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On some applications of graph theory, I

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Abstract

In a series of papers, of which the present one is Part I, it is shown that solutions to a variety of problems in distance geometry, potential theory and theory of metric spaces are provided by appropriate applications of graph theoretic results.

§1

In what follows we are going to discuss systematic applications of graph theory – among others – to geometry, to potential theory, and to the theory of function-spaces. This sounds perhaps surprising to those who still think of graph theory as the "slum of topology". These applications show that suitably devised graph theorems act as flexible logical tools (essentially as generalizations of the pigeon hole principle) and have nothing to do with topology at all. We believe that the applications given in this sequence of papers do not exhaust all possibilities of applications of graph theory to other branches of mathematics. Scattered applications of graph theory to geometry and number theory (mostly via Ramsey's theorem) existed already in the papers of Erdös and Szekeres [6] and Erdös [2,5]. The inherence of graph theoretic methods in the problems we are dealing with is indicated also by the fact that it leads often to best possible results.

Several parts of the results contained in this sequence of papers were subject to lectures given by the authors. The first lecture was given by the last named author on Aug. 30, 1968 in Calgary. The first printed account, reproducing lectures of the last named two authors at the Conference on Combinatorial Structures and Their Applications in June 1969 at the University of Calgary, appeared in the Proceedings of this Conference (see [17] and [20]). The second paper of this series which was written much earlier than the present one, appeared already in [7]. Accounts were given also by the second named author in a lecture at Imperial College, London, in 1970.

The first group of applications refers to the distance distribution of point sets in a complete metric space. Let (X, d) be a complete metric space and let F be a family of point sets f in X satisfying the following restrictions:

(1.1) For a sufficiently large R, all sets of F are in a sphere of radius R.
(1.2) If $f \in F$ and f_1 is a finite subset of f, then $f_1 \in F$.

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(1.3) If $f \in F$ is finite, $P \in f$, then for arbitrary $\epsilon > 0$ there exists a P_1 in X such that

$$P_1 \neq P, \quad d(P, P_1) < \epsilon,$$

and the set

$$f_1 = f \cup \{P_1\}$$

belongs to the family F too.

Important examples of such families in case of finite-dimensional euclidean spaces R^k, which interest us in this paper almost exclusively, are:

(a). The family F of all closed domains in R^k with maximal chord 1 (taking in account that in R^k translation does not change distances).
(b). All closed subsets of the closure of a fixed bounded domain D in $R^l (l \leq k)$,
(c). All closed sets in R^k whose projection to all hyperplanes in R^l $(l < k)$ can be translated into the closure of a fixed bounded l-dimensional domain.

For given F let F_n denote the subfamily of F whose elements f satisfy the additional restriction

$$(1.4) \quad |f| = n.$$

We are interested in the distribution of distances

$$(1.5) \quad d(P_\mu, P_\nu), \quad 1 \leq \mu < \nu \leq n,$$

in sets f which belong to F_n.

The families F are so general that at first glance it seems hopeless to assert anything nontrivial for the distribution in this generality. Nevertheless we have found that by introducing “the packing-constants[1] belonging to the family F” a great deal can be said about the distribution. These constants are defined for $\nu \geq 2$ by

$$(1.6) \quad \delta_\nu = \delta_\nu(F) \sup_{(P_1,\ldots,P_\nu)\in F_\nu} \min_{1\leq i<j\leq \nu} d(P_i, P_j).$$

