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# Paradoxical Decompositions and Growth Conditions

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This paper is an extended version of the lecture given by the second author delivered at the conference 'Combinatorics: Walter Deuber Memorial Meeting' held on 7–8 October 2002 at Humboldt-Universität zu Berlin. Regretfully, this topic was to be the last that fascinated Walter Deuber's mind. We wrote this article in remembrance of that.

## 1. Introduction

### 1.1. Measurability and paradoxical decompositions in $\mathbb{R}^n$

The theory of paradoxical decompositions arose in connection with the existence of non-Lebesgue measurable sets. Vitali [63] used a decomposition of the unit circle $S^1 = \{x \in \mathbb{R}^2,\ |x| = 1\}$ to prove that there is no nontrivial isometry-invariant and $\sigma$-additive measure defined on all subsets of the unit circle $S^1$. Motivated by this, Hausdorff asked: What is the situation if one requires the measure to be only finitely additive? The answer was that there exists a finitely additive isometry-invariant measure on $S^1$, but there is no such measure on the two-sphere $S^2 = \{x \in \mathbb{R}^3,\ |x| = 1\}$. Hausdorff [34] also proved this by means of some sort of paradoxical decomposition of $S^2$. However, Banach proved in [2] the existence of isometry-invariant finitely additive measures in $\mathbb{R}^1$ and $\mathbb{R}^2$.

The non-existence of isometry-invariant finitely additive measures in $\mathbb{R}^3$ was proved by Banach and Tarski [4], also by means of paradoxical decomposition. They proved that it is possible to partition the unit ball in $\mathbb{R}^3$ into finitely many pieces and to rearrange them by rigid motions (using isometric transformations) to form two unit balls. This 'duplication', this 'paradoxical decomposition' of the ball, at first seems to be impossible. We will discuss it below in more detail.

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The analysis of this surprising phenomenon led to the concept of amenable groups, introduced and first studied by von Neumann [49]. Since that time the subject has developed into a field that has importance besides analysis, group theory and geometry in discrete mathematics and computer science (*e.g.*, in the theory of random walks, percolation, and expanders).

The objective of the present paper is to give some illustrations and indications of the wide range of topics that have developed from the subject mentioned above, providing some motivation for the particular problem considered in the paper [17] and some of its consequences. We emphasize that this is not a survey of any part of these topics.

For detailed information and references about the extremely wide area the reader is referred to the excellent books of Gromov [30], de la Harpe [33], Lubotzky [45], Paterson [51], Wagon [64] and Woess [68], and the many survey papers on these subjects, *e.g.*, by Ceccherini-Silberstein, Grigorchuk and de la Harpe [12], Laczkovich [42, 44], Thomassen and Woess [59] and Woess [67].

## 2. Invariant measure and paradoxicity

### 2.1. Banach–Tarski paradox

The 'duplication' of the ball in $\mathbb{R}^3$ means the following.

**Definition 2.1.** $A, B \subseteq \mathbb{R}^3$ are *piecewise congruent* if there exist decompositions

$$A = \dot{\bigcup_{i=1}^{k}} A_i, \qquad B = \dot{\bigcup_{i=1}^{k}} B_i,$$

such that $A_i$ is congruent to $B_i$ for $1 \leqslant i \leqslant k$.

(Here 'congruent' means that there is an isometric transformation $t : B_i \to A_i$ such that $A_i = t(B_i)$.)

**Theorem 2.2. ([4])** *The unit ball $B \subseteq \mathbb{R}^3$ is paradoxical in the following sense: there exists a decomposition of $B$,*

$$B = B_1 \bigcup B_2,$$

*such that*

$$B,\ B_1,\ B_2$$

*are pairwise piecewise congruent.*

### 2.2. Amenability and paradoxicity

As we mentioned in the Introduction, the Hausdorff–Banach–Tarski paradoxical decompositions of the ball (or of the sphere) in $\mathbb{R}^d$ exist for $d = 3$ (and also for $d > 3$), but do not exist for $d = 1$ and $d = 2$.

It was von Neumann [49] who discovered that these different phenomena are due to the difference between the isometry groups of $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$; the latter is more'rich'.

He considered a general setting where the basic notions are the finitely additive group-invariant measure (or invariant mean) and the paradoxical groups (or amenable groups = non-paradoxical groups).

Here we give a very short introduction to the definitions and properties which are the most relevant, from our point of view.

Throughout this paper all graphs and all groups are infinite.

**Definition 2.3.** Let $\Gamma$ be a group acting on a set $X$ as bijections. The subsets $A, B \subset X$ are $\Gamma$-*equidecomposable* if there exist partitions

$$A = \bigcup_{i=1}^{k} A_i, \qquad B = \bigcup_{i=1}^{k} B_i$$

and elements $g_i \in \Gamma$ such that

$$g_i A_i = B_i \qquad \text{for any} \quad 1 \leqslant i \leqslant k.$$

We denote this by $A \overset{\Gamma}{\sim} B$.

**Definition 2.4.** The set $A \subset X$ is called $\Gamma$-paradoxical if there exists a decomposition

$$A = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset$$

such that

$$A \overset{\Gamma}{\sim} A_1 \overset{\Gamma}{\sim} A_2.$$

**Definition 2.5.** Let $\Gamma$ be a group acting on the set $X$ as bijections. A finitely additive $\Gamma$-invariant (probability) measure on the set $X$ is a function $\mu : P(X) \to [0, 1]$ such that

(i) $\mu(X) = 1$,
(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$,
(iii) $\mu(A) = \mu(gA)$ for all $g \in \Gamma$, $A \subseteq X$.

Let $E \subseteq X$. If $\mu : P(X) \to [0, \infty]$ satisfies (ii), (iii) and $\mu(E) = 1$ then it is called a *Γ-invariant measure normalized on $E$*.

Here $P(X)$ denotes the power set of $X$.

Every group acts naturally on itself by left multiplication: if $X = \Gamma$, we identify every element $g \in \Gamma$ with the map $x \to g\ x$ ($x \in \Gamma$). So we identify $\Gamma$ with this transformation group acting on the set $\Gamma$. According to this we have the following.

**Definition 2.6.** $A, B \subseteq \Gamma$ are equidecomposable if there are partitions

$$A = \bigcup_{i=1}^{k} A_i, \qquad B = \bigcup_{i=1}^{k} B_i$$

and elements $g_i \in \Gamma$ such that

$$g_i A_i = B_i \qquad \text{for} \quad i = 1, \ldots, k.$$

We denote this by $A \sim B$.

**Definition 2.7.** The set $A \subseteq \Gamma$ is paradoxical if there exists a decomposition

$$A = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset$$

such that

$$A \sim A_1 \sim A_2.$$

**Definition 2.8.** A group $\Gamma$ is called *amenable* if there is a finitely additive $\Gamma$-invariant probability measure $\mu : P(\Gamma) \to [0,1]$.

A group is called *supramenable* if, for any $\emptyset \neq A \subseteq X$, there is a finitely additive $\Gamma$-invariant measure which is normalized on $A$, whenever $\Gamma$ acts on $X$.

(The word amenable was introduced only later, by M. M. Day in 1950.)

**Remark 2.9.** Obviously, if $X$ is paradoxical then there is no finitely additive $\Gamma$-invariant measure on $X$. In other words, suppose

$$X = A \cup B, \qquad A \cap B = \emptyset$$

and

$$X \overset{\Gamma}{\sim} A \overset{\Gamma}{\sim} B.$$

Then

$$1 = \mu(X) = \mu(A \cup B) = \mu(A) + \mu(B) = 2,$$

a contradiction.

Tarski proved the basic result that the converse of the above remark is also true.

**Theorem 2.10. ([57])** *Let $\Gamma$ be a group acting on a set $X$. Then there is a $\Gamma$-invariant finitely additive measure $\mu$ normalized on $A \subset X$ if and only if $A$ is not $\Gamma$-paradoxical.*

**Corollary 2.11.** *Every group $\Gamma$ is either amenable or paradoxical.*

The most important example of an amenable group is the group of integers $\mathbb{Z}$. Let us recall a direct proof of this fact.

