L_v[B] \subseteq L_v$ is mapped into $G_n[Y]$ appropriately; the last two factors express the probability that $\psi(A) \subseteq X$ is joined to $\psi(B) \subseteq Y$ according to the bipartite subgraph $L(A, B)$. Here the sum is just the one in definition (4.1) of $\mathbb{P}^k_{u,v}(s)$.

The sum $\sum S_k$ can be obtained in two different ways: considering the whole graph $G[X \cup Y]$ or the separate S_ks, as we did above, and then summing them up. Condition (3.1) holds if and only if, for all the possible choices of x, y $(x + y \leqslant n)$,

$$\sum_k S_k = u^\eta(1-u)^{\binom{v}{2}-\eta}(x+y)^v + o(n^v) = \sum_k \binom{v}{k} u^\eta (1-u)^{\binom{v}{2}-\eta} x^k y^{v-k} + o(n^v).$$

(Here, in the middle, we used that $\beta_L(p) = u^\eta(1-u)^{\binom{v}{2}-\eta}$.) If $c^* > 0$ is fixed and $x, y > c^* n$, then the $o(n^v)$ term is negligible: by (4.2), the above equation holds if and only if

$$S_k = \left(\sum_{\substack{|A|=k \\ A \subseteq V(L_v)}} \cdots \right) x^k y^{v-k} = \binom{v}{k} u^\eta (1-u)^{\binom{v}{2}-\eta} x^k y^{v-k} \quad \text{for} \quad k = 0, \ldots, v,$$

for any x, y. This proves that s is the root of the copy polynomial system, given by (4.1), for $k = 0, \ldots, v$. □

Observe that the above argument also showed that, if $\mathbb{E}(\cdot)$ denotes the expected value, then (for $|X|, |Y| > cn$ and $n \to \infty$)

$$\mathbb{E}(\mathbf{N}^*(L_v \stackrel{*}{\subseteq} G(X, Y, u, v, s))) \sim \sum_{k=0}^{v} \mathbb{P}^k_{u,v}(s)|X|^k|Y|^{v-k}. \tag{4.3}$$

5. Proof of Theorem 3.2

Outline of the proof. This somewhat involved proof is based on the idea that, if we have a sequence (G_n), then we take the Szemerédi partition of the graphs G_n (see below) with the classes $U_1, \ldots, U_k$, and – using an extremal graph theorem and Ramsey's theorem – we can select two groups of classes, $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ (where the A_is and B_js are from the U_qs, and $|I| = |J| = t \to \infty$), so that basically only three densities occur: with appropriate u_0, v_0, s_0,

$$d(A_i, A_{i'}) \approx u_0, \quad d(B_j, B_{j'}) \approx v_0, \quad \text{and} \quad d(A_i, B_j) \approx s_0. \tag{5.1}$$

If the graph sequence $\mathcal{G} = (G_n)$ cannot be partitioned into two quasi-random sequences, then, by Theorem A, the densities in its regular partition cannot be roughly the same (depending only on G_n) and therefore – for some subsequence of (G_n) – we may select the A_i, B_js so that the densities u_0, v_0, s_0 are not all the same. By (5.1) and the regularity of the pairs (A_i, A_j), (B_i, B_j) and (A_i, B_j), if we take the generalized random graph $W_{2m} = G(V_1, V_2, u_0, v_0, s_0)$ for $V_1 = \cup A_i$, and $V_2 = \cup B_j$, then for the spanned subgraphs $W^*_\mu \in \mathcal{G}(V'_1, V'_2, u_0, v_0, s_0)$ ($V'_i \subseteq V_i$, for $i = 1, 2$), we have $\mathbf{N}^*(L_v \stackrel{*}{\subseteq} W^*_\mu) \approx \mathbf{N}^*(L_v \stackrel{*}{\subseteq} G_n[V'_1 \cup V'_2])$ and this enables us to prove that the corresponding structure $\mathcal{G}(u_0, v_0, s_0)$ yields a strong counterexample sequence for (L_v, p).

However, many technical difficulties need to be overcome: among others, we will have two parameters $\tau > 0$ and $\varepsilon > 0$, where $\tau \gg \varepsilon$ will help us to construct the counterexample sequences; $\varepsilon \to 0$ will ensure that we can take the limits of the original edge densities. On the other hand, ε should tend to 0 sufficiently slowly to use condition (3.1), which becomes uninformative if we take over-small subgraphs $F_h \subseteq G_n$. Some technical difficulties are overcome by a hidden diagonalization.

5.1. Regularity Lemma and Szemerédi partitions

An important tool in the proof of our theorem is Szemerédi's Regularity Lemma, which will enable us to apply Theorem A to prove that some graph sequences are quasi-random. We have defined the edge density $d(X, Y)$ and the 'regular pairs' in Definition 1.

Regularity Lemma. (Szemerédi [10]) *For every $\varepsilon > 0$ and integer κ there exist an $n_0(\varepsilon, \kappa)$ and an $\Omega(\varepsilon, \kappa)$ such that, for $n > n_0$, and for every graph G_n, the vertex set $V(G_n)$ can be partitioned into k subsets $U_1, \dots, U_k$ with $\kappa < k < \Omega(\varepsilon, \kappa)$ so that $||U_i| - n/k| < 1$, and all but at most εk^2 pairs (U_i, U_j) are ε-regular.*

Such partitions will be called Szemerédi partitions, κ will be called the *lower bound* on the number of classes, ε the *precision* and (a minimum) $\Omega(\varepsilon, \kappa)$ the *upper bound function*.

5.2. Approximate counting

We first apply a standard counting technique, connected to the Regularity Lemma.