These constants obviously exist and are monotonic in ν:

$$(1.7) \quad \delta_2 \geq \delta_3 \geq \ldots .$$

Moreover they are also “monotonic in F” in the sense that $F_1 \subset F_2$ implies obviously

$$(1.8) \quad \delta_\nu(F_1) \leq \delta_\nu(F_2), \quad \nu \geq 2.$$

In the case of R^k, we have also

$$(1.9) \quad \lim_{\nu\to\infty} \delta_\nu = 0.$$

We found that in the general case, in addition to the packing constants, the “critical indices” $i_2, i_3, \ldots$ play a decisive role in the distribution of distances of the sets of F_n. They are defined by

$$(1.10) \quad \delta_2 = \ldots = \delta_{i_2} > \delta_{i_2+1} = \ldots = \delta_{i_3} > \delta_{i_3+1} = \ldots;$$

and for convenience we define

$$(1.11) \quad i_1 = 1.$$

We can formulate now

[1] The name can be justified the easiest when the family F consists of the point sets on the unit sphere. Having spherical distance, for each $\nu \geq 2$ suitably placed disjoint caps with spherical radii $\delta_{\nu/2}$ realise the densest packing by ν congruent spherical caps of the unit sphere.

Theorem 1. *For each fixed $v \geq 2$ and $n > i_2$, the number of distances $d(P_i, P_j)$ $(i \neq j)$ in each set f of F_n satisfying the inequality*

$$(1.12)\quad d(P_i, P_j) \leq \delta_{i_{v+1}} (=\delta_{i_v+1})$$

is at least

$$(1.13)\quad \tfrac{1}{2} n^2/i_v - \tfrac{1}{2} n.$$

The theorem is best possible in a very strong sense. Equality in (1.13) can be attained for *all* F-families, for *all* $v \geq 2$ and $n > i_v$, $n \equiv 0 \pmod{i_v}$.

§2

In order to show that Theorem 1 leads to genuine geometrical results, let first F be the family of sets on the periphery of the unit circle. Then evidently

$$\delta_l = 2\sin(\pi/l) \quad (l = 2, 3, \ldots)$$

and $i_l = l$. Hence by Theorem 1 for $v \geq 2$ we have: If $n > v$ and n points lie on the periphery of the unit circle, then at least $\left(\frac{1}{2} n^2/v - \frac{1}{2} n\right)$ distances are $\leq 2\sin(\pi/(v+1))$. Putting m points on the periphery very close to each vertex of a regular v-gon, we see at once that the number of distances $\leq 2\sin(\pi/(v+1))$ (even the number of distances $\leq 2\sin(\pi/v) - \eta$, η small positive) equals $\frac{1}{2} n^2/v - \frac{1}{2} n$ indeed.

Another important case when all packing constants can be determined is given by the subsets of an arc AB having the property that if P is fixed on it and Q moves along it off P then

$$(2.1)\quad \overline{QP} \text{ decreases strictly monotonically.}$$

In this case – as is easy to see – δ_v is furnished by the side length b_v of the "inscribed quasi-regular v-gon $AP_2P_3 \ldots P_{v-1}B$" defined by

$$(2.2)\quad \overline{AP_2} = \overline{P_2P_3} = \ldots = \overline{P_{v-2}P_{v-1}} = \overline{P_{v-1}B} = b_v$$

the points P_j being on the arc.

All packing constants belonging to circular arcs can be explicitly determined. Several packing constants belonging to the unit-square in R^2 and unit cube in R^3 have been determined in the papers of Meir and Schaer [12] and Schaer [14]. Probably all packing constants belonging to a convex curve can be explicitly determined (somewhat in the sense of (2.2)).

In the case when the family F consists of plane sets with maximal chord length 1, the packing constants δ_v, $2 \leq v \leq 7$ were determined for a different purpose by Bateman and Erdös in 1951 [1]; they are

$$(2.3)\quad \begin{aligned} &\delta_2 = \delta_3 = 1, \quad \delta_4 = \tfrac{1}{2}\sqrt{2}, \quad \delta_5 = \tfrac{1}{2}(\sqrt{5} - 1), \\ &\delta_6 = 1/(2\sin 72^\circ), \quad \delta_7 = \tfrac{1}{2}. \end{aligned}$$