**Proposition 2.12.** *$\mathbb{Z}$ is amenable.*

Pick a set $A \subset \mathbb{Z}$. The frequency of $A$ in $\{-n, -(n-1), \ldots, n\}$ is defined by

$$f_n(A) = \frac{|A \cap \{-n, -(n-1), \ldots, n\}|}{2n+1}.$$

The density of $A$, if it exists, would be the limit of the $f_n$s. Unfortunately, this limit in general does not exist. However, it is always true that $f_n(A)$ is a bounded sequence, and if $A \cap B = \emptyset$ then $f_n(A \cup B) = f_n(A) + f_n(B)$. By the Hahn–Banach theorem, there exists a continuous linear functional BLim on the space of bounded real sequences $l^\infty$ that coincides with the limit on the space of convergent sequences $c \subset l^\infty$. So we can define $\mu(A)$ as $\mathrm{BLim}\{f_n(A)\}_{n\geqslant 1}$. By linearity, $\mu(A\cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$. Also, $\mu(\mathbb{Z}) = 1$. Note that, for any natural number $k$, $\lim_{n\to\infty}(f_n(A+k) - f_n(A)) = 0$. Hence $\mathrm{BLim}\{f_n(A+k) - f_n(A)\}_{n\geqslant 1}$=0. Thus $\mu$ is a finitely additive measure which is invariant with respect to the $\mathbb{Z}$-action. □

Von Neumann [49] gave the simplest and most important example for paradoxical group: he proved the following.

**Proposition 2.13.** *$F_2$, the free group generated by $(a, b)$, is paradoxical.*

Below, 'r.w.s.w.' means 'reduced words starting with'. Consider

$$\begin{aligned} A &= \{\text{r.w.s.w. } a\} \\ A^- &= \{\text{r.w.s.w. } a^{-1}\} \\ B &= \{1, b^{-1}, b^{-2}, \dots\} \bigcup \{\text{r.w.s.w. } b\} \\ B^- &= \{\text{r.w.s.w. } b^{-1}\} \ \text{ but not in } \ \{b^{-1}, b^{-2}, \dots\}. \end{aligned}$$

Using these sets we get the following decomposition:

$$F_2 = A \cup A^- \cup B \cup B^- = A \cup (a\ A^-) = B \cup (b\ B^-).$$

Hence $F_2$ is paradoxical and consequently non-amenable. □

Here we list some important results about amenable groups.

**Proposition 2.14.** *All elementary groups, in particular all Abelian groups, all nilpotent groups, all solvable groups are amenable, or non-paradoxical:*

- *a subgroup of an amenable group is amenable,*
- *a group $\Gamma$ is amenable if and only if every finitely generated subgroup of $\Gamma$ is amenable,*
- *as we mentioned, $F_2$ (or $F_k$) is paradoxical,*
- *if $F_2$ is a subgroup of $\Gamma$, then $\Gamma$ is paradoxical.*

It was conjectured for more than 40 years that the converse of the last statement is also true. However Ol'shanskii [50] constructed a paradoxical group that has no elements of infinite order and consequently has no free subgroup of any rank.

Under certain restriction (*e.g.*, linear groups) on $\Gamma$ it is true that $\Gamma$ is paradoxical if and only if $F_2$ is a subgroup of $\Gamma$. For references see [64].

**Remark 2.15.** It is not always simple to decide whether a group is amenable or not amenable. There are some famous examples for which it is still undecided whether the group is amenable or not, such as the Thompson group.

**2.3. Paradoxicity of $X$ and $\Gamma$**

(1) If $X$ is $\Gamma$-paradoxical, then $\Gamma$ is paradoxical as well.

Consequently:

(2) if an amenable group $\Gamma$ acts on $X$ as bijections, then there exists a finitely additive, $\Gamma$-invariant measure on $X$.

The converse is not true. However,

(3) if $\Gamma$ acts on $X$ without nontrivial fixed points, then $X$ is $\Gamma$-paradoxical if and only if $\Gamma$ is paradoxical.

More generally, the following theorem was proved by Rosenblatt.

**Theorem 2.16. ([52])** *Let $\Gamma_x$ denote the group $\{g \in \Gamma : g(x) = x\}$. Suppose $\Gamma_x$ is amenable for every $x \in \Gamma$. Then $\Gamma$ is amenable if and only if $X$ is not $\Gamma$-paradoxical.*

**2.4. Back to $\mathbb{R}^d$**

Let $\Gamma_d$ be the group of isometries of $\mathbb{R}^d$ and let $SO_d$ be the group of rotations of the sphere $S^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ (or of the ball $B^d$).

Then, for $d \geqslant 3$:

(i) $F_2$ is a subgroup of $SO_d$,
(ii) $SO_d$, $\Gamma_d$ are paradoxical and non-amenable,
(iii) the ball $B_d$ and the sphere $S^{d-1}$ are $SO_d$, $\Gamma_d$-paradoxical.

For $d = 1, 2$:

(iv) $F_2$ is not a subgroup of $SO_d$,
(v) $SO_d$, $\Gamma_d$ are amenable and not paradoxical (*i.e.*, they are solvable),
(vi) $B^d$, $S^{d-1}$ are not $SO_d$, $\Gamma_d$-paradoxical.

**Remark 2.17.** Since isometries of $\mathbb{R}^d$ can have many fixed points, the Banach–Tarski paradox for (iii) is not just a consequence of (ii).

In fact, Banach and Tarski proved the following stronger version of Theorem 2.2.

**Theorem 2.18. ([4])** *Let $A, B \subset \mathbb{R}^3$ be bounded sets with nonempty interiors. Then $A \overset{\Gamma_3}{\sim} B$. Consequently, every bounded subset of $\mathbb{R}^3$ with nonempty interior is paradoxical.*

**2.5. Growth conditions and amenability**

Above we gave a very short outline of the start of the theory of amenable groups, finitely additive, invariant measures, and paradoxicity. The properties described above are strongly related to certain growth properties and this indicates already the importance of all these in the theory of random walks, percolation and expanders. (Below we restrict our considerations to random walks.)

The first explicit growth condition which gives an important and useful characterization of amenability was formulated by Følner.

**Theorem 2.19. ([23])** *A group $\Gamma$ is amenable if and only if, for every finite subset $K \subset \Gamma$ and every $\varepsilon > 0$, there is another finite set $F \subset \Gamma$ such that, for any $g \in K$,*

$$\frac{|gF \triangle F|}{|F|} < \varepsilon.$$

**Remark 2.20.** There are different variations of the Følner condition, *e.g.*, when, instead of the symmetric difference, only the difference is considered, or when the 'test' sets $F_n$ form a nested sequence as $\epsilon_n \to 0$.

Adel'son-Vel'skii and Sreider [1] defined a growth function providing a sufficient condition for amenability.

**Definition 2.21.** Let $S$ be a finite subset of the group $\Gamma$ (not necessarily a generating set!) and let

$$\gamma_S^*(n) =: \{g_1 \dots g_n : g_i \in S \cup S^{-1} \quad \text{for} \quad 1 \leqslant i \leqslant n\}.$$

Here $|\gamma_S^*(n)|$ is the number of elements of $\Gamma$ obtainable as a *reduced* word of length at most $n$ using elements of $S \cup S^{-1}$ as letters.

Let $\gamma^*(\Gamma) = \sup_S \lim |\gamma_S^*(n)|^{1/n}$ where the supremum is taken over all finite sets $S \subset \Gamma$. It is easy to see that either $\gamma^*(\Gamma) = 1$ or $\gamma^*(\Gamma) = \infty$.

The group $\Gamma$ is called *exponentially bounded* if $\lim_{n\to\infty} |\gamma_S^*(n)|^{1/n} = 1$ for every finite $S \subset \Gamma$.

**Theorem 2.22. ([1])** *If $\Gamma$ is exponentially bounded, then $\Gamma$ is amenable (even supramenable). If $\Gamma$ is exponentially bounded, $\Gamma$ acts on $X$ and $\phi \neq A \subset X$, then $A$ is not $\Gamma$-paradoxical.*

The converse of this theorem is not true.

**Remark 2.23.** $|\gamma_S^*(n)|$ measures in some way the noncommutativity of $\Gamma$, the system of relations in $\Gamma$.