Lemma 5.1. (Approximate counting) *There is a function $f_v(\varepsilon) \to 0$ (as $\varepsilon \to 0$) with the following property. Let $(U_1, \dots, U_k)$ be an ε-regular partition of G_n, $k > \frac{1}{\varepsilon}$, and, for some index set $I \subseteq [1, k]$, let $Z_M \subseteq G_n$ be spanned by $\bigcup_{i \in I} U_i$. Let all the pairs (U_i, U_j) for $i, j \in I$ $(i \neq j)$ be ε-regular. If we replace the edges in Z_M between U_i and U_j independently by random edges of probability $d(U_i, U_j)$, and arbitrarily change the edges with end-vertices in the same classes U_i, then almost surely, in the resulting W_M,*

$$|\mathbf{N}^*(L_v \stackrel{*}{\subseteq} W_M) - \mathbf{N}^*(L_v \stackrel{*}{\subseteq} Z_M)| < \left(\frac{\binom{v}{2}}{|I|} + f_v(\varepsilon) \right) M^v. \tag{5.2}$$

Observe that (5.2) is trivial if $|I| < \binom{v}{2}$. We may and shall assume that

$$f_v(\varepsilon) > v^2 \varepsilon. \tag{5.3}$$

Here $f_v(\varepsilon)$ corresponds to the 'errors' coming from the application of the Regularity Lemma and the approximation by generalized random graphs, and $\binom{v}{2} M^v / |I|$ estimates the number of copies of L_vs having at least two points in the same U_q, $(q = 1, \dots, |I|)$: if $a_\xi, a_\zeta \in V(L_v)$ are mapped into $b_\xi, b_\zeta \in U_q$ (*i.e.*, into the same class), then each b_i $(i = 1, \dots, v)$ can be selected in at most M ways; selecting b_ξ fixes U_q, so b_ζ can be selected in at most $|U_q| \leqslant M/|I|$ ways.

5.3. The proof

We now have the tools to prove the theorem. Assume that $\mathscr{G} = (G_n)$ is a strong counterexample sequence for (L_v, p). We shall apply the Regularity Lemma to these G_n with some $\varepsilon > 0$ and *lower bound* $\kappa = R_\varepsilon$ *on the number of classes*, and use the obtained regular partitions to build a strong 2-class counterexample structure $\mathscr{G}(u, v, s)$.[12] In the proof we shall concentrate on the distance of the densities of $d(U_i, U_j)$ from p and $\overline{p}$: we shall keep checking whether these distances are smaller than a constant $\tau \gg \varepsilon$.

The parameters $r(\varepsilon)$, $\omega(\varepsilon)$ and R_ε. Define $t := \lceil 1/\varepsilon \rceil$, and $T := \lceil 1/\tau \rceil$. The proof will (implicitly) use a diagonalization procedure: we apply the Regularity Lemma to each of our graphs G_n with an $\varepsilon \to 0$ *very slowly*. At the end, we shall let $\tau \to 0$ as well.

When applying the Regularity Lemma to a G_n with a fixed $\varepsilon > 0$, we shall use a lower bound R_ε on the number of classes, defined as follows.

The Kővári–T. Sós–Turán theorem [7] asserts that

$$\text{if } K(a,b) \nsubseteq H_k \quad \text{then } e(H_k) < \frac{1}{2}\sqrt[a]{b-1}\, k^{2-1/a} + \frac{a-1}{2}k. \tag{5.4}$$

We shall need only that, for fixed ω,[13]

$$\text{if } K(\omega,\omega) \nsubseteq H_k \quad \text{then } e(H_k) = o(k^2), \quad \text{as } k \to \infty. \tag{5.5}$$

Using (5.5), define the (generalized Ramsey) number $\omega(\varepsilon)$ in order that, if we t-colour $K(\omega(\varepsilon), \omega(\varepsilon))$, then we must always have a monochromatic $K(t,t)$. Choose (the Ramsey number) $r(\varepsilon)$ such that every edge colouring of a $K_{r(\varepsilon)}$ by $t+1$ colours contains a monochromatic $K_{\omega(\varepsilon)}$.

Now we fix R_ε so that, for *any graph* H_k, if $k > R_\varepsilon$ and $e(H_k) > \varepsilon^2 k^2$, then $H_k \supseteq K_2(r(\varepsilon), r(\varepsilon))$. This can be done, again, by (5.5).

Types of regular partition. For $\varepsilon > 0$ and R_ε – defined above – we determine two constants n_ε and Ω_ε, so that, by the Regularity Lemma, for each G_n of a sequence $\mathscr{G} = (G_n)$ we have an ε-regular partition,

$$V(G_n) := \bigcup_{i=1}^{k} U_i, \quad \text{with } R_\varepsilon < k < \Omega_\varepsilon, \tag{5.6}$$

for $n > n_\varepsilon$. This partition of $V(G_n)$ can have (at most) εk^2 non-ε-regular pairs (U_i, U_j). For each such pair we delete all the edges joining U_i to U_j. That may change $\mathbf{N}^*(L_v \overset{*}{\subseteq} G_n)$ by at most εn^v. We shall apply (3.1) to a situation where $\varepsilon \to \infty$. Therefore, deleting these edges will not change the validity of (3.1). After this, all the pairs become ε-regular.

[12] Recall the use of this notation when we wish to emphasize that the structure is important, not the sets V_1, V_2.
[13] Actually, this is a subcase of the Erdős–Stone theorem.

Now (L_v, p) is fixed, and therefore $\overline{p}$ is also fixed. An ε-regular partition $(U_1, \ldots, U_k)$ of a graph G_n will be classified as follows.[14]

(a) 'Bad densities': at least τk^2 of the ε-regular pairs (U_i, U_j) satisfy

$$|d(U_i, U_j) - p| > \tau \quad \text{and} \quad |d(U_i, U_j) - \overline{p}| > \tau. \tag{5.7}$$

(b) 'Mixed densities': assume that (a) does not hold, but

$$\begin{cases} |d(U_i, U_j) - p| < \tau & \text{for at least } \tau k^2 \text{ pairs } (U_i, U_j)\text{, and} \\ |d(U_i, U_j) - \overline{p}| < \tau & \text{for some other } \tau k^2 \text{ pairs } (U_i, U_j). \end{cases} \tag{5.8}$$

(c) Neither (a) nor (b) holds.

Classification of graph sequences. Let $\mathscr{G}$ (as we assumed) be a strong counterexample sequence. Below, certain subgraphs and subsets obtained from some graphs $G_n \in \mathscr{G}$ may depend on n, but we shall not indicate this dependence in our notation. We shall also mostly neglect to indicate the dependence on ε, τ or t.

Also, ignoring the simpler case $p = \overline{p}$, we assume that $p \neq \overline{p}$. (In the general case $p \neq \overline{p}$ we may have two types of counterexample, one corresponding to (a), the other to (b) above. If $p = \overline{p}$, then case (b) disappears: it yields a quasi-random sequence, and the analysis of case (a) becomes 'one step' shorter.)

