Explicit Constructions of Triple Systems for Ramsey—Turán Problems

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Abstract: We explicitly construct four infinite families of irreducible triple systems with Ramsey-Turán density less than the Turán density. Two of our families generalize isolated examples of Sidorenko [14], and the first author and Rödl [12]. Our constructions also yield two infinite families of irreducible triple systems whose Ramsey-Turán densities are exactly determined. © 2006 Wiley Periodicals, Inc. J Graph Theory 52: 211–216, 2006

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For an r-graph $\mathcal{F}$, the Turán number ex$(n, \mathcal{F})$ is the maximum number of edges in an n vertex r-graph containing no copy of $\mathcal{F}$. It is well known that $\pi(\mathcal{F}) =$

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$\lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$ exists, but these numbers are very hard to determine when $r \geq 3$. For example, until very recently [10] no nontrivial infinite family $\{\mathcal{F}_i\}$ of triple systems has been constructed for which $\pi(\mathcal{F}_i)$ is known (by "nontrivial" we mean that for every $\mathcal{F}_i$, there are no two vertices x, y of $\mathcal{F}_i$ for which (1) no edge contains both x and y, and (2) xuv is an edge iff yuv is an edge). Two examples that are known, and used in this note, are $\pi(F_5) = 2/9$, and $\pi(F(3, 2)) = 4/9$, where $F_5 = \{123, 124, 345\}$ and $F(3, 2) = \{123, 124, 125, 345\}$.

Many of the (conjectured) extremal examples for (hyper)graph Turán problems have large independent sets. Motivated by this observation, Erdős and Sós imposed a restriction on the underlying r graphs in this problem, namely that they should not have large independent sets. This new class of problems has become known as the Ramsey–Turán problems. More precisely, for $0 < \delta \leq 1$,

$$\mathrm{ex}(n, \mathcal{F}, \delta) = \max\{|\mathcal{G}| : \mathcal{G} \text{ is an } r\text{-graph with } \mathcal{F} \not\subseteq \mathcal{G} \text{ and } \alpha(\mathcal{G}) < \delta n\},$$

or zero if no such hypergraph exists. The Ramsey–Turán number $\rho(\mathcal{F})$ is defined as

$$\sup_{\delta(n)} \left\{ \limsup_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F}, \delta(n))}{\binom{n}{r}} : \delta(n) \to 0 \text{ as } n \to \infty \right\}.$$

Since obviously $\rho(\mathcal{F}) \leq \pi(\mathcal{F})$ for every $\mathcal{F}$, a fundamental question is whether equality holds. A sequence of papers [1,3,16] showed that in the case that $\mathcal{F}$ is a complete (2-uniform) graph, $\rho(\mathcal{F}) < \pi(\mathcal{F})$. It was therefore a surprise when Erdős and Sós [4] proved that for r graphs when $r \geq 3$, this does not hold. Call an r-graph $\mathcal{H}$ *locally dense* if for every edge $E \in \mathcal{H}$, there is another edge $E' \in \mathcal{H}$ with $|E \cap E'| \geq 2$.

Theorem 1 (Erdős-Sós [4]). *Let $r \geq 3$ and $\mathcal{H}$ be a locally dense r-graph. Then $\rho(\mathcal{H}) = \pi(\mathcal{H})$.*

On the other hand, it is proved in [4] that r graphs $\mathcal{F}$ exist ($r \geq 3$) for which $0 = \rho(\mathcal{F}) < \pi(\mathcal{F})$. Motivated by these examples, Erdős and Sós asked whether there exist r graphs ($r > 2$) with

$$0 < \rho(\mathcal{F}) < \pi(\mathcal{F}). \tag{1}$$

This was answered positively by Frankl and Rödl [8] for every $r > 2$, who showed that there exist infinitely many r graphs for which (1) holds however, they did not obtain a single explicit example. Subsequently, Sidorenko [14], using ideas from [8] proved that for the 3-graph $F_7 = \{123, 145, 167, 245, 267, 345, 367, 467, 567\}$, inequality (1) holds. Recently, the first author and Rödl [12] proved that $F(3, 3) = \{124, 125, 126, 134, 135, 136, 234, 235, 236, 456\}$ is another example.

Call an r-graph $\mathcal{H}$ *reducible* if

(1) it is disconnected, or

(2) its vertex set can be partitioned into $X \cup Y$, such that no edges of $\mathcal{H}$ are contained in Y, and all edges E of $\mathcal{H}$ with $E \cap X$ and $E \cap Y$ both nonempty have the form $\{x, y_1, \ldots, y_{r-1}\}$, where $x \in X$ is fixed, $\{y_1, \ldots, y_r\} \subset Y$, and $\{\{y_1, \ldots, y_{r-1}\} : \{x, y_1, \ldots, y_{r-1}\} \in \mathcal{H}\}$ is $(r-1)$-partite.