If $\Gamma$ is an Abelian group, then many words give the same group element, and $\gamma_S^*(n)$ is polynomially bounded. If $S = (a_1, \dots, a_k)$ then $\gamma_S^*(n) \leqslant n(2n+1)^k$. Hence the theorem above gives a simple proof that Abelian groups are amenable (even supramenable).

On the other hand, for the free group $F_k$ with free generators $S = (a_1, \dots, a_k)$, $\gamma_S^*(n) = (k(2k-1)^n - 1)/k - 1$, and hence $F_k$ is not exponentially bounded.

In Section 4 we will discuss some further results on growth functions and their relations to each other and to random walks.

The so-called *cogrowth* function of a group $\Gamma$ introduced by Grigorchuk [27] provides a characterization of amenability – a necessary and sufficient condition in terms of the number of words of length at most $n$ that vanish when interpreted in $\Gamma$. See also Cohen [13].

### 2.6. Squaring the disc

Since finitely additive isometry-invariant measure (Banach measure) $\mu$ exists in $\mathbb{R}^2$, two sets $A, B \subset \mathbb{R}^2$ can be equidecomposable only if $\mu(A) = \mu(B)$.

In 1925 Tarski asked whether a disc and a square of the same measure are equidecomposable.

Tarski's question was answered affirmatively only after more than 60 years by Laczkovich [39], who proved the following.

**Theorem 2.24.** **([39])** *The disc is $\Gamma_2$-equidecomposable to a square of the same area. Moreover, it is true using translations only.*

Generalizations in several directions are given in [40, 41]. In the paper [39], Laczkovich introduced the notion of uniformly spread sets. This was a basic tool to prove the above theorem; he established the connection between uniformly spread sets, discrepancies and measure theory.

**Definition 2.25.** $X \subseteq \mathbb{R}^d$ is uniformly spread if there is an $f : X \to \mathbb{Z}^d$ such that

$$\sup_{x \in X} d(x, f(x)) < \infty.$$

This provided the motivation for considering transformations with this property in general. We shall discuss this in the next section.

## 3. Paradoxicity of metric spaces

To speak about the paradoxicity of a space $X$ one has to specify a group $\Gamma$ which acts on $X$. In the classical theory outlined above in the case $X = \mathbb{R}^d$, the group was $\Gamma = \Gamma_d$, the group of isometries of $\mathbb{R}^d$ or some subgroup of it.

The paradoxicity with respect to the group of isometries can be considered for more general metric spaces, *e.g.*, for hyperbolic spaces.

### 3.1. Wobbling paradoxicity

In the paper of Deuber, Simonovits and Sós [17] – generalizing the above-mentioned concept introduced by Laczkovich – the concept of *wobbling transformations* (called more recently *bounded perturbation of the identity*) for an arbitrary metric space is introduced.

**Definition 3.1.** Let $(X, d)$ be a metric space, $A, B \subseteq X$. A bijection $f : A \to B$ is called a *wobbling bijection* if

$$\sup_{x \in A} d(x, f(x)) < \infty.$$

$A, B \subseteq X$ are called *wobbling-equivalent* if there is a wobbling bijection $f : A \to B$.

**Definition 3.2.** The set $A \subseteq X$ is called *wobbling-paradoxical* if there is a decomposition

$$A = A_1 \cup A_2, \qquad A_1 \cap A_2 = \emptyset$$

such that $A, A_1, A_2$ are pairwise wobbling-equivalent.

Let $\Gamma_w = \Gamma_w(X)$ denote the group of wobbling bijections $f : X \to X$.

Apparently wobbling equivalence is a stronger requirement than $\Gamma_w$-equidecomposability. However, Laczkovich proved that the two properties are equivalent.

**Proposition 3.3. ([43])** *The sets $A, B \subseteq X$ are wobbling-equivalent if and only if $A \overset{\Gamma_w}{\sim} B$.*

Consequently, a set $A \subseteq X$ is wobbling-paradoxical if and only if it is $\Gamma_w$-paradoxical.

In the paper [17], wobbling paradoxicity is characterized by the following growth condition. For $A \subset X$, $k > 0$, let $N_k(A)$ denote the $k$-neighbourhood of $A$:

$$N_k(A) = \{x \in X : d(x, A) \leqslant k\}.$$

**Definition 3.4.** The metric space $(X, d)$ is *doubling* if there is a $k > 0$ such that

$$|N_k(A)| \geqslant 2|A| \text{ for every finite } A \subset X.$$

**Theorem 3.5. ([17])** *Let $(X, d)$ be a discrete and countable metric space. $(X, d)$ is wobbling-paradoxical if and only if it is doubling.*

**Proof.** Let $G_k(X, X)$ denote the bipartite $k$-distance graph of $(X, d)$, that is, for $x, y \in X$, $(x, y)$ is an edge in $G_k(X, X)$ if and only if $d(x, y) \leqslant k$.

If $(X, d)$ is a discrete and countable metric space, then $G_k(X, X)$ is a countable, locally finite bipartite graph.

$(X, d)$ is wobbling-paradoxical if and only if, for some $k > 0$, $G_k(X, X)$ contains a perfect $(2, 1)$ matching. According to the generalized Hall–Rado theorem, the doubling property of $(X, d)$ implies that $G_k(X, X)$ has a $(2, 1)$ matching, hence $(X, d)$ is wobbling-paradoxical.

The converse statement, that wobbling paradoxicity implies doubling, is obvious. □

**Remark 3.6.** The doubling property is strongly related to the so-called Gromov condition (see [32] and [12]).

Laczkovich proved that an analogous result is true in more general (so-called bounded) spaces. As a special case he obtained the following generalization of Theorem 3.5.

**Theorem 3.7. ([43])** *An arbitrary metric space $(X, d)$ is $\Gamma_w$-paradoxical if and only if it is doubling.*

**Example 3.8.** **(1)** Let $X = \{\log n;\ n \in \mathcal{N}^+\}$ and $d$ be the usual metric in $R$. Then the decomposition with

$$X_1 = \{\log(2n+1)\}, \quad X_2 = \{\log 2n\}$$

shows that $X$ is wobbling-paradoxical.

**(2)** $\mathbb{R}^2$ is wobbling-paradoxical. Take a checkerboard tiling of the plane. A translation moves the black tiles into the white ones. Moreover, any single square is equivalent to a domino. This shows that $\mathbb{R}^2$ and the set of black tiles are equivalent.

Observe that these two metric spaces are not isometry ($\Gamma_1$ resp. $\Gamma_2$)-paradoxical. Note also that the lattices $Z^d$ are not wobbling-paradoxical.

### 3.2. Wobbling paradoxicity of graphs

As an interesting and important special case of a metric space we consider graphs.

Let $G(V;E)$ be an infinite, locally finite, connected graph.

$(V,d)$ is a discrete, countable metric space where, for $x, y \in V$, $d(x,y)$ is the length of the shortest path in $G$ between $x$ and $y$. For short, we call $G$ paradoxical, resp. non-paradoxical, depending on whether $(V,d)$ is wobbling-paradoxical, resp. non-wobbling-paradoxical.

An interesting question to ask is which graph properties, besides the doubling property, are relevant in the paradoxicity.

For trees there is a simple characterization. A path $P_k \subseteq G$ is a hanging chain if all its inner vertices have degree 2.

**Theorem 3.9.** **([17])** *A locally finite, infinite tree $T$ without end-vertices is wobbling-paradoxical if and only if the length of hanging chains in $T$ is bounded.*

**Corollary 3.10.** *An infinite tree $T$ is wobbling-paradoxical if its minimum degree* $\geqslant 3$.

For further reference on paradoxicity of graphs see Fon-der-Flaass [24], Carstens, Deuber, Thumser and Koppenrade [9], and Deuber and Kostochka [16]. See also the expository paper of Walter Deuber, 'Paradoxe Zerlegung Euklidischer Räume' [15], where Theorem 3.5 [17] is already quoted, even with a proof; however, because of some technical reasons the paper [17] did not appear until 1995.

## 4. Growth in groups and on graphs

Different variations of growth properties, expanding properties are the most determinant in the theory of random walks. Hence it is not surprising that there is a strong connection between the properties of random walks and amenability. There are different definitions and concepts to measure these properties.