For a given $\varepsilon > 0$, we defined R_ε in the previous subsection, and Ω_ε (the 'upper bound function' in the Regularity Lemma) in (5.6). Finally, let $\mathscr{G}[\varepsilon]$ denote the subsequence of graphs $G_n \in \mathscr{G}$ for which

$$\begin{aligned} &\text{for every} \quad F_h \stackrel{*}{\subseteq} G_n \quad \text{with} \quad h > n/\Omega_\varepsilon, \\ &|\mathbf{N}^*(L_v \stackrel{*}{\subseteq} F_h) - \beta_L(p)h^v| < \varepsilon h^v. \end{aligned} \tag{5.9}$$

(Note that, for fixed ε, by $h > n/\Omega_\varepsilon$, and, by (3.1), we have $|\mathbf{N}^*\big(L_v \stackrel{*}{\subseteq} F_h\big) - \beta_L(p)h^v| = o(n^v)$, and therefore (5.3) holds for every $n > n_0(\varepsilon)$. See also Lemma 3.1.)

For the given (L_v, p), we shall create a 2-dimensional, infinite matrix of graphs, $\mathbb{B}$. For every pair (τ, ε) (where $\tau = 1, \frac{1}{2}, \frac{1}{3} \ldots$ and $\varepsilon = 1, \frac{1}{2}, \frac{1}{3} \ldots$) our matrix will have a 'box', $\mathbb{B}_{\tau,\varepsilon}$ which may be empty or may contain a graph $G_n \in \mathscr{G}$, with an ε-regular partition.

For $\varepsilon \geqslant \tau$ we agree to set $\mathbb{B}_{\tau,\varepsilon} = \emptyset$. For each (τ, ε), with $\varepsilon < \tau$, we check if there exist graphs $G_n \in \mathscr{G}[\varepsilon]$ having ε-regular partitions for $\kappa = R_\varepsilon$, satisfying either (a) or (b). If there are no such graphs, we define $\mathbb{B}_{\tau,\varepsilon} = \emptyset$.

If there exist such graphs G_n, we put one of them into $\mathbb{B}_{\tau,\varepsilon}$, and also fix a corresponding ε-regular partition of it, $(U_1, \ldots, U_k) = (U_1^{n,\varepsilon}, \ldots, U_{k(n,\varepsilon)}^{n,\varepsilon})$. The same graph G_n may occur in several rows.

[14] Here the full notation would be $(U_1^{n,\varepsilon}, \ldots, U_{k(n,\varepsilon)}^{n,\varepsilon})$.

We distinguish two cases.

(i) There exists a τ for which *infinitely many* $\mathbb{B}_{\tau,\varepsilon}$ are non-empty.

(ii) For any τ, we have only finitely many non-empty boxes $\mathbb{B}_{\tau,\varepsilon}$: if $\varepsilon < \varepsilon_0(\tau)$, then, for any $G_n \in \mathcal{G}$ with sufficiently large n, every ε-regular partition $(U_1, \ldots, U_k)$ has at most τk^2 pairs (U_i, U_j) satisfying (5.7) or (5.8).[15]

In case (i) we shall provide the 2-class counterexamples; in case (ii) we shall prove that our $\mathcal{G} = (G_n)$ – satisfying (3.1) – is the union of p-quasi-random and $\overline{p}$-quasi-random graph sequences. We start with the simpler case (ii).

Settling case (ii). Let $\mathcal{G} = (G_n)$ be a graph sequence satisfying (3.1). Decompose $\mathcal{G}$ into $\mathcal{G}_1 \cup \mathcal{G}_2$ by the densities

$$G_n \in \mathcal{G}_1 \quad \text{if} \left| \frac{2e(G_n)}{n^2} - p \right| < \frac{|p - \overline{p}|}{2}; \quad \text{otherwise } G_n \in \mathcal{G}_2. \tag{5.10}$$

The partition $\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2$ does not depend on τ. Assuming that $\mathcal{G}_1$ is infinite, we show that $\mathbf{P}_S(p)$ holds for $\mathcal{G}_1$. Therefore, by Theorem A, $\mathcal{G}_1$ is p-quasi-random.

Fix a $\tau < \frac{p-\overline{p}}{10}$. Assume that $\tau > 0$ is sufficiently small. Case (ii) means that $\mathbb{B}_{\tau,\varepsilon} = \emptyset$ for any sufficiently small $\varepsilon > 0$. In other words, every ε-regular partition of G_n is of type (c) in our classification. If n is sufficiently large, then, by (3.1), we also have (5.3). So the only reason we have not put our $G_n \in \mathcal{G}_1$ into $\mathbb{B}_{\tau,\varepsilon}$ is that G_n does not satisfy (5.7), or (5.8). Hence, for all but at most $2\tau k^2$ ε-regular pairs (U_i, U_j), either

(∗) $|d(U_i, U_j) - p| \leqslant \tau$, or
(∗∗) $|d(U_i, U_j) - \overline{p}| \leqslant \tau$.

Case (∗∗) would imply that $2e(G_n)/n^2$ is nearer to $\overline{p}$ than to p, violating (5.10), and contradicting the definition of $\mathcal{G}_1$.

The above argument works for arbitrary small values of τ. Hence $\mathbf{P}_S(p)$ holds for $\mathcal{G}_1$, and therefore $\mathcal{G}_1$ is p-quasi-random. Similarly, $\mathcal{G}_2$ is also either finite or $\overline{p}$-quasi-random. This proves that $\mathcal{G}$ is the union of two quasi-random graph sequences.

(This was the part where we used an 'implicit diagonalization'.)

Settling case (i). Now we know that there is a τ for which $\mathbb{B}_{\tau,\varepsilon} \neq \emptyset$, for *infinitely many* $\varepsilon = \frac{1}{t}$. Fix this τ. Choose infinitely many graphs $G_n \in \left\{\mathbb{B}_{\tau,\varepsilon} : \varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$: they form a sequence $\mathcal{G}^* = (G_n : n \in N_1)$.[16] Recall that a regular partition is also fixed for each G_n. We shall distinguish two subcases.

(α) There exists an infinite subsequence $\mathcal{G}^{**} \subseteq \mathcal{G}^*$ for which the corresponding graphs G_{n_t} and their $\frac{1}{t}$-regular partitions are of type (a) (see (5.7)). (This is the more important case.)

(β) There exists an infinite $\mathcal{G}^{**} \subseteq \mathcal{G}^*$ for which the corresponding G_{n_t} and their $\frac{1}{t}$-regular partitions are of type (b) (see (5.8)).

[15] Formally we should restrict ourselves to $G_n \in \mathcal{G}[\varepsilon]$ but this does not make any difference here: all but finitely many $G_n \in \mathcal{G}[\varepsilon]$.

[16] This $\mathcal{G}^*$ forms a strong counterexample, but we shall not need this directly.