As proved by Thue (see [18]), δ_v is, for large v, asymptotically

$$\left(\tfrac{1}{12}\pi^2\right)^{1/4} v^{-1/2}.$$

Since in this case $i_v = v + 1$ $(2 \leq v \leq 5)$, for $v = 2$ Theorem 1 yields that if $n > 3$ points are located on a plane with maximal distance 1, then at least $\frac{1}{6} n^2 - \frac{1}{2} n$ distances are $\leq \frac{1}{2}\sqrt{2}$. This was the only known case of Theorem 1 (see Erdös [3]). A classical case of the determination of the packing constants is known since Newton and Gregory. If F

consists of all subsets of the unit sphere in R^3, their known dispute (see [8], p. 236) boils down to the question whether or not in this case $\delta_{13} < 1 = \delta_{12}$ or $\delta_{13} = \delta_{12}$. Since now, this time in nonspherical metric,

$$(2.5) \quad \delta_2 = 2, \quad \delta_3 = \sqrt{3}, \quad \delta_4 = \sqrt{\tfrac{8}{3}}, \quad \delta_5 = \delta_6 = \sqrt{2} > \delta_7,$$

we have $i_v = v$ for $2 \le v \le 4$ and $i_5 = 6$. Theorem 1 now yields, e.g. for $v = 4$, that if $n > 4$ points lie on the unit sphere, at least $\frac{1}{8}n^2 - \frac{1}{2}n$ euclidean distances between them are $\le \sqrt{2}$ (and generally no more). [2] Schoenberg [15] and Seidel [16] found that choosing F to be the family of all sets in R^k with maximal chord length 1, beside the trivial $\delta_2 = \delta_3 = \ldots = \delta_{k+1} = 1$, we have

$$(2.4) \quad \delta_{k+2} = \begin{cases} \sqrt{k/(k+2)} & \text{if } k \text{ is even,} \\ \sqrt{(k^2+2k-1)/(k^2+4k+3)} & \text{if } k \text{ is odd,} \end{cases}$$

and hence $i_2 = k + 1$. Theorem 1 gives e.g. that if $n > k + 1$ points in R^k (k even) have maximal distance 1 then at most $kn^2/(2k + 2)$ distances can be greater than $\sqrt{k/(k+2)}$. In other words, if we have (for some even k) a system of $n > k + 1$ points with maximal distance 1 and *more* than $kn^2/(2k + 2)$ of the distances are $> \sqrt{k/(k+2)}$, then the system cannot be isometrically embedded in R^k (again best possible). Such type of non-embeddability criteria seem not to be observed before.

All these motivate the interest in the general problems of prescribability, uniqueness and geometrical realizability of the sequence of packing constants (as mentioned already in [20]).

§3

Theorem 1 gives sharp lower bounds for the number of distances not exceeding $\delta_{i_{v+1}}$ in $F_n \subset F$. What can be said of the number of distances not exceeding δ for a fixed δ? We are going to prove

Theorem 2. *If* $0 < \delta \le \delta_2$ *and* $v \ge 2$ *is (uniquely) determined by*

$$(3.1) \quad \delta_{i_{v+1}} \le \delta < \delta_{i_v},$$

then for $n > i_v$ *the number of distances* $d(P_i, P_j)$, $i \ne j$, *in each set* f *of* F_n *satisfying the inequality*

$$(3.2) \quad d(P_i, P_j) \le \delta$$

is at least

$$(3.3) \quad n^2/(2i_v) - \tfrac{1}{2}n.$$

This lower bound is best possible for all F-families, for all $\delta \le \delta_2$ and $n > i_v$, $n \equiv 0 \pmod{i_v}$.