**4.1. Isoperimetric constant**

Let $(X, d)$ be a discrete, countable metric space. For a finite $A \subset X$ let $\partial(A) := N_1(A) \setminus A$. The (*vertex*)-*isoperimetric constant* of $(X, d)$ is defined by

$$i(X) = \inf_{\substack{A \subset X \\ |A| < \infty}} \frac{|\partial(A)|}{|A|}.$$

Evidently there is a connection between the isoperimetric constant and the doubling property, though at the isoperimetric constant we consider only 1-neighbourhoods, whereas at the doubling property $k$-neighbourhoods for $k > 1$ are also considered.

Obviously $i(X) > 0$ implies that $X$ is doubling. The converse is not always true. $i(X)$ can be zero for $(X, d)$ spaces which are doubling.

**Example 4.1.** Let $G(V, E)$ be the following infinite graph: take the disjoint union of the complete graphs $K_{2^n}$ and connect $K_{2^n}$ and $K_{2^{n+1}}$ by a single edge for every $n \in N$.

$G$ is doubling (hence paradoxical); however, $i(G) = 0$. Nevertheless, for graphs of bounded degree we have the following.

**Proposition 4.2.** *If the infinite, connected graph G is of bounded degree, then* $i(G) > 0$ *if and only if G is doubling, or equivalently wobbling-paradoxical.*

Define the $k$-isoperimetric constant:

$$i_k(G) = \inf_{\substack{A \subset V(G) \\ |A| < \infty}} \frac{|N_k(A) \setminus A|}{|A|}.$$

In the general case the following proposition holds.

**Proposition 4.3.** *Let G be an infinite, connected, locally finite graph. G is doubling, or equivalently wobbling-paradoxical, if and only if* $i_k(G) > 0$ *for at least one positive integer k.*

**Question 4.4.** Note that wobbling paradoxicity of a graph $G$ does not change, whether we add or delete finitely many edges, or substitute some edges with hanging chains of bounded length, as long as we keep connectedness. An infinite tree without end-vertices and with minimal degree three is in some sense 'irreducible' (or minimal) wobbling-paradoxical. Which are the 'irreducible' wobbling-paradoxical graphs? What are the basic or characteristic properties?

The next problem is in this direction.

In [17] the following problem was formulated.

**Problem 4.5.** Is it true that if an infinite graph is paradoxical, then there is an infinite spanning tree $T \subseteq G$ which is paradoxical?

Example 4.1 shows that this is not always true.

It is interesting that, independently from Deuber, Simonovits and Sós, and led by different motivation, Benjamini and Schramm considered a similar problem. They proved the following.

**Theorem 4.6. ([6])** *Let $G$ be an infinite, locally finite graph. If $i(G) > 0$ then $G$ contains a* tree *$T$ with $i(T) > 0$.*

*Moreover, if $i(G) \geqslant 1$ then $G$ contains a* spanning tree *$T$ with $i(T) \geqslant 1$.*

**Corollary 4.7.** *By Proposition 4.2, if $G$ is of bounded degree and wobbling-paradoxical, then it contains a wobbling-paradoxical tree.*

The following problem is still open.

**Problem 4.8.** Suppose that an infinite, connected, locally finite (or of bounded degree) graph $G$ is wobbling-paradoxical and $1 > i(G) > 0$. Does it imply that it contains a wobbling-paradoxical spanning tree? Example 4.1 shows that one should consider the case $i(G) = 0$ separately.

Now we have some reason to think that for this question the answer is 'no'.

Some more results about the existence of transient subtrees in transient graphs will be mentioned in Section 5 (Theorem 5.10).

Motivated by Følner's condition, a graph $G$ is called *amenable* if $i(G) = 0$.

### 4.2. Groups as metric spaces, Cayley graphs

The representation of groups by graphs was introduced by Cayley at the end of the 19th century. This representation provides a way to consider groups as metric spaces.

For detailed reference on geometric group theory see the monographs [32] and [33]. Here we give a very brief introduction, simply to indicate the relevance of the topic to the present subject.

In this section let $\Gamma$ be a finitely generated group, and let $S$ be a symmetric set of generators: $S = S \cup S^{-1}$ and $1 \notin S$.

**Definition 4.9. (Cayley graph)** The Cayley graph of $\Gamma$ with respect to $S$ has vertex set $V(G) = \Gamma$ and $(x, y) \in E(G)$ if and only if $xy^{-1} \in S$.

Denote by $d_S$ the resulting (shortest path) graph metric on $G$.

**Example 4.10. (1)** The lattice $Z^d$ is a Cayley graph of the free Abelian group of rank $d$.

**(2)** The regular tree $T_d$ of degree $d + 1$ is the Cayley graph of the free group with $(d + 1)/2$ generators if $d + 1$ is even. If $d + 1$ is odd, then $T_d$ is the Cayley graph of the group with $d/2$ free generators and one which is identical to its inverse.

Note that the lattice $Z^d$ is not wobbling-paradoxical, but $T_d$ is wobbling-paradoxical.

**Remark 4.11.** Every Cayley graph is connected and regular, and hence of bounded degree.

Every Cayley graph is vertex-transitive. The converse is not true. See the paper of Diestel and Leader [18]. The Cayley graph does not determine $\Gamma$: there are different groups with the same Cayley graph.

It is important to see that most of the properties of the Cayley graphs which will be considered here are independent of the choice of the generating set.

One of the reasons is the connection between the metrics belonging to different generating sets.

**Definition 4.12.** Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces. The two spaces are called Lipschitz-equivalent if there is a bijection $\varphi : X_1 \to X_2$ and constants $C_1, C_2 > 0$ such that, for any $x, y \in X_1$,

$$C_1 d_1(x, y) \leqslant d_2(\varphi(x), \varphi(y)) \leqslant C_2\, d_1(x, y).$$

**Proposition 4.13.** *Let $G_{S_1}$ and $G_{S_2}$ be two Cayley graphs of the same finitely generated group $\Gamma$. Then the two metric spaces, the two metrics $d_{S_1}$ and $d_{S_2}$ are Lipschitz-eqivalent.*

Let $G_{S_1}$ and $G_{S_2}$ be two Cayley graphs of the same finitely generated group $\Gamma$. Then the isoperimetric constant $i(G_{S_1}) > 0$ if and only if $i(G_{S_2}) > 0$.

The next result is a consequence of Følner's Theorem 2.19 and of Proposition 4.2.

**Proposition 4.14.** *A finitely generated group $\Gamma$ is amenable if and only if some Cayley graph $G_S$ of $\Gamma$ has isoperimetric constant $i(G_S) = 0$ or equivalently is not wobbling-paradoxical.*

More generally, the results above imply the following result.

**Proposition 4.15.** *The group $\Gamma$ is amenable if and only if all Cayley graphs belonging to finitely generated subgroups of $\Gamma$ are not wobbling-paradoxical.*

Clearly, for a finitely generated group there is a direct way (without considering the Cayley graph) to define the metric.

**Definition 4.16.** The word length $l_S(g)$ of $g \in \Gamma$ is defined as $\min\{n : g \in S^n\}$. The left, resp. right, word metric with respect to $S$ in $\Gamma$ is defined by

$$d_L(x, y) = l_S(x^{-1}y), \qquad \text{resp.}\ \ d_R(x, y) = l_S(xy^{-1}).$$

Obviously $d_S$ is the same as the $d_L$ word metric on $\Gamma$ belonging to $S$.

**Remark 4.17.** The group action on the group itself is defined by left multiplication, $g \to gx$; hence the two metrics in general have different features from certain points of view, as we will see below. Obviously, if $\Gamma$ is Abelian then $d_L = d_R$.

**Definition 4.18.** Let $A, B \subset \Gamma$. $f : A \to B$ is a piecewise left, resp. right, multiplication, if there are partitions

$$A = \bigcup_{i=1}^{k} A_i, \qquad B = \bigcup_{i=1}^{k} B_i$$

and elements $g_i \in \Gamma$ such that $g_i A_i = B_i$, resp. $A_i g_i = B_i$, for $1 \leqslant i \leqslant k$.