It may happen that both (α) and (β) hold. In both cases we shall find the promised 2-class counterexample sequences of type $\mathscr{G}(u, v, s)$ (with (3.4)).

Details of case (α).[17]

(α_1) Now we assume that there exists a $\tau > 0$ and infinitely many G_n, each one with an ε-regular partition

$$V(G_n) = U_1 \cup \cdots \cup U_k,$$

having at least τk^2 ε-regular pairs (U_i, U_j) satisfying (5.7). We know that the classes U_i have sizes $\ell \sim n/k \geqslant n/\Omega_\varepsilon$, where Ω_ε was fixed when we applied the Regularity Lemma (see (5.6)).

We restrict ourselves to a fixed $G_n \in \mathbb{B}_{\tau,\varepsilon}$ and the corresponding regular partition $(U_1, \ldots, U_k)$. Consider the graph $H_k := H_k(\tau, \varepsilon)$, the vertices of which are the classes U_i $(i = 1, \ldots, k)$, and the edges of which are the regular pairs satisfying (5.7).[18] We shall colour these edges (= pairs) with their *rounded densities*: (U_i, U_j) gets colour $\chi(U_i, U_j) := \frac{1}{t}\lceil t \cdot d(U_i, U_j)\rceil$. Hence we get a $\leqslant (t+1)$-coloured H_k. In fact, H_k has fewer than t colours, since the densities near to p or $\overline{p}$ are excluded, by (5.7).

Let H_k^* denote the *monochromatic* subgraph of H_k having the *most* edges. By (5.7), $e(H_k) \geqslant \tau k^2$. Clearly,

$$e(H_k^*) > \frac{1}{t} e(H_k) \geqslant \frac{\tau}{t} k^2 > \varepsilon^2 k^2.$$

So, by $k > R_\varepsilon$, we have a monochromatic $K(r(\varepsilon), r(\varepsilon)) \subseteq H_k$, that is, two sets of classes, $A_1, \ldots, A_{r(\varepsilon)}$ and $B_1, \ldots, B_{r(\varepsilon)}$ (where each of them is some U_i) such that all the pairs (A_i, B_j) satisfy (5.7) and are of the same colour, say $s = s_{n,\tau}$.[19]

(α_2) Here the pairs $(A_i, A_{i'})$ and $(B_j, B_{j'})$ $(1 \leqslant i, i', j, j' \leqslant r(\varepsilon))$ may have many different rounded densities. We apply Ramsey's theorem to the complete graph $K_{r(\varepsilon)}$, which is defined on $A_1, \ldots, A_{r(\varepsilon)}$ and coloured by the (at most) $t+1$ rounded densities. This complete graph contains a *monochromatic* $K_{\omega(\varepsilon)}$ spanned by some classes $\{A_i : i \in I\}$, $(|I| = \omega(\varepsilon))$. Similarly, the $(t+1)$-coloured $K_{r(\varepsilon)}$ defined on $B_1, \ldots, B_{r(\varepsilon)}$ contains a *monochromatic* subgraph, spanned by some classes $\{B_j : j \in J\}$, $(|J| = \omega(\varepsilon))$. We may assume that $I = J = \{1, \ldots, \omega(\varepsilon)\}$. Let the colour used for $(A_i, A_{i'})$ be $u = u_n$, and for $(B_j, B_{j'})$ $v = v_n$.

Since the colours encode densities, we used altogether at most 3 (rounded) densities. These define a 'structure' $\mathscr{G}(u_n, v_n, s_n)$. Clearly, the densities (u_n, v_n, s_n) satisfy

$$|d(A_i, A_{i'}) - u_n| \leqslant \frac{1}{t} = \varepsilon, \quad |d(B_j, B_{j'}) - v_n| \leqslant \varepsilon,$$

and

$$|d(A_i, B_j) - s_n| \leqslant \varepsilon.$$

[17] In this case a single application of the Ramsey theorem to define R_ε would suffice.

[18] Generally we would call this graph the *coloured reduced graph* or *coloured cluster graph*.

[19] The same G_n may occur in many rows ($\mathbb{B}_{\tau,\varepsilon}$).

Here we have infinitely many graphs G_n, and each of them corresponds to smaller and smaller $\varepsilon = \varepsilon_t = \frac{1}{t} \to 0$. If, by chance, (u_n, v_n, s_n) is not convergent, then we take a *convergent subsequence*: $(u_n, v_n, s_n) \to (u_0, v_0, s_0)$.

(α_3) We shall prove that, if $|V_1| = |V_2| = m = \omega(\varepsilon)\ell$ and we take a W_{2m} from $\mathscr{G}(V_1, V_2, u_0, v_0, s_0)$, then (W_{2m}) is almost surely a strong counterexample sequence.

The proof proceeds by comparing the subgraph $Z_{2m} \subseteq G_n$ spanned by the classes $A_1, \dots, A_\omega$ and $B_1, \dots, B_\omega$ to its 'limit randomization' W_{2m}, *i.e.*, to the graph *built on the same vertex set* but joining the vertices in $V_1 = \cup A_i$ with probability u_0, in $V_2 = \cup B_j$ with probability v_0, and the vertices of $\cup A_i$ to the vertices of $\cup B_j$ with probability s_0, independently. So W_{2m} is taken from $\mathscr{G}(V_1, V_2, u_0, v_0, s_0)$ and $V(W_{2m}) = V(Z_{2m})$.

To prove that (W_{2m}) is a strong counterexample sequence, we need to prove that, if $\vartheta > 0$ is a constant, $\mu = v(W^o_\mu) > \vartheta m$, and $W^o_\mu \overset{*}{\subseteq} W_{2m}$, then

$$\mathbf{N}^*\big(L_\nu \overset{*}{\subseteq} W^o_\mu\big) = \beta_L(p)\mu^\nu + o(m^\nu). \tag{5.11}$$

Applying (3.1) to subgraphs of G_n spanned by V_1 or V_2, we get that $u_0, v_0 \in \{p, \overline{p}\}$. We will need that $s \neq p, \overline{p}$: we would not get a counterexample if (and only if) either $u_0 = v_0 = s_0 = p$ or $u_0 = v_0 = s_0 = \overline{p}$ were valid. But we know $s \neq p, \overline{p}$, by (5.7):

$$|s_0 - p| \geqslant \tau, \quad \text{and} \quad |s_0 - \overline{p}| \geqslant \tau. \tag{5.12}$$

(α_4) To prove (5.11) we compare $\mathbf{N}^*(L_\nu \overset{*}{\subseteq} W^o_\mu)$ with $\mathbf{N}^*(L_\nu \overset{*}{\subseteq} Z^o_\mu)$, where $Z^o_\mu \subseteq Z_{2m} \subseteq G_n$ is spanned by the vertices of W^o_μ. By (5.3), we know that

$$\mathbf{N}^*\big(L_\nu \overset{*}{\subseteq} Z^o_\mu\big) = \beta_L(p)\mu^\nu + o(m^\nu). \tag{5.13}$$

We shall prove that

$$\mathbf{N}^*\big(L_\nu \overset{*}{\subseteq} W^o_\mu\big) - \mathbf{N}^*\big(L_\nu \overset{*}{\subseteq} Z^o_\mu\big) = o(m^\nu). \tag{5.14}$$

To prove (5.14) we transform Z^o_μ into W^o_μ in five steps and estimate the corresponding errors.

