Paradoxical Decompositions and Growth Properties

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The theory of paradoxical decompositions arose in connection with the existence of non-Lebesgue measurable sets.

The non-existence of isometry-invariant finitely additive measure in $\mathbb{R}^3$ was proved by Banach and Tarski (1924) [1] by means of paradoxical decomposition. They proved that it is possible to partition the unit ball in $\mathbb{R}^3$ into finitely many pieces and to rearrange them by rigid motions (using isometric transformations) to form two unit balls. This "duplication", this "paradoxical decomposition" of the ball at first seems to be impossible.

The analysis of this surprising phenomenon led to the concept of amenable groups introduced and studied first by von Neumann (1929) [10]. Since that time the subject developed into a field which has importance beside analysis, group theory and geometry in discrete mathematics and computer science (e.g., in the theory of random walks, percolation, expanders).

The Hausdorff-Banach-Tarski paradoxical decompositions of the ball (or of the sphere) in $\mathbb{R}^d$ exist for $d = 3$ (and also for $d > 3$), but do not exist for $d = 1$ and $d = 2$.

Von Neumann discovered that these different phenomena are due to the difference between the isometry groups of $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$, the latter one is more *"rich"*. He considered a general setting where the basic notions are the finitely additive group invariant measure (or invariant mean) and the paradoxical groups (or amenable groups=non-paradoxical groups).

The objective of the present talk is to give some illustrations and indications of the wide range of topics which developed from the subject mentioned above, providing some motivation of the particular problem considered in the paper of Deuber, Simonovits and Sós [3] and some of its aftermath.

In the paper [3] - - for an arbitrary metric space the concept of *wobbling transformations* (called more recently also *bounded perturbation of the identity*) is introduced.

Definition. Let (X, d) be a metric space, $A, B \subseteq X$. A bijection $f : A \to B$ is called a *wobbling bijection* if

$$\sup_{x \in A} d(x, f(x)) < \infty \ .$$

$A, B \subseteq X$ are called *wobbling equivalent* if there is a wobbling bijection $f : A \to B$.

Definition. The set $A \subseteq X$ is called *wobbling paradoxical* if there is a decomposition

$$A = A_1 \cup A_2, \qquad A_1 \cap A_2 = \emptyset$$

such that A, A_1, A_2 are pairwise wobbling equivalent.

In [3] wobbling paradoxicity is characterized by the following growth condition: For $A \subset X$, $k > 0$ let $N_k(A)$ denote the k-neighbourhood of A:

$$N_k(A) = \{x \in X : d(x, A) \leq k\} \ .$$

Definition. The metric space (X, d) is *doubling*, if there is a $k > 0$ such that

$$|N_k(A)| \geq 2|A| \text{for every finite} A \subset X.$$

Theorem 1 *Let (X, d) be a discrete and countable metric space. (X, d) is wobbling paradoxical if and only if it is doubling.*

In the lecture we surveyed the connection of wobbling paradoxicity to the amenability of groups, to theory of random walks on graphs and groups and some recent applications of the doubling property and wobbling paradoxicity.

A survey paper written jointly with Gábor Elek will appear in a Volume dedicated to the memory of Walter Deuber.

For detailed information and references about the extremely wide area the reader is referred to the excellent books like of Gromov [5], de la Harpe [6], Lubotzky [9], Paterson[11], Wagon [13], Woess [14], and the many survey papers on these subjects, e.g., by Ceccherini-Silberstein, Grigorchuk and de la Harpe [2], Laczkovich [7], [8], Thomassen and Woess [12].

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On the Sparse Regularity Lemma

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(joint work with S. Gerke, Y. Kohayakawa, V. Rödl)

Over the last decades Szemerédi's regularity lemma [17] has proven to be a very powerful tool in modern graph theory. Roughly speaking, the regularity lemma asserts that one can partition a graph G into a constant number of equal-size parts in such a way that most parts are pairwise ε-regular; see [1, 2, 14] for the precise