**Proposition 4.19. ([12])** *Any piecewise left, resp. right, multiplication in $\Gamma$ is a piecewise isometry of $(\Gamma, d_L)$, resp. of $(\Gamma, d_R)$. This follows from*

$$d_L(gx, gy) = d_L(x, y), \quad \textit{and} \quad d_R(xg, yg) = d_R(x, y).$$

It is easy to see that the converse is not true. However, we do have the following result.

**Proposition 4.20. ([12])** *Let $A, B \subset \Gamma$. A bijection $f : A \to B$ is a wobbling bijection on $(\Gamma, d_R)$, resp. on $(\Gamma, d_L)$, if and only if $f$ is a piecewise left, resp. right, multiplication on $A$.*

The 'if' part follows simply by

$$d_R(x, gx) = l_S(g), \qquad d_L(x, xg) = l_S(g).$$

**Remark 4.21.** In the above-quoted paper of Ceccherini-Silberstein, Grigorchuk and de la Harpe [12], different aspects of amenability, resp. paradoxicity of groups and pseudogroups of transformation of a set $X$ are considered. An important particular pseudogroup of transformations is the set of wobbling transformations which have the property formulated in Proposition 3.3. This pseudogroup plays an specific role in the discussions. The propositions above provide a possibility of using wobbling paradoxicity of a metric space in the study of different questions of amenability and paradoxicity of pseudogroups, and as a special case, of groups. As a side result in [12] there is a simple proof of Tarski's theorem (Theorem 2.10) using the theorem of [17]. For an elegant exposition of this proof see Ceccherini-Silberstein [10].

## 5. Random walks on graphs and on groups

In order to illustrate the relation of paradoxicity of a graph to random walks, we recall some definitions.

**Definition 5.1.** Let $G$ be an infinite, locally finite, connected graph. The simple random walk on $G$ is given by the transition probabilities

$$p(x, y) = \begin{cases} 1/d(x) & \text{if } (x, y) \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

where $d(x)$ is the degree of the vertex $x \in V$.

For the matrix $P = (p(x, y))$ let $P^n = (p_n(x, y))$, where $p_n(x, y)$ is the probability that a random walk starting in $x$ will be at $y$ after $n$ steps. The *spectral radius* of $G$ belonging to $P$ is defined by

$$\rho(G) := \limsup_{n\to\infty}(p_n(x, y))^{1/n}.$$

It is easy to see that $\rho(G)$ does not depend on $x$ and $y$.

**Definition 5.2.** The random walk $(G, P)$ is called *recurrent* if

$$\sum_{n=1}^{\infty} p_n(x, y) = \infty$$

for some (and hence for all) $(x, y) \in V$. If this does not hold, then $(G, P)$ is called *transient*.

Note that in the recurrent case, with probability 1 the random walk starting at $x$ returns to $x$ infinitely often; in the transient case, this event has probability 0.

The following theorem of Dodziuk gives a basic relation between the isoperimetric constant and the spectral radius.

**Theorem 5.3. ([19])** *Let $G$ be an infinite, connected graph of bounded degree. Then $\rho(G) < 1$ if and only if $i(G) > 0$.*

Obviously, if $\rho(G) < 1$ then $(G, P)$ is transient. Consequently we have the following result.

**Corollary 5.4.** *Let $G$ be an infinite graph of bounded degree. By Proposition 4.2, if it is wobbling-paradoxical, then the simple random walk on $G$ is transient.*

The converse is not true. As has already been proved by Polya, the simple random walk on the $d$-dimensional lattice is transient for $d \geqslant 3$ (and recurrent for $d = 1, d = 2$). It is easy to see that at the same time $i(G) = 0$ (and it is not doubling).

Note that the groups $\mathbb{Z}$ and $\mathbb{Z}^2$ are amenable. The Cayley graphs – the lattices $Z$ and $Z^2$ – are not wobbling-paradoxical.

For locally finite graphs which are not necessarily of bounded degree, Kaimanovich [35] proved an analogue of Theorem 5.3 which gives the connection between the spectral radius and a properly modified version of the isoperimetric constant.

**Definition 5.5.** Let

$$\partial_e A = \{(x, y) \in E(G), x \in A,\ y \in V \setminus A\}$$

and $\mathrm{Vol}(A) := \sum_{v\in A} d(v)$. Then the *strong isoperimetric constant* is defined by

$$i^*(G) := \inf_{\substack{|A|<\infty \\ A\subset V}} \frac{|\partial_e A|}{\mathrm{Vol}(A)}.$$

**Theorem 5.6. ([35])** *Let G be an infinite, connected, locally finite graph. Then $\rho(G) < 1$ if and only if $i^*(G) > 0$. Hence $i^*(G) > 0$ implies transience.*

**Remark 5.7.** Note that the two theorems above show that, for locally finite graphs, the condition on the strong isoperimetric constant can have the same implication as the corresponding condition on the vertex-isoperimetric constant for graphs of bounded degree. See the same phenomena, *e.g.*, in Theorem 5.9, Theorem 5.10 and Remark 5.25.

Now we consider the critical case when $i(G) = 0$.

Varopoulos [60] proved that any graph of bounded degree that 'grows' just slightly stronger than the 2-dimensional lattice is transient.

**Theorem 5.8. ([60])** *Let G be an infinite, connected graph of bounded degree. If, for some fixed $\epsilon > 0$,*

$$|\partial A| > |A|^{1/2+\epsilon}$$

*for every finite $A \subset V(G)$, then the G is transient.*

Thomassen [58] proved a stronger version of this theorem.

Let

$$\partial_- A = \{x \in A \ \text{ such that } \ \exists y \notin A : (x, y) \in E(G)\}$$

denote the inner boundary of $G$. Let $f$ be a nondecreasing positive real function defined on the natural numbers. $G$ satisfies a (rooted connected) $f$-isoperimetric inequality, if there is a constant $c > 0$ such that, for each finite $A \subset V$ (which contains a fixed vertex $v$ and induces connected subgraphs in $G$),

$$|\partial_- A| > cf(|A|).$$

**Theorem 5.9. ([58])** *Each connected graph of bounded degree satisfying the rooted connected f-isoperimetric inequality with $f(k) = k^{1/2+\epsilon}$ for some $\epsilon > 0$ is transient.*

In order to formulate the more general theorem for locally finite graphs, let $\partial_e A$ and Vol($A$) be as in Definition 5.5. $G$ satisfies a rooted, connected $f$-isoperimetric edge-inequality if there is a vertex $v \in V(G)$ and a positive constant $c > 0$ such that, for each finite $A \subset V$ containing $v$ and inducing a connected graph in $G$, we have

$$|\partial_e A| > cf(\mathrm{Vol}(A)).$$

**Theorem 5.10. ([58])** *Let G be a connected, locally finite connected graph satisfying a rooted, connected f-isoperimetric edge-inequality. If*

$$\sum_{k=1}^{\infty} f^{-2}(k) < \infty,$$

*then G is transient.*

*Moreover, the condition above implies that G contains a transient subtree with maximum degree three.*

**Remark 5.11.** As noted in the paper [58], not every transient graph contains a transient tree [46].

**Question 5.12.** By Theorem 4.3, if $G$ is of bounded degree, then we know that a graph $G$ is amenable ($i(G) = 0$) if and only if on $V(G)$ there is a finitely additive, wobbling-invariant measure $\mu$. Does this measure have any relation to the simple random walk on $G$?

### 5.1. The growth function belonging to balls

**Definition 5.13.** Let

$$B(x, n) = \{y \in V(G) : d(x, y) \leqslant n\}.$$

$G$ has exponential growth if

$$\liminf_{n\to\infty} |B(x, n)|^{1/n} > 1$$

and subexponential growth if

$$\limsup_{n\to\infty} |B(x, n)|^{1/n} = 1;$$

$G$ has polynomial growth with degree at most $k \geqslant 1$ if

$$\limsup_{n\to\infty} \frac{|B(x, n)|}{n^k} < \infty;$$

$G$ has intermediate growth if it has subexponential growth but it is not of polynomial growth.

The property of having exponential growth, subexponential growth or polynomial growth is independent of the choice of $x$.

For quasi-transitive graphs (when the automorphism group of $G$ acts on $G$ with finitely many orbits) the growth function characterizes the simple random walk on $G$.