A particularly elegant (but somewhat weaker) form can be given to Theorem 2 by observing that, together with its best possibility concerning n, it implies the existence of

$$(3.4) \quad \lim_{n\to\infty} \binom{n}{2}^{-1} \min_{f \in F_n} \sigma(f),$$

where $\sigma(f)$ denotes the number of pairs P_i, P_j in f with

$$(3.5) \quad d(P_i, P_j) \le \delta, \quad i < j.$$

Denoting the limit in (3.4) by $H_F(\delta)$ and calling it "the lower distance distribution function of the family F", an alternative form of Theorem 2 is

[2] Since the Newton–Gregory dispute, the sequence of packing-constants is intensely investigated from the point of view of strict monotonicity. Using Theorem 1 the other way around, one can devise a general method to show $\delta_v > \delta_{v+1}$ if it is true (see [21]).

Theorem 2′. *For all F-families of sets in* (X, d), *the lower distance distribution function* $H_F(\delta)$ *is a right-continuous step function with jumps only at* $\delta = \delta_{i_\nu}$ *and*

$$\lim_{\delta \to \delta_{i_\nu}+0} H_F(\delta) = 1/(i_{\nu-1}), \quad \nu = 2, 3, \dots .$$

(Note the definition of i_1 in (1.11)).

§4

Next we turn to some applications in complex function theory. Let B be a bounded and closed set on the plane with boundary ∂B. Then we assert for the capacity (see [10]) of B

Theorem 3. *If* ∂B *belongs to an F-family satisfying* (1.1), (1.2), (1.3) *with packing constants* δ_ν^* *and*

$$(4.1) \quad \sum_\nu \nu^{-2} \log(1/\delta_\nu^*)$$

diverges, *then the capacity of B is* 0.

The theorem is best possible in the sense that for arbitrary small $\epsilon > 0$ there exist sets B with *positive* capacity such that

$$(4.2) \quad \sum_\nu \nu^{-2+\epsilon} \log(1/\delta_\nu^*) \quad \text{diverges.}$$

More generally, we shall prove

Theorem 4. *If B is a bounded and closed continuum* (*whose complement is simply connected*) *and* ∂B *belongs to an F-family of sets satisfying* (1.1), (1.2), (1.3) *and having the packing constants* δ_ν^*, *then the outer conformal radius* $r = r(B)$ *obeys the inequality*

$$(4.3) \quad r(B) \le \prod_{\nu=2}^{\infty} (\delta_\nu^*)^{1/(\nu-1)\nu}.$$

Since $\sum_{\nu=2}^{\infty} 1/(\nu-1)\nu = 1$, both sides of (4.3) are linear in a magnifying constant, hence without loss of generality we may assume

$$\delta_2^* = 1.$$

If we retain in (4.3) only the first few packing-constants we can get upper bounds for $r(B)$.

Obviously (4.3) can be used as a *system* of inequalities, giving upper bounds for the outer conformal radius via various geometrical properties of the set (expressed by our F-families).

As is well known, Pólya proved the inequality

$$F \le \pi d(B)^2,$$

where F stands for the outer Jordan measure of B and $d(B)$ for its transfinite diameter (supposing now only that B is bounded and closed). Connecting this with the real content of Theorem 4 we get

Corollary 4.1. *If B is a bounded and closed set in* R^2 *with outer Jordan measure F so that* ∂B *belongs to one of our set-families H with packing constants* $\delta_\nu(H)$, *then we have the inequality*

$$(4.4) \quad F \le \pi \prod_{\nu=2}^{\infty} \delta_\nu(H)^{2/(\nu-1)\nu}.$$

This is a purely geometrical inequality between certain geometrical constants of B. It would be of interest to find a geometrical proof for it and also to find the higher-dimensional analogues, mainly for R^3.