(1) One error comes from the fact that, to prove the hereditary property, we should take an arbitrary $W^o_\mu \subseteq W_{2m}$, but we shall restrict ourselves to those W^o_μ which are the unions of complete classes A_i and B_j.

We show that the error coming from this is $o(m^\nu)$. The vertices of the original W^o_μ can be scattered around in Z_{2m}. Recall that the size of the Us (or A_is and A_js) was ℓ (or $\ell + 1$). First we delete at most 2ℓ vertices to get a multiple of ℓ both in V'_1 and in V'_2. Then we replace them by other vertices so that the new vertices fill up complete classes A_i or B_j, but keep the sizes of (the new) $V'_1 = V(W^o_\mu) \cap (\cup A_i)$ and $V'_2 = V(W^o_\mu) \cap (\cup B_j)$.

The 'moving around' does not change the expected number of the L_νs in the subgraph, since that depends only on the sizes of V'_1 and V'_2. Deleting the $\leqslant 2\ell$ vertices results in an error at most

$$2\ell\mu^{\nu-1} \leqslant 2\varepsilon m^\nu, \tag{5.15}$$

since $\ell \leqslant m/t = \varepsilon m$.

(2) Consider now some W^o_μ whose $V(W^o_\mu)$ is the union of full classes: for some I' and J', we set

$$V_1' := \bigcup_{i\in I'} A_i \quad \text{and} \quad V_2' := \bigcup_{j\in J'} B_j'. \tag{5.16}$$

Assuming above that $\mu \geqslant \vartheta m$ is equivalent to assuming that $\lambda = |I' \cup J'| \geqslant \vartheta\omega(\varepsilon)$. (One of I' or J' may be empty.) Now we have 4 graphs: W_{2m}, the subgraph $W^o_\mu = G(V_1', V_2', u_0, v_0, s_0)$ of W_{2m}, Z_{2m}, and the corresponding subgraph $Z^o_\mu \subseteq Z_{2m} \subseteq G_n$.

We get W^o_μ from Z^o_μ in the same way as we get W_{2m} from Z_{2m}.

(2a) In most applications of the Regularity Lemma, we delete the edges within the classes; here we replace them by randomly selected edges, with edge probabilities u_0 for the A_is and v_0 for the B_js. This changes the number of L_νs by at most $o(\mu^\nu)$, since we have $\lambda \geqslant \vartheta\omega(\varepsilon) \to \infty$ classes in W^o_μ and less than μ^2/λ edges inside these classes. This creates or ruins at most $O(\mu^\nu/\lambda)$ L_νs.

(2b) We wish to apply Lemma 5.1 to Z^o_μ. We operate with three types of density for (U_i, U_j): the original $d(U_i, U_j)$, the rounded one, $\chi(U_i, U_j)$, and the three limit densities u_0, v_0, s_0. In Lemma 5.1 we should use the original densities $d(A_i, A_j)$, $d(A_i, B_j)$, and $d(B_i, B_j)$, instead of (u_0, v_0, s_0).

We could easily generalize the lemma, or use the following trick of adding or subtracting edges randomly: for each pair (A_i, A_j) we may add/delete $|u_0 - d(A_i, A_j)|h^2$ random edges to get the density u_0. Applying the corresponding modification to the pairs (B_i, B_j) and (A_i, B_j) we get two graphs, Q_{2m} and Q^o_μ, to which we can apply Lemma 5.1. (So most of the edges of Q_{2m} are the original edges of $Z_{2m} \subseteq G_m$.) This approximation includes two steps: rounding $d(U_i, U_j)$ up, to a multiple of ε, and taking the limit. Replacing the actual densities by the rounded densities (u_n, v_n, s_n) changes at most $\varepsilon\mu^2$ edges. This yields again only an error at most $\varepsilon\mu^\nu$. Then, replacing (u_n, v_n, s_n) by (u_0, v_0, s_0) we have, almost surely,

$$\begin{aligned}\left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Q^o_\mu\right) - \mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Z^o_\mu\right)\right| &\leqslant 2\mu^\nu \max\{|u_n - u_0|, |v_n - v_0|, |s_n - s_0|\} \\ &= o(\mu^\nu), \quad \text{as} \quad n \to \infty. \end{aligned} \tag{5.17}$$

(We used the factor 2 to compensate for the randomness.)

(3) Finally, we apply Lemma 5.1 to Q^o_μ. Its randomization (described in Lemma 5.1) is just $W^o_\mu := G(V_1', V_2', u_0, v_0, s_0)$, where we used (5.16). By Lemma 5.1,

$$\left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Q^o_\mu\right) - \mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} W^o_\mu\right)\right| < \left(f_\nu(\varepsilon) + \frac{\nu^2}{\lambda}\right)\mu^\nu. \tag{5.18}$$

Since $G_n \in \mathcal{G}[\varepsilon]$, it satisfies (5.3). Therefore

$$\left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Z^o_\mu\right) - \beta_L(p)\mu^\nu\right| \leqslant \varepsilon\mu^\nu \quad \text{as} \quad n \to \infty. \tag{5.19}$$

Using (5.15), (5.17), (5.18) and (5.19), we get for every $W^o_\mu \overset{*}{\subseteq} W_{2m}$ that

$$\begin{aligned}\left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} W^o_\mu\right) - \beta_L(p)\mu^\nu\right| &\leqslant \left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} W^o_\mu\right) - \mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Q^o_\mu\right)\right| \\ &\quad + \left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Q^o_\mu\right) - \mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Z^o_\mu\right)\right| \\ &\quad + \left|\mathbf{N}^*\left(L_\nu \overset{*}{\subseteq} Z^o_\mu\right) - \beta_L(p)\mu^\nu\right| = o(m^\nu),\end{aligned}$$

as $n \to \infty$, *i.e.*, we get (5.11). Since, by (5.12), $s_0 \neq p, \overline{p}$, this concludes the proof that the generalized random graph structure $\mathcal{G}(u_0, v_0, s_0)$ really gives a strong counterexample sequence.