If $\mathcal{H}$ is not reducible, then $\mathcal{H}$ is *irreducible*. It follows [13] from the definition of ρ that for every reducible $\mathcal{H}$, there is an irreducible $\mathcal{H}' \subset \mathcal{H}$ for which $\rho(\mathcal{H}') = \rho(\mathcal{H})$ (in Case 1, $\mathcal{H}'$ is an appropriately chosen component, and in Case 2, $\mathcal{H}' = \mathcal{H} - Y$). Therefore, it makes sense to ask for $\rho(\mathcal{H})$ only when $\mathcal{H}$ is irreducible.

In this note, we use essentially the same ideas from [8] to explicitly construct four infinite families of irreducible triple systems for which $0 < \rho < \pi$. Our constructions are in the spirit of [14], but we obtain more variety (in particular, infinite families) with no extra effort. Although the underlying principle behind our construction is a rather general phenomenon (see (3)), our lack of understanding of π limits our approach.

One of our families (see Example 1) contains $F(3,3)$, and another (see Example 3) contains F_7. Thus our contribution can be viewed as a generalization of results in [12,14]. Our constructions also yield two infinite families of irreducible triple systems $\mathcal{F}_i$ for which $\rho(\mathcal{F}_i)$ is determined. The values in the two cases are 2/9 and 4/9 (see Examples 2 and 3). Thus in this sense, our understanding of ρ for hypergraphs is greater than that for π (we can think of the notion of *irreducible* for ρ as analogous to the notion of *nontrivial* for π).

Given an r-graph $\mathcal{H}$, let $\mathcal{H}^*$ be an r-graph obtained from $\mathcal{H}$ by replacing a vertex v with r vertices $v_1, \ldots, v_r$, replacing each edge E containing v with r edges $E_1, \ldots, E_r$, where $E_i = E - v \cup \{v_i\}$, and adding the edge $\{v_1, \ldots, v_r\}$.

The main tool for the constructions is the following theorem. Although we proved it independently, later we noticed that the main part of it is proved in ([8] Lemma 2.3).

Theorem 2. *Let $\mathcal{H}$ be an r-graph, and $\mathcal{H}^*$ be obtained from $\mathcal{H}$ by replacing any vertex v. Then $\rho(\mathcal{H}^*) \le \pi(\mathcal{H})$. If, in addition, $\mathcal{H}$ is locally dense, then $\rho(\mathcal{H}^*) = \pi(\mathcal{H})$.*

Proof. The first part is proved in [8]. For the last statement, Theorem 1 yields $\rho(\mathcal{H}^*) \le \pi(\mathcal{H}) = \rho(\mathcal{H})$. Since $\mathcal{H} \subset \mathcal{H}^*$, the result follows. ■

A vertex multiplication in a hypergraph $\mathcal{H}$ is the replacement of a vertex v in $\mathcal{H}$ by a finite set of vertices $\{v_1, \ldots, v_k\}$, and the replacement of every edge E containing v by the k edges $E - v \cup \{v_i\}$. If $\mathcal{H}'$ is obtained from $\mathcal{H}$ by a finite sequence of vertex multiplications, then we say that $\mathcal{H}'$ is a *blowup* of $\mathcal{H}$. It is easy to see that if $\mathcal{H}$ is locally dense, then $\mathcal{H}'$ is locally dense as well. Also, it is well known (see, e.g. [15]) that

$$\pi(\mathcal{H}) = \pi(\mathcal{H}'). \tag{2}$$

Note that if $\mathcal{F}^*$ is obtained from $\mathcal{F}$ by replacing a nonisolated vertex of $\mathcal{F}$, then $\mathcal{F}^*$ contains the hypergraph $F(3,2) = \{567, 467, 367, 345\}$, and it is known [11] that $\pi(F(3,2)) \geq 4/9$. Therefore $\pi(\mathcal{F}^*) \geq 4/9$.

Our constructions below yield infinite families of irreducible 3 graphs, since in each case (except Example 2) we begin with an arbitrary blowup $\mathcal{F}$ of an irreducible 3-graph $\mathcal{H}$ (with $\pi(\mathcal{H}) > 0$). After this we form $\mathcal{F}^*$ by replacing any vertex from $\mathcal{F}$, except in Example 3, where we are more specific. Usually $\mathcal{H}$ is locally dense, and hence $\mathcal{F}$ is also locally dense. Consequently, Theorems 1, 2 and (2) yield

$$0 < \pi(\mathcal{H}) = \rho(\mathcal{H}) \leq \rho(\mathcal{F}^*) = \pi(\mathcal{F}) \leq \pi(\mathcal{F}^*). \tag{3}$$

One only needs to verify that $\pi(\mathcal{F}) < \pi(\mathcal{F}^*)$ to obtain $0 < \rho(\mathcal{F}^*) < \pi(\mathcal{F}^*)$. Although this may be true in general, we are only able to show it for the few examples below.