§5

Next we turn to some applications which yield bounds for energy integrals. Let D be a bounded and closed set in R^k with positive finite l-dimensional Jordan measure $|D|$, $l \leq k$. We consider integrals of the form

$$(5.1) \quad I(g) = \int_{(D)} \int_{(D)} g(\overline{PQ})\, \mathrm{d}\mu_P\, \mathrm{d}v_Q$$

connected to a mass distribution with density 1 on D. Here $\overline{PQ}$ means euclidean distance and $g(x)$ is any function satisfying

$$(5.2) \quad \begin{array}{ll} \text{(i)} & g(x) \text{ is monotonically decreasing} \\ \text{(ii)} & g(x) \text{ is bounded from below in } (0, \delta_2). \end{array}$$

The cases

$$g(x) = \log x^{-1}, \qquad g(x) = x^{-\alpha}, \quad \alpha > 1,$$

are obviously included. Now we choose as the family F all subsets of D. Then we assert

Theorem 5. *Denoting by δ_v the packing constants of the family F, the inequality*

$$(5.3) \quad |D|^{-2} I(g) \geq \sum_{v=2}^{\infty} g(\delta_v)/(v-1)v$$

holds for all $g(x)$ satisfying (5.2).

Equality holds in (5.3) for $g(x) \equiv 1$. It is perhaps of interest to note that the evaluation points on the right-hand side do not depend on g, reminding the classical formulae of mechanical quadrature.

Denoting the potential at P generated by $g(x)$ (with uniform mass distribution) by

$$(5.4) \quad G(P) = \int_{(D)} g(\overline{PQ})\, \mathrm{d}v_Q,$$

Theorem 5 yields at once the inequality

$$(5.5) \quad \sup_{P \in D} G(P) \geq |D| \sum_{v=2}^{\infty} g(\delta_v)/(v-1)v.$$

It is therefore a plausible conjecture that for every γ, $0 \leq \gamma \leq 1$, the inequality

$$(5.6) \quad G(P) \geq \gamma |D| \sum_{v=2}^{\infty} g(\delta_v)/(v-1)v$$

holds in D with the possible exception of a set of measure $\leq \gamma|D|$. This would be an interesting counterpart of the classical upper bound of Ahlfors–Cartan (see [13]). We could prove so far a weaker theorem only. Let $g(x)$ be positive and monotonically decreasing for $x > 0$ and let the index r be defined (if it exists) by

$$(5.7) \quad \max_v \{g(\delta_v)/(v-1)v\} = g(\delta_v)/(r-1)r.$$

Then we have

Theorem 6. *For the potential $G(P)$ generated by $g(x)$ in* (5.4) *the inequality*

$$G(P) \geq \gamma |D| \sum_{v=2}^{r} g(\delta_v)/(v-1)v$$

holds in D for every γ, $0 \leq \gamma \leq 1$, with exception of a set of measure $\gamma(1-(r-1)^{-1})|D|$ at most.

§6

Next we turn to the proof of Theorem 1. It is based on the following graph theorem [17]:

For given $K, N, 3 \leq K \leq N$, let

$$(6.1) \quad N = (K-1)t + s, \quad 0 \leq s \leq K-2.$$

A graph T_N (admitting only simple edges and no loops) with N vertices which does not contain a complete subgraph of order K cannot have more than

$$(6.2) \quad \frac{K-2}{2(K-1)}(N^2 - s^2) + \binom{s}{2}$$

edges. In the unique extremal graph (where equality can be attained) the vertices can be divided into $K-1$ disjoint classes each containing $t+1$ or t vertices so that each pair of vertices from different classes is connected by an edge whereas pairs from identical classes are not connected.

Let now v be fixed and $\{P_1, P_2, \ldots, P_n\}$ be in F_n. We make correspond to it a graph Δ'_n with vertices $P'_1, P'_2, \ldots, P'_n$ as follows: P'_j and P'_k $(j<k)$ be connected by an edge in Δ'_n if and only if

$$(6.3) \quad d(P_j, P_k) \geq \delta_{i_{v+1}} (=\delta_{i_v+1}).$$

Let

$$n = i_v m + h, \quad 0 \leq h \leq i_v - 1.$$

We assert that the number of pairs satisfying (6.3) cannot exceed

$$(6.4) \quad \frac{i_v - 1}{2i_v}(n^2 - h^2) + \binom{h}{2} \stackrel{\text{def}}{=} U.$$