Details of case (β). We have to modify the previous argument just a little bit. We know that there are at least τk^2 pairs (U_i, U_j) with densities around p, and also at least τk^2 with densities around $\overline{p}$, and that $0 < \tau < \frac{|p-\overline{p}|}{10}$.

We first consider the pairs (U_i, U_j) of density approximately p, and repeat the argument of (α). We get a graph $Z_{2\omega(\varepsilon)\ell}$ as above. Then, taking the limit, we get a 2-class graph $G(V_1, V_2, u, v, s)$ which is a counterexample, unless it is a $G(V_1, V_2, p, p, p)$: we know that $s_n \to p$. Then we can consider the pairs of density $\sim \overline{p}$, and either get a counterexample or a $G(V_1, V_2, \overline{p}, \overline{p}, \overline{p})$.

If we have not yet obtained the desired 2-class counterexample, then we proceed as follows. In the original graph G_n we have the classes $A_1, \dots, A_{\omega(\varepsilon)}, B_1, \dots, B_{\omega(\varepsilon)}$ corresponding to $G(V_1, V_2, p, p, p)$, and $C_1, \dots, C_{\omega(\varepsilon)}$, $D_1, \dots, D_{\omega(\varepsilon)}$ corresponding to $G(V_1, V_2, \overline{p}, \overline{p}, \overline{p})$. $(\cup A_i) \cup (\cup B_j)$ and $(\cup C_i) \cup (\cup D_j)$ intersect in at most one cluster, since the densities in the first case are near to p, and in the second case near to $\overline{p}$. So we may assume that $A_1, \dots, A_{\omega(\varepsilon)}$ and $C_1, \dots, C_{\omega(\varepsilon)}$ are all different. The densities satisfy $d(A_i, A_j) \sim p$ and $d(C_i, C_j) \sim \overline{p}$. The densities $d(A_i, C_j)$ could be arbitrary, but applying (5.5) to them we can get (apart from the indexing) $A_1, \dots, A_t$ and $C_1, \dots, C_t$, where $d(A_i, C_j) = s_n \to s_0$ (for a subsequence). So, taking the limit, we get the structure $G(V_1, V_2, p, \overline{p}, s)$ as in (α), *now* satisfying (5.8) and again giving a strong counterexample sequence.

Perhaps just one additional remark should be made here. If the reader dislikes the fact that in the last step we only got $2t$ classes instead of $2\omega(\varepsilon)$ classes, then we could change the above proof in several ways. We notice, for instance, that we needed only two things: (5.3), and the number of classes tending to ∞. □

5.4. The algorithm

Tarski's theorem [11] asserts – in a very special case – that if we have a polynomial of d variables and wish to decide if this polynomial has zeros in the d-dimensional unit cube, where not all the coordinates of this solution are equal, then that can be decided algorithmically: there is an algorithm which will give a 'yes' or 'no' to this question in finitely many steps.

This means that, for a given L_v, we can decide algorithmically whether there exists a $p \in (0,1)$ for which there exist strong counterexamples. Similarly, for a given (L_v, p) we can decide algorithmically whether there exist strong counterexamples.

We shall not explain Tarski's theorem here, for it is beyond our scope, but basically it asserts that, if some assertion of Peano arithmetic is given by polynomials and inequalities, then it can be proved or disproved algorithmically. In our case the inequalities are that the variables u, v, s are restricted to the unit interval and that they are not all equal.

6. Remarks on the copy polynomials

In this section we have collected a few easy facts about the copy polynomials. Some of them will be used in the next section.

The vertices of L_v are labelled. Each $L_v \stackrel{*}{\subseteq} G(V_1, V_2, u, v, s)$ defines a partition of $V(L_v)$. The partitions correspond to the 2^v 0-1 sequences, and $\binom{v}{k}$ of them contribute to $\mathbb{P}^k_{u,v}(s)$. Replacing k by $v-k$ is equivalent to switching to the complementary set of A. Hence the system of copy polynomials is symmetric in the following sense.

Proposition 6.1. $\mathbb{P}^k_{u,v}(s) = \mathbb{P}^{v-k}_{v,u}(s)$.

So, in the *symmetric case*, when $u = v \in \{p, \overline{p}\}$, we have $\lfloor \frac{v}{2} \rfloor$ equations.

Breaking the symmetry p versus $\overline{p}$. Using Remark 2 and replacing the original p by $\overline{p}$, if necessary, we get the following.

Proposition 6.2. *If, for a given L_v, there are no counterexamples of the structure $\mathcal{G}(p, p, s)$ or of the structure $\mathcal{G}(p, \overline{p}, s)$, for $p \in (0, 1)$, then there are no counterexamples at all.*

Generally we shall be interested in the solutions of the system of polynomial equations

$$\mathbb{P}^k_{u,v}(s) = 0, \quad k = 0, \dots, v, \quad \text{where} \quad u \in \{p, \overline{p}\}. \tag{6.1}$$

We may forget $k = v$, since $\mathbb{P}^v_{u,v}(s) = 0$ for $s \in (0, 1)$. It is worth considering the cases $k = 0$ and $k = v - 1$ separately. For $k = 0$ we retrieve (3.3),

$$u^\eta (1-u)^{\binom{v}{2}-\eta} = v^\eta (1-v)^{\binom{v}{2}-\eta},$$

where $\eta = e(L_v)$, *i.e.*, $v \in \{p, \overline{p}\}$ as well. More importantly, $\mathbb{P}^{v-1}_{u,v}(s)$ does not contain v, since $|B| = 1$. Actually, if $V(L_v) = \{a_1, \dots, a_v\}$ and d_i denotes the degree of a_i in L_v, then (taking $B := \{a_i\}$) we get the following lemma.

Lemma 6.3. $\mathbb{P}^{v-1}_{u,v}(s) = 0$ *is equivalent to*

$$1 = \frac{1}{v} \sum_i \left(\frac{s}{u}\right)^{d_i} \left(\frac{1-s}{1-u}\right)^{v-1-d_i}. \tag{6.2}$$

Therefore, for a given s we can choose u only in finitely many ways.