Example 1. Let $H(4,3)$ be the (unique) four vertex triple system with three edges, and let $\mathcal{F}$ be a blowup of $H(4,3)$. Then Theorem 2 implies that $\rho(\mathcal{F}^*) = \pi(\mathcal{F})$. We also have $\pi(\mathcal{F}) = \pi(H(4,3))$, and $2/7 \leq \pi(H(4,3)) < 1/3 - 10^{-6}$ (see [6,9]). Therefore

$$2/7 \leq \rho(\mathcal{F}^*) < 1/3 - 10^{-6} < 4/9 \leq \pi(\mathcal{F}^*).$$

Note that in the case $\mathcal{F} = H(4,3)$, and the vertex used to form $\mathcal{F}^*$ is the unique vertex of degree three in $H(4,3)$, we obtain $\mathcal{F}^* = F(3,3)$, thus retrieving the example of [12].

Example 2. Let F_5 be the five vertex triple system with edges 123, 124, 345. Let $\mathcal{F}$ be a blowup of F_5, where the vertex labeled 5 is replaced by at least two vertices. Then Theorem 2 implies that $\rho(\mathcal{F}^*) = \pi(\mathcal{F})$. We also have $\pi(\mathcal{F}) = \pi(F_5) = 2/9$ [5]. Therefore

$$2/9 = \rho(\mathcal{F}^*) < 4/9 \leq \pi(\mathcal{F}^*).$$

Example 3. Recall that $F(3,2) = \{567, 467, 367, 345\}$. Let $\mathcal{F}$ be a blowup of $F(3,2)$. Let $\mathcal{F}^*$ be obtained from $\mathcal{F}$ by replacing one of the vertices playing the role of a vertex in $\{3,4,5\}$ (say 3), and then adding an edge among the three new vertices. Note that in the case $\mathcal{F} = F(3,2)$, we have $\mathcal{F}^* = \{567, 467, 367, 345, 267, 167, 245, 145, 123\}$, where 1 and 2 are the two new vertices, thus $\mathcal{F}^* = F_7$. Now Theorem 2 implies that $\rho(\mathcal{F}^*) \leq \pi(\mathcal{F})$. The last part of Theorem 2 implies that $\rho(\mathcal{F}^*) = \pi(\mathcal{F})$ except possibly in the case that $\mathcal{F}$ was obtained from $F(3,2)$ without blowing up any of the vertices 3, 4, 5 (this includes the case $\mathcal{F}^* = F_7$).

In this case, let F^- be obtained from $\mathcal{F}^*$ by deleting the edge (in the labeling above) 123. Then F^- is locally dense and so $\rho(F^-) = \pi(F^-)$ by Theorem 1.

Consequently,

$$4/9 \leq \pi(F(3,2)) = \pi(F^-) = \rho(F^-) \leq \rho(\mathcal{F}^*) \leq \pi(\mathcal{F}) = \pi(F(3,2)) \leq 4/9,$$

where the first inequality is from [11], the first and last equalities are from (2), and the last inequality was recently proved by Füredi, Pikhurko, and Simonovits [7]. Thus even in this case, $\rho(\mathcal{F}^*) = \pi(\mathcal{F}) = 4/9$.

On the other hand, a short case analysis shows that $\mathcal{F}^*$ is absent in the hypergraph $\mathcal{G}$ with vertex partition $A_1 \cup A_2 \cup A_3$ ($\| A_i| - |A_j\| \leq 1$ for $i \neq j$), and all edges of the form abc, where $a, b \in A_i, c \in A_{i+1}$ (indices modulo 3), or $a \in A_1, b \in A_2, c \in A_3$. Since $\mathcal{G}$ has density 5/9, $\pi(\mathcal{F}^*) \geq 5/9$. Therefore,

$$4/9 = \rho(\mathcal{F}^*) < 5/9 \leq \pi(\mathcal{F}^*).$$

Example 4. Let K_4^3 be the complete triple system on four points, and let $\mathcal{F}$ be a blowup of K_4^3. Then Theorem 2 implies that $\rho(\mathcal{F}^*) = \pi(\mathcal{F})$. We also have $\pi(\mathcal{F}) = \pi(K_4^3)$, and from [2,17], $5/9 \leq \pi(K_4^3) < 0.592$. It is easy to see that $\mathcal{F}^*$ is not 2-colorable, therefore $\pi(\mathcal{F}^*) \geq 3/4$. Consequently,

$$5/9 \leq \rho(\mathcal{F}^*) < 0.592 < 3/4 \leq \pi(\mathcal{F}^*).$$

We end by remarking that our examples are nontrivial not only in the sense that the hypergraphs produced are irreducible, but also because one notes that ρ is not preserved in general under the blowup operation (as π is). Consequently, one cannot hope to just take blowups and produce an infinite family from a single 3-graph satisfying (1). One well-known problem in this regard is to determine $\rho(K_{2,2,2})$, where $K_{2,2,2}$ is the complete 3-partite graph with two vertices in each part. It is trivial that $\rho(K_3) = 0$, but it is unknown whether $\rho(K_{2,2,2}) > 0$.

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