**Theorem 5.14. ([62])** *Let G be an infinite, connected quasi-transitive graph. The simple random walk on G is recurrent if and only if it has polynomial growth with degree at most* 2.

**Remark 5.15.** In the definition of the growth function of a graph $G$, the neighbourhoods of points $x \in V$,

$$B(x, n) = \{y \in V(G) : d(x, y) \leqslant n\},$$

are considered, while at the isoperimetric constants and also at the doubling condition, the neighbourhoods of arbitrary finite subsets $A \subset V(G)$,

$$N_1(A) = \{y \in V(G) : d(y, A) \leqslant 1\},$$

or more generally,

$$N_k(A) = \{y \in V(G) : d(y, A) \leqslant k\}$$

(or different variations of these) are considered. There are some results on the connection of the growth of $|B(x,n)|$ and the growth of $|N_1(A)|$. See, *e.g.*, [33] or the very recent [7].

**Remark 5.16.** Clearly, if $G$ has the doubling property, then it has exponential growth. The converse is not true. Take, for example, the binary tree $T_3$ and for any $k$ insert a path of length $k$ instead of one edge on the $k$ level.

In the paper [17] we (Deuber, Simonovits and T. Sós) used the expression 'exponential growth' differently to how it is used above (and in the literature). For a finite set $A \subset V$ the growth of $|N_r(A)|/|A|$ (which determines the wobbling paradoxicity) was considered. This leads to some misunderstandings. Since the doubling property is stronger than the exponential growth property, one could call it (and it is sometimes called) 'strict exponential growth'.

### 5.2. Random walks on groups

Random walks on groups were initiated in the seminal paper of Kesten [38]. He considered the important special case of random walks on graphs when the graph $G_S$ is a Cayley graph of a finitely generated group. A basic result is that a finitely generated group $\Gamma$ is amenable if and only if $\rho(G_S(\Gamma)) = 1$ for all $S$.

Let $S$ be a generating set of $\Gamma$.

**Definition 5.17.** The growth function $\gamma_S(n) = |\{g : l_S(g) \leqslant n\}|$ belonging to the generating set $S$ is the number of elements of $\Gamma$ obtainable as a *reduced* word of length at most $n$ using elements of $S \cup S^{-1}$ as letters.

$\gamma_S(n)$ is the same as $|B(g,n)|$ in the Cayley graph $G_S$, which – since $G_S$ is vertex-transitive – is independent of $g \in \Gamma$.

For a Cayley graph $G_S$ being of exponential growth, intermediate growth or polynomial growth depends only on the group $\Gamma$ and is independent of the choice of the generating set $S$. (See also Proposition 4.13.)

We mention just one important result on the rate of growth of groups proved by Gromov.

**Theorem 5.18. ([29])** *A group has polynomial growth rate if and only if it is virtually nilpotent, i.e., it has a nilpotent subgroup of finite index.*

For previous results see, *e.g.*, [48] and [69].

The results above on graphs imply the following.

**Corollary 5.19.** *Let $\Gamma$ be a finitely generated group. The following are equivalent:*

- *$\Gamma$ is non-amenable,*
- *for some Cayley graph $G_S$ (for all Cayley graphs) $i(G_S) > 0$,*

- *for some Cayley graph $G_S$ (for all Cayley graphs) $\rho(G_S) < 1$,*
- *a Cayley graph (all Cayley graphs) are wobbling-paradoxical.*

*Furthermore:*

- *if $\Gamma$ is non-amenable, then all its Cayley graphs have exponential growth.*

**Remark 5.20.** The converse of this last statement is not true. There are amenable groups of exponential growth, like the so-called Lamplighter group. (See [33].)

**Corollary 5.21.** *By Theorem* 5.14*, if $\Gamma$ is amenable, the random walk on $G_S$ is recurrent if and only if it has polynomial growth of degree at most* 2*.*

*Moreover, it is an important consequence of Theorem* 5.14 *that the only recurrent groups are the finite extensions of* $0$, $\mathbb{Z}$ *and* $\mathbb{Z}^2$.

**Remark 5.22.** Suppose now that the Cayley graph $G_S$ of $\Gamma$ is transient. If $G_S$ has subexponential growth, then $\Gamma$ is amenable. If $G_S$ has exponential growth, then it can be amenable or non-amenable.

**Remark 5.23.** Let $\Gamma$ be a finitely generated group, and let $G_S$ be a Cayley graph of $\Gamma$. An interesting question is: Which functions can be growth functions of finitely generated groups?

Answering a question of Milnor [48], Grigorchuk [28] proved that there are finitely generated groups of intermediate growth.

It is easy to see that

$$1 \leqslant \liminf_{n\to\infty} |\gamma_S(n)|^{1/n} \leqslant (2|S|-1),$$

and the type of the growth rate is independent of $S$. Further, $\liminf_{n\to\infty} |\gamma_S(n)|^{1/n} = (2|S|-1)$ if and only if $\Gamma$ is the free group of rank $|S|$.

In the monograph of de la Harpe [33] a large variety of problems on the growth function can be found. Here we quote just one: Is there a $c(|S|) \in (1, 2|S|-1)$ such that $G$ is non-amenable if $\liminf_{n\to\infty} |\gamma_S(n)|^{1/n} > c(|S|)$?

Considering the other extreme, a negative answer was proved recently by Wilson [66], answering a question of Gromov.

**Theorem 5.24.** *There is group $\Gamma$ which has non-Abelian free subgroups and which has a sequence $S_n$ of generating sets such that:*

$$\liminf_{n\to\infty}\Big(\lim_{k\to\infty} |\gamma_{S_n}(k)|^{\frac{1}{k}}\Big) = 1.$$

### 5.3. Quasi-isometry

**Remark 5.25.** Here we considered the following classes of infinite connected graphs: locally finite graphs, graphs of bounded degree, quasi-transitive graphs (the automorphism group has finitely many orbits), transitive graphs, Cayley graphs. Obviously each class is a proper subset of the previous one. The theorems refer to one or another class. In some cases the theorem was originally formulated for groups or in a different setting for Cayley graphs, and was then generalized to some of the larger classes, while in other cases it was formulated for one of the classes of graphs above, which applied to Cayley graphs and led to results for groups.

One of the properties that helps in this transference is quasi-isometry (which is a weaker property than Lipschitz equivalence).

**Definition 5.26.** Two metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ are quasi-isometric if there exist $l \geqslant 1, c \geqslant 0$ and mapping $f : X_1 \to X_2$ such that, for all $x, y \in X_1$,

$$l^{-1} d_1(x, y) - c \leqslant d_2(f(x), f(y)) \leqslant l d_1(x, y) + c$$

and

$$d_2(x, f(X_1)) \leqslant c,$$

for all $x \in X_2$.

The map $f : X_1 \to X_2$ that satisfies the above conditions is called quasi-isometry.

We illustrate the consequence of quasi-isometry by the following theorems.

**Theorem 5.27. ([37, 47])** *Let $G_1$ and $G_2$ be two quasi-isometric graphs with bounded vertex degree. Then $G_1$ is transient if and only if $G_2$ is transient.*

**Proposition 5.28.** *Every quasi-transitive graph is quasi-isometric with a transitive graph.*

**Proposition 5.29.** *Let $G_1$ and $G_2$ be two quasi-isometric graphs of bounded degree* (*with the shortest path metric*). *Then $G_1$ is wobbling-paradoxical if and only if $G_2$ is wobbling-paradoxical.*

## 6. Some further applications of doubling

In this section we show some applications of the Theorem 3.5 [17]. We will call a graph $G(V, E)$ *nice* if it is infinite, connected and has bounded vertex degrees. Instead of 'wobbling-paradoxical' or 'doubling property' we also use the term 'strict exponential growth'.

### 6.1. Quasi-isometries and bi-Lipschitz maps

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be nice graphs with path-metrics $d_1$ and $d_2$. If $\varphi_1$ and $\varphi_2$ are maps from $V_1$ to $V_2$, then their distance is defined by

$$\mathrm{dist}(\varphi_1, \varphi_2) = \sup_{x \in V_1} d_2\big(\varphi_1(x), \varphi_2(x)\big).$$

Let $\varphi_1$ be a bi-Lipschitz map and suppose that $\mathrm{dist}(\varphi_1, \varphi_2) < \infty$. What can be said about $\varphi_2$? Such a map $\varphi_2$ does not need to be either one-to-one or onto; however, it must be a quasi-isometry.