For, otherwise, the graph Δ'_n had more than U edges and thus the graph (6.1)–(6.2) with

$$N = n, \quad K = i_v + 1, \quad t = m, \quad s = h$$

would imply the existence of a complete subgraph of order $i_v + 1$. Returning to distances, however, this would mean that for suitable points $P_1, P_2, \ldots, P_{i_v+1}$ from $\{P_1, \ldots, P_n\}$, all distances were $> \delta_{i_v+1}$. Since the number of points is finite, for a sufficiently small $\eta > 0$ *all* these distances are even $> \delta_{i_v+1} + \eta$. But this is in contradiction to the definition of the δ_v's in (1.6). Hence our assertion (6.4) is correct. But then the number of pairs with

$$d(P_j, P_k) \leq \delta_{i_v+1}, \qquad 1 \leq j < k \leq n,$$

is at least

$$\binom{n}{2} - \tfrac{1}{2}\{(i_v - 1)/i_v\}(n^2 - h^2) - \binom{h}{2}$$
$$= \tfrac{1}{2}n^2/i_v - \tfrac{1}{2}n + \tfrac{1}{2}h(1 - h/i_v) \geq \tfrac{1}{2}n^2/i_v - \tfrac{1}{2}n,$$

as claimed.

§**7**

In order to prove that Theorem 1 is best possible for all F-families and for all $v \geq 2$, let for arbitrarily small $\epsilon > 0$ the set

$$(7.1) \quad \{P_1^*, P_2^*, \ldots, P_{i_v}^*\}$$

be in F_{i_v} such that

$$(7.2) \quad \min_{i \leq j < k \leq i_v} d(P_j^*, P_k^*) > \delta_{i_v} - \epsilon.$$

Let M be an arbitrary positive integer. Repeated use of (1.3) results that arbitrarily close to each of the points P_j^*, $M-1$ different points can be found so that the resulting system Π of $n = Mi_v$ points belongs to F_n and the distance of two points located "close" to *different* points is

$$> \delta_{i_v} - 2\epsilon.$$

Since $\delta_{i_v} > \delta_{i_v+1}$, ϵ can be chosen so small that

$$\delta_{i_v} - 2\epsilon > \delta_{i_v+1}.$$

Hence the number of distances between points of Π not exceeding δ_{i_v+1} is indeed

$$i_v \binom{M}{2} = i_v \left(\tfrac{1}{2} M^2 - \tfrac{1}{2} M\right) = i_v \left(\tfrac{1}{2} n^2 / i_v^2 - \tfrac{1}{2} n / i_v\right) = \tfrac{1}{2} n^2 / i_v - \tfrac{1}{2} n.$$

In order to prove Theorem 2, we observe that the estimation (3.3) follows at once from monotonicity with respect to δ. Hence we have only to show that it is best possible also for $\delta_{i_v+1} < \delta < \delta_{i_v}$. Fixing such a δ, and choosing an $\eta > 0$ so small that

$$\delta < \delta_{i_v} - 2\eta,$$

the reasoning of §7 repeated with $\epsilon = \frac{1}{2}\eta$ yields the desired conclusion.

§**8**

Before proving Theorems 3 and 4, we shall prove Theorem 5. We remark first that without loss of generality we may assume

$$(8.1) \quad g(x) \geq 0, \quad 0 < x \leq \delta_2.$$

Namely if (5.3) holds for this case and $g_1(x)$ decreases monotonically for $x > 0$ so that

$$g_1(x) \geq -c_1, \quad 0 < x \leq \delta_2,$$

then applying the result to $g(x) = g_1(x) + c_1$, we get

$$c_1 |D|^{-2} \int_{(D)} \int_{(D)} \mathrm{d}v_P \, \mathrm{d}v_Q + |D|^{-2} \int_{(D)} \int_{(D)} g_1(\overline{PQ}) \, \mathrm{d}v_P \, \mathrm{d}v_Q$$
$$\geq \sum_{v=2}^{\infty} \frac{g_1(\delta_v) + c_1}{(v-1)v},$$

hence the theorem follows for the general case. So we may suppose (8.1).