Proof of Lemma 6.3. If the degree sequence of L_v is $(d_1, d_2, \dots, d_v)$, then, since $e(B) = 0$, equation (4.1) reduces to

$$\mathbb{P}^{v-1}_{u,v}(s) = v u^{\frac{1}{2}\sum d_i} (1-u)^{\binom{v}{2} - \frac{1}{2}\sum d_i} - \sum_{i=1}^{v} u^{\frac{1}{2}\sum d_i - d_i} (1-u)^{\binom{v-1}{2} - \frac{1}{2}\sum d_i + d_i} s^{d_i} (1-s)^{v-d_i-1}$$

So $\mathbb{P}^{v-1}_{u,v}(s) = 0$ is equivalent to

$$0 = v(1-u)^{v-1} - \sum_{i=1}^{v} u^{-d_i} (1-u)^{d_i} s^{d_i} (1-s)^{v-d_i-1}. \tag{6.3}$$

This proves Lemma 6.3. □

Taking the two sides of (3.3) to the power $\frac{2}{v-1}$, we get the following.

Lemma 6.4. *If $D = 2e(L_v)/v$ (i.e., D is the average degree of L_v), then the conjugacy relation* (3.3) *is described by*

$$p^D(1-p)^{v-D-1} = \overline{p}^D(1-\overline{p})^{v-D-1}. \tag{6.4}$$

An easy but useful consequence of these assertions is as follows.

Corollary 6.5. *If $G_n \in \mathscr{G}(u, v, s)$, for $u, v, s \in \{p, \overline{p}\}$, satisfies* (3.1)*, then $u = v$, too.*

Proof. By Lemma 4.1, the conditions of the corollary imply (6.2). Using (6.2) and the inequality between the arithmetic and geometric means, we get, for $s = v, u = \overline{v}$, that

$$1 = \frac{1}{v}\sum_i \left(\frac{s}{u}\right)^{d_i}\left(\frac{1-s}{1-u}\right)^{v-1-d_i} \geqslant \prod_i \left(\frac{s}{u}\right)^{\frac{1}{v}\sum d_i}\left(\frac{1-s}{1-u}\right)^{\frac{1}{v}\sum(v-1-d_i)}$$
$$= \frac{s^D(1-s)^{v-1-D}}{u^D(1-u)^{v-1-D}} = 1.$$

Hence all the terms $(\frac{s}{u})^{d_i}(\frac{1-s}{1-u})^{v-1-d_i}$ must be equal, implying that if L_v is not regular then $s = u$. The remaining case is when L_v is regular. Then we use Theorem 3.6 (proved below) according to which, if L_v is regular and $G_n \in \mathscr{G}(u, v, s)$ satisfies (3.1), then $u = v = s$. □

7. Proof of Theorem 3.6

Using Theorem 3.2 and Lemma 4.1, below we show that, if L_v is d-regular, $u, v, s \in (0, 1)$, and

$$\begin{aligned} \mathbb{P}^1_{u,v}(s) &= 0, \\ \mathbb{P}^2_{u,v}(s) &= 0, \end{aligned} \tag{7.1}$$

then $u = v = s = p$ or $u = v = s = \overline{p}$.

Using $\mathbb{P}^{v-1}_{u,v}(s) = 0$. We know that, for any L_v, if $\mathscr{G}(u, v, s)$ is a counterexample, then $u, v \in \{p, \overline{p}\}$. For d-regular graphs we can easily see that even $s \in \{p, \overline{p}\}$. Indeed, we can use (6.3) from the proof of Lemma 6.3. It yields

$$u^d(1-u)^{v-1-d} = s^d(1-s)^{v-d-1}. \tag{7.2}$$

Observe that (by Lemma 6.4) $s = u$ or $s = \overline{u}$. Further, by (7.2),

$$1 = \left(\frac{s}{u}\right)^d\left(\frac{1-s}{1-u}\right)^{v-1-d}, \tag{7.3}$$

Taking the logarithms, and dividing by $v - 1$,

$$\frac{d}{v-1}\log\left(\frac{s}{u}\right) + \left(1 - \frac{d}{v-1}\right)\log\left(\frac{1-s}{1-u}\right) = 0.$$

Set

$$\alpha := 1 - \frac{d}{v-1}, \quad \beta := \frac{d}{v-1}, \quad x := \left(\frac{1-u}{1-s}\right)^2, \quad y := \left(\frac{u}{s}\right)^2, \tag{7.4}$$

Then

$$\alpha \log x + \beta \log y = 0. \tag{7.5}$$

Using $\mathbb{P}_{u,v}^{v-2}(s) = 0$. Let us calculate $\mathbb{P}_{u,v}^{v-2}(s) = 0$. By the *d-regularity* (!) $\mathbb{P}_{u,v}^{v-2}(s)$ has (at most) three distinct terms:

- a fixed term, that is,
$$\binom{v}{2} u^{\frac{1}{2}vd}(1-u)^{\binom{v}{2}-\frac{1}{2}vd};$$
- a term corresponding to the case when the two points in B are independent, that is, $e(B) = 0$, $e(A,B) = 2d$, $e(A) = \frac{1}{2}vd - 2d$; and
- a term corresponding to when B contains an edge, that is, $e(B) = 1$, $e(A,B) = 2d-2$, $e(A) = \frac{1}{2}vd - 2d + 1$.

Hence

$$\begin{aligned}\mathbb{P}_{u,v}^{v-2}(s) = {} & \binom{v}{2} u^{\frac{1}{2}vd}(1-u)^{\binom{v}{2}-\frac{1}{2}vd} \\ & - \left(\binom{v}{2} - \frac{1}{2}vd\right)(1-v)u^{\frac{1}{2}vd-2d}(1-u)^{\binom{v-2}{2}-\frac{1}{2}vd+2d}s^{2d}(1-s)^{2v-2d-4} \\ & - \frac{1}{2}vdvu^{\frac{1}{2}vd-2d+1}(1-u)^{\binom{v-2}{2}-\frac{1}{2}vd+2d-1}s^{2d-2}(1-s)^{2v-2d-2}.\end{aligned}$$

Squaring (7.2) and plugging it into the equation $\mathbb{P}_{u,v}^{v-2}(s) = 0$ and then simplifying, we get

$$0 = (v-1) - \big(v-1-d\big)(1-v)(1-u)(1-s)^{-2} - dvus^{-2}. \tag{7.6}$$

From here on, we distinguish two cases:

- $u = v$, $s = \overline{u}$, and
- $u \neq v$.