Is it always true that if $\varphi$ is a quasi-isometry, then there is always a bi-Lipschitz map $\psi$ such that $\operatorname{dist}(\varphi,\psi) < \infty$?

The answer is no. Let $G(V,E)$ be the line graph on the natural numbers. Define $\varphi(n) = n/2$ if $n$ is even and $\varphi(n) = \frac{n+1}{2}$ if $n$ is odd.

Then $\varphi$ is a quasi-isometry and it is easy to see that there exists no bi-Lipschitz map in bounded distance of $\varphi$.

The following proposition is a corollary of Theorem 3.5 [17]; see also [65].

**Proposition 6.1.** *Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be nice graphs which have the doubling property, and let $\varphi\colon V_1 \to V_2$ be a quasi-isometry. Then there exists a bi-Lipschitz map $\psi : V_1 \to V_2$ such that* $\operatorname{dist}(\varphi,\psi) < \infty$.

### 6.2. Analysis on nice graphs

Let $G(V,E)$ be a nice directed graph. Let $\ell^\infty(E)$ denote the space of bounded functions on the edges, and let $\ell^\infty(V)$ be the space of bounded functions on the vertices. The boundary map $\partial : \ell^\infty(E) \to \ell^\infty(V)$ is defined by

$$(\partial f)(p) = \sum_{\overrightarrow{(q,p)}\in E} f(e) - \sum_{\overrightarrow{(p,q)}\in E} f(e).$$

Block and Weinberger [8] gave the following characterization of the doubling property.

**Proposition 6.2.** *$\partial$ is onto if and only if $G$ is wobbling-paradoxical.*

Proposition 6.2 was also proved in [20] using Theorem 3.5 together with the following result, originally due to Gerl.

**Proposition 6.3.** *Let $G(V,E)$ be a nice graph. Let $\ell^2(E)$ denote the space of square-summable functions on the edges, and let $\ell^2(V)$ be the space of square-summable functions on the vertices. Let $\partial : \ell^2(E) \to \ell^2(V)$ be given as above. Then $\partial$ is onto if and only if $G$ has the doubling property.*

### 6.3. Infinite matrix rings

Again, let $G(V,E)$ be a nice graph. Following Gromov [32] one can define the translation algebra $T_G$.

The elements of $T_G$ are matrices $M$ with rows and columns indexed by $V$, such that the entries $M_{(p,q)}$ satisfy the following conditions.

(1) There exists a constant $Q_M$, depending only on $M$, such that, for any $p,q \in V$, $|M_{(p,q)}| \leqslant Q_M$.

(2) There exists a constant $W_M$ depending only on $M$ such that $M_{(p,q)} = 0$ if $d(p,q) > W_M$.

The addition and multiplication is defined in $T_G$ the usual way:

$$(M+N)_{(p,q)} = M_{(p,q)} + N_{(p,q)}, \qquad (MN)_{(p,q)} = \sum_{r\in V} M_{(p,r)} N_{(r,q)}.$$

If $S$ is a unital ring then a trace $\mathrm{Tr}: S \to \mathbb{R}$ is an additive homomorphism such that

$$\mathrm{Tr}(\mathbf{1}) = 1, \qquad \mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

The following proposition [21] characterizes the doubling property via translation algebras.

**Proposition 6.4. ([21])** *$G$ has the doubling property if and only if there exists a trace on the ring $T_G$.*

### 6.4. Harmonic functions on graphs

Let $G$ be a locally finite graph. A function $f$ on $V(G)$ is called harmonic if its value at any vertex is the average of the values at its neighbours.

A function $f$ has finite energy if

$$\sum_{e \in E(G), (x,y)=e} |f(x) - f(y)|^2 < \infty.$$

Soardi [53] proved that if $G$ is the Cayley graph of an amenable group then all harmonic functions of finite energy are constant.

Now let $G$ be a nice graph. It is called roughly transitive if, for any $x, y \in V(G)$, there exists a quasi-isometry $T_{x,y}$, such that $T_{xy}(x) = y$ and all the constants $c_{x,y}, n_{x,y}$ in the definition of quasi-isometry are uniformly bounded.

Obviously, any transitive or quasi-transitive graph is roughly transitive. However, there exist roughly transitive graphs that are not even quasi-isometric to a transitive graph.

**Proposition 6.5. (Elek and Tardos [22])** *If $G$ is a roughly transitive graph and non-doubling, then any harmonic function of finite energy on $V(G)$ is constant.*

Note, however, that there exist non-constant harmonic functions of finite energy on the Cayley graph of a free group.

Harmonic functions have an interesting relation to random walks.

Let $G$ be a nice graph and consider the symmetric random walk on $G$. Let $Z_n$ be the integer-valued random variable that measures the distance from the origin after the $n$th step. Then

$$D = \liminf_{n \to \infty} \frac{E(Z_n)}{n}$$

is called the *drift* of the random walk.

Obviously, if the walk is recurrent then $D = 0$. If $G$ is the Cayley graph of a non-amenable group then $D > 0$. If $G$ is the Cayley graph of an amenable group of subexponential growth then $D = 0$. For the Cayley graphs of amenable groups of exponential growth $D$ is sometimes zero and sometimes positive [36, 61]. It is known [36] that $D > 0$ if and only if there exists a non-constant bounded harmonic function on $V(G)$.

It is important to note that Bartholdi and Virág [5] proved the amenability of the so-called basilica group by showing that its drift is zero.

## References


[1] Adel'son-Vel'skii, G. M. and Sreider, Y. A. (1957) The Banach mean on groups. *Mat. Nauk.* (N.S.) **12** 131–136.

[2] Banach, S. (1923) Sur le problème de la mesure. *Fund. Math.* **4** 7–33.

[3] Banach, S. (1924) Un théorème sur les transformations biunivoques. *Fund. Math.* **6** 236–239.

[4] Banach, S. and Tarski, A. (1924) Sur la decomposition des ensembles de points en parties respectivement congruents. *Fund. Math.* **6** 244–277.

[5] Bartholdi, L. and Virág, B. Amenability via random walks. Preprint.

[6] Benjamini, I. and Schramm, O. (1997) Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant. *GAFA, Geom. Funct. Anal.* **7**. 403–419.

[7] Benjamini, I. and Schramm, O. (2003) Pinched exponential volume growth implies on infinite dimensional isoperimetric inequality.

[8] Block, J. and Weinberger, S. (1997) Large scale homology theories and geometry. In *Geometric Topology, 1993 Georgia Int. Top. Conf.* (W. H. Kazez, ed.), AMS, pp. 522–569.

[9] Carsten, H-G., Deuber, W. A., Thumser, W. and Koppenrade, E. (1999) Geometrical bijections in discrete lattices. *Combin. Probab. Comput.* **8** 109–129.

[10] Ceccherini-Silberstein, T. G. (1999) An interplay between ergodic theory and combinatorics. In *Paul Erdős and his Mathematics*, János Bolyai Math. Soc., Budapest, pp. 45–49.

[11] Ceccherini-Silberstein, T., Grigorchuk, R. I. and de la Harpe, P. (1998) Décompositions paradoxales des groupes de Burnside. *CR Acad. Sci. Paris* **327** 127–132.

[12] Ceccherini-Silberstein, T., Grigorchuk, R. I. and de la Harpe, P. (1999) Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces (Russian). *Tr. Mat. Inst. Steklova* **224** 68–111. The web-site `http://www.unige.ch/math/biblio/preprint(1998)` contains a summary in French also including a one-page abridged English version. The file's name is `cgh98.ps`.

[13] Cohen, J. (1982) Cogrowth and amenability of discrete groups. *J. Funct. Anal.* **48** 301–349.

[14] Day, M. M. (1950) Amenable groups. *Bull. Amer. Math. Soc.* **56** 1054–1072.

[15] Deuber, W. (1993) 'Paradoxe' Zerlegung Euklidischer Räume. *Elem. Math.* **48** 62–75.

[16] Deuber, W. and Kostochka, A. (2000) On degrees of vertices in paradoxical trees. *Discrete Math.* **212** 53–59.

[17] Deuber, W. A., Simonovits, M. and Sós, V. T. (1995) A note on paradoxical metric spaces. *Studia Sci. Math. Hungar.* **30** 17–23.

[18] Diestel, R. and Leader, I. (2001) A conjecture concerning a limit of non-Cayley graphs. *J. Algebraic Comb.* **14** 17–25.

[19] Dodziuk, J. (1984) Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.* **284** 787–794.

[20] Elek, G. (1995) $l_p$-homologies and translation invariant functionals. *J. Austral. Math. Soc. Ser. A* **65** 111–119.

[21] Elek, G. (1997) The $K$-theory of Gromov's translations algebras and the amenability of discrete groups. *Proc. Amer. Math. Soc.* **125** 2551–2553.

[22] Elek, G. and Tardos, G. (2000) On roughly transitive amenable graphs and harmonic Dirichlet function. *Proc. Amer. Math. Soc.* **128** 2479–2485.

[23] Følner, E. (1955) On groups with full Banach mean value. *Math. Scand.* **3** 243–254.

[24] Fon-Der-Flaass, D. (1998) On exponential trees. *Europ. J. Combin.* **19** 25–27.

[25] Gerl, P. (1987) *Amenable Groups and Amenable Graphs*, Vol. 1359 of *Lecture Notes in Math.*, Springer.

[26] Gerl, P. (1988) Random walks on graphs with a strong isoperimetric inequality. *J. Theoret. Probab.* **1** 171–187.

[27] Grigorchuk, R. I. (1980) Symmetrical random walks on discrete groups. Vol. 6 of *Multicomponent Random systems, Adv. Prob. Related Topics*, Dekker, New York.

[28] Grigorchuk, R. I. (1984) Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk. SSSR Ser. Mat.* **48** 939–985.
[29] Gromov, M. (1981) Groups of polynomial growth and expanding maps. *Inst. Hautes Etudes Sci. Publ. Math.* **53** 53–73.
[30] Gromov, M. (1984) Infinite groups and geometric objects. In *Proc. Intern. Congr. Math., Warsaw, 1983*, Vol. 1, pp. 385–392.
[31] Gromov, M. (1987) Hyperbolic groups. In *Essays in Group Theory* (G. M. Gersten, ed.), MSRI publ. Vol. 8, pp. 75–263.
[32] Gromov, M. (1993) *Asymptotic Invariants of Infinite Groups*, *Geometric Group Theory*, Vol. 2 (G. A. Niblo and M. A. Roller, eds), Vol. 182 of *London Math. Soc. Lecture Notes.*
[33] de la Harpe, P. (2000) *Topics in Geometric Group Theory*, The University of Chicago Press.
[34] Hausdorff, F. (1914) Bemerkung über den Inhalt von Punktmengen. *Math. Ann.* **75** 428–433.
[35] Kaimanovich, V. (1992) Dirichlet norms, capacities and generalised isoperimetric inequalities for Markov operators. *Potential Analysis* **1** 61–92.
[36] Kaimanovich, V. A. and Vershik, A. M. (1983) Random walks on discrete groups, boundary and entropy. *Ann. Probab.* **11** 457–490.
[37] Kanai, M. (1986) Rough isometries and the parabolicity of Riemannian manifolds. *J. Math. Soc Japan* **38** 227–238.
[38] Kesten, H. (1959) Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92** 336–354.
[39] Laczkovich, M. (1990) Equidecomposability and discrepancy: A solution of Tarski's circle-squaring problem. *J. Reine und Angew. Math.* (*Crelle's J.*) **404** 77–117.
[40] Laczkovich, M. (1992) Uniformly spread discrete sets in $\mathbf{R}^d$. *J. London Math. Soc.* **46** 39–57.
[41] Laczkovich, M. (1992) Decompositions of sets with small boundary. *J. London Math. Soc.* **45** 39–57.
[42] Laczkovich, M. (1994) Paradoxical decompositions: A survey of recent results. In *First European Congress of Mathematics* (*Paris, July 6–10, 1992*), Vol. 120 of *Progress in Mathematics*, Birkhäuser, Vol. II, pp. 159–184.
[43] Laczkovich, M. (2001) On paradoxical spaces. *Studia Sci. Math. Hungar.* **38** 267–271.
[44] Laczkovich, M. (2002) Paradoxes in measure theory. In *Handbook of Measure Theory*, Vol. I, II, North-Holland, Amsterdam, pp. 83–123.
[45] Lubotzky, A. (1994) *Discrete Groups, Expanding Graphs and Invariant Measures.* Vol. 125 of *Progress in Mathematics*, Birkhäuser.
[46] McGuinness, S. (1989) Random walks on graphs and directed graphs. PhD Dissertation, Univ. Waterloo.
[47] Markvorsen, St., McGuinness, S. and Thomassen, C. (1992) Transient random walks on graphs metric spaces, with application to hyperbolic surfaces. *Proc. London Math. Soc.* **64** 1–20.
[48] Milnor, J. (1968) Growth of finitely generated solvable groups. *J. Differential Geom.* **2** 447–449.
[49] von Neumann, J. (1929) Zur allgemeinen Theorie des Masses. *Fund. Math.* **13** 73–116.
[50] Ol'shanskii, A. Y. (1980) On the problem of existence of an invariant mean on a group. *Russian Math. Surveys* (*Uspekhi*) **35** 180–181.
[51] Paterson, A. L. T. (1988) *Amenability*, Vol. 29 of *Mathematical Surveys and Monographs*, AMS, Providence, RI.
[52] Rosenblatt, J. M. (1981) Uniqueness of invariant means for measure-preserving transformations. *Trans. Amer. Math. Soc.* **265** 623–636.
[53] Soardi, P. M. (1993) Rough isometries and Dirichlet finite harmonic functions on graphs. *Proc. Amer. Math. Soc.* **119** 1239–1248.
[54] Tarski, A. (1925) Problème 38. *Fund. Math.* **7** 381.
[55] Tarski, A. (1929) Sur les fonctions additives dans les classes abstraites et leur applications au problème de la mesure. *CR Séances Soc. Sci. Lettres Varsovie, Cl.* III **22** 114–117.
[56] Tarski, A. (1938) Über das absolute Mass linearer Punktmengen. *Fund. Math.* **30** 218–234.
[57] Tarski, A. (1938) Algebraische Fassung des Massproblems. *Fund. Math.* **31** 47–66.

[58] Thomassen, C. (1992) Isoperimetric inequalities and transient random walks on groups. *Ann. Prob.* **29** 1592–1600.
[59] Thomassen, C. and Woess, W. (1994) Vertex transitive graphs and accessibility. *J. Combin. Theory Ser. B* **26** 1–60.
[60] Varopoulos, N. T. (1985) Isoperimetric inequalities and Markov chains. *J. Functional Anal.* **63** 215–239.
[61] Varopoulos, N. (1985) Long range estimates for Markov-chains. *Bull. Sci. Math.* **109** 225–252.
[62] Varopoulos, N. (1986) Théorie du potentiel sur des groupes et des varietés. *CR Acad. Sci. Paris Ser. I* **302** 203–205.
[63] Vitali, G. (1905) Sul problema della mesura dei gruppi di punti di una retta. *Bologna.*
[64] Wagon, S. (1986) *The Banach–Tarski Paradox*, 2nd edn, Cambridge University Press. First paperback edn (1993).
[65] Whyte, K. (1999) Amenability, bi-Lipschitz equivalence and the Von Neumann conjecture. *Duke Math. J.* **99** 93–112.
[66] Wilson, J. S. (2003) On exponential growth and uniformly exponential growth for groups. Preprint.
[67] Woess, W. (1994) Random walks on infinite graphs and groups: A survey on selected topics. *Bull. London Math. Soc.* **26** 1–60.
[68] Woess, W. (2001) *Random Walks on Infinite Graphs and Groups*, Vol. 138 of *Cambridge Tracts in Mathematics*, Cambridge University Press.
[69] Wolf, J. A. (1968) Growth of finitely generated solvable groups and curvature of Riemannian manifolds. *J. Differential Geom.* **2** 421–446.