The symmetric case: $v = u$. Now (7.6) gives

$$0 = (v-1) - \big(v-1-d\big)(1-u)^2(1-s)^{-2} - du^2s^{-2}. \tag{7.7}$$

Rearranging, we get

$$\left(1 - \frac{d}{v-1}\right)\left(\frac{1-u}{1-s}\right)^2 + \frac{d}{v-1}\left(\frac{u}{s}\right)^2 = 1.$$

Hence we get

$$\alpha x + \beta y = 1,$$

but this and the concavity of $\log t$ contradicts (7.5), unless $x = y$, implying that $u = s$, and completing this part of the proof.

The asymmetric case: $v \neq u$. (ii) We start again with (7.6):

$$0 = (v-1) - \big(v-1-d\big)(1-v)(1-u)(1-s)^{-2} - dvus^{-2}.$$

By $v \neq u$, either $s = u$ or $s = v$. By symmetry, we may assume that $s = v$,[20] that is,

$$\begin{aligned}(v-1) &= \big(v-1-d\big)(1-u)(1-s)^{-1} + dus^{-1} \\ \left(1-\frac{d}{v-1}\right)\left(\frac{1-u}{1-s}\right) &+ \left(\frac{d}{v-1}\right)\left(\frac{u}{s}\right) = 1.\end{aligned} \tag{7.8}$$

Here we can use the same convexity argument used in the previous subsection, with

$$x := \left(\frac{1-u}{1-s}\right), \quad y := \left(\frac{u}{s}\right).$$

(The squaring is missing!) This completes the whole proof. □

8. Proof of Theorem 3.3 on induced P₃s

We shall use Lemma 4.1. First we calculate the copy polynomials of P_3 (using (4.1)), and then solve the corresponding system of equations.

Clearly,

$$\mathbb{P}^0_{u,v}(s) := u^2(1-u) - v^2(1-v). \tag{8.1}$$

To get $\mathbb{P}^1_{u,v}(s)$ we use $k = 1$ in (4.1): $e(A) = 0$ and either $e(B) = 0$ or $e(B) = 1$. If $e(B) = 0$, then $e(A, B) = 2$. So we get $(1-v)s^2$. In the other case, when $e(B) = 1$, then $e(A, B) = 1$; we get $vs(1-s)$, but we get this term twice:

$$\mathbb{P}^1_{u,v}(s) = s^2(1-v) + 2vs(1-s) - 3u^2(1-u). \tag{8.2}$$

Exchanging u and v in the first two terms, we get

$$\mathbb{P}^2_{u,v}(s) = s^2(1-u) + 2us(1-s) - 3u^2(1-u). \tag{8.3}$$

(i) $\mathbb{P}^2_{u,v}(s)$ does not contain v. Since $s = u$ is a trivial root of $\mathbb{P}^2_{u,v}(s) = 0$, we can decompose $\mathbb{P}^2_{u,v}(s)$:

$$\mathbb{P}^2_{u,v}(s) = (s-u)((1-3u)(s+u) + 2u).$$

For each value of u, the equation $\mathbb{P}^2_{u,v}(s) = 0$ yields two values of s. One of them is $s = u$ (of course!) but we are interested in the other one,

$$s = 3u\frac{1-u}{3u-1},$$

which is negative for $u \in (0, \frac{1}{3})$ and exceeds 1 for $u \in (\frac{1}{3}, \frac{1}{\sqrt{3}})$ and $s \in (0, 1)$ for $u > \frac{1}{\sqrt{3}}$. Observe that $3u\frac{1-u}{3u-1} = u$, if and only if $u = 2/3$.

(ii) To verify Theorem 3.3, observe that $G(V_1, V_2, u, v, s)$ is a strong counterexample (for $u = p$ or $u = \overline{p}$) if and only if the corresponding copy polynomials vanish and $s \in [0, 1]$ and $u = v = s$ does not hold.

[20] Below we shall use (7.3), which uses u, and this may seem to create some asymmetry. However, (7.3) remains valid if u is replaced by v.

In parts (a) and (b) we assumed that either $u = v = p$ or $u = v = \overline{p}$. So we assumed that $u = v$. Hence (8.1) is automatically satisfied, and (8.2) and (8.3) coincide. So we have to satisfy only that (8.3) vanishes and ensure that $u, s \in [0, 1]$. But the formula in Theorem 3.3(a) is just the solution of $(1 - 3p)(s + p) + 2p = 0$, providing, by (i), an $s \in (0, 1)$ for every $p > \frac{1}{\sqrt{3}}$. The same holds for the formula in Theorem 3.3(b), for s^*. This proves (a) and (b).

(iii) To prove (c) we have to solve the system of equations provided by (8.1), (8.2) and (8.3). Subtract $\mathbb{P}^2_{u,v}(s)$ from $\mathbb{P}^1_{u,v}(s)$:

$$\begin{aligned}\mathbb{P}^1_{u,v}(s) - \mathbb{P}^2_{u,v}(s) &= s^2(1 - v) + 2vs(1 - s) - s^2(1 - v) - 2us(1 - s)\\ &= s(3s - 2)(u - v).\end{aligned}$$

This means that $\mathbb{P}^1_{u,v}(s)$ and $\mathbb{P}^2_{u,v}(s)$ can vanish at the same time if and only if $v = u$ or $s = 0$ or $s = 2/3$. Here $s = 0$ is excluded, since $\mathbb{P}^2_{u,v}(s) = 0$ would then imply $u = 0$ or $u = 1$. For $s = 2/3$, (8.3) yields

$$4 - 27u^2(1 - u) = 0.$$

Then $u^2(1 - u) = \frac{4}{27}$, which is the maximum of the conjugacy curve $u^2(1 - u)$ at $u = 2/3$. Thus $u = 2/3$ and $v = u$. Hence the case $s = 2/3$ is completely settled.

So $u = v$. Then $\mathbb{P}^0_{u,v}(s) = 0$ automatically holds, and $\mathbb{P}^1_{u,v}(s) = \mathbb{P}^2_{u,v}(s) = 0$ follows from (i). By (ii), this completes the proof of (c). □

Corrigendum to [9]. *In* [9] *we forgot to state explicitly that* $e(L_v) > 0$. *If* $e(L_v) = 0$, *then* $\mathbf{N}(L_v \subseteq F_h) = \binom{h}{v}$ *is independent of the structure of* G_n, *and of* p*: the theorem trivially does not hold.*

Acknowledgement

We thank the referee for his valuable work.

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