Let $f = \{P_1, P_2, \ldots, P_n\} \in F_n$ and consider the sum

$$(8.2) \quad S_n = n^{-2} \sum_{i \leq j \neq k \leq n} g(P_j, P_k) \stackrel{\text{def}}{=} n^{-2} \sum_{1 \leq j \neq k \leq n} g(r_{jk}).$$

This can be split into partial sums according to

$$\delta_{h+1} < r_{jk} \le \delta_h, \quad h = 2, 3, \ldots$$

(permitting also empty sums if it happens that $\delta_{h+1} = \delta_h$). Owing to the monotonicity of $g(x)$ this gives

$$\text{(8.3)} \quad S_n \ge 2n^{-2} \sum_{h=2}^{\infty} g(\delta_h) W_h,$$

where W_h denotes the number of pairs (j, k), $j \ne k$, with

$$\delta_{h+1} < r_{jk} \le \delta_h.$$

Using the notation

$$\text{(8.4)} \quad Z_h \stackrel{\text{def}}{=} \sum_{m=h}^{\infty} W_m,$$

it follows from Theorem 1 that for $n \ge h - 1$,

$$\text{(8.5)} \quad Z_h \ge \tfrac{1}{2} n^2/(h-1) - \tfrac{1}{2} n.$$

Further, clearly, for certain integer L

$$Z_h = 0 \quad \text{for } h \ge L.$$

Using partial summation we get from (8.3)

$$\begin{aligned} S_n &\ge 2n^{-2}\{g(\delta_2)(Z_2 - Z_3) + g(\delta_3)(Z_3 - Z_4) + \ldots + g(\delta_{L-1})(Z_{L-1} - Z_L)\} \\ &= 2n^{-2}\left\{ g(\delta_2) Z_2 + \sum_{h=3}^{L-1} (g(\delta_h) - g(\delta_{h-1})) Z_h \right\} \\ &= 2n^{-2}\left\{ g(\delta_2) Z_2 + \sum_{h=3}^{\infty} (g(\delta_h) - g(\delta_{h-1})) Z_h \right\}. \end{aligned}$$

All terms of the last sum are nonnegative; hence retaining only the terms with $h \le n + 1$ and applying (8.5) we get

$$\begin{aligned} S_n &\ge 2n^{-2}\left\{ g(\delta_2)\left(\tfrac{1}{2} n^2 - \tfrac{1}{2} n\right) + \sum_{h=3}^{n+1} (g(\delta_h) - g(\delta_{h-1}))\left(\tfrac{1}{2} n^2/(h-1) - \tfrac{1}{2} n\right) \right\} \\ &= 2n^{-2}\left\{ -\tfrac{1}{2} n g(\delta_{n+1}) + \tfrac{1}{2} n^2 (g(\delta_2) + \sum_{h=3}^{n+1} (g(\delta_h) - g(\delta_{h-1}))/(h-1)) \right\} \\ &= \sum_{h=2}^{n} g(\delta_h)/(h-1)h. \end{aligned}$$

Now Theorem 5 follows from (8.6) by usual passage to limit.

§9

Next we turn to the proof of Theorem 6. Let $z_1, z_2, \ldots, z_n$ be in B and

$$E(z_1, \ldots z_n) = \binom{n}{2}^{-1} \sum_{i \le j < k \le n} \log |z_j - z_k|^{-1} \stackrel{\text{def}}{=} \binom{n}{2}^{-1} \sum_{1 \le j < k \le n} \log(1/r_{jk}).$$

The minimum on B of $E(z_1, z_2, \ldots, z_n)$ for fixed n exists and is attained for a system of points $\{z_1^*, z_2^*, \ldots, z_n^*\}$ on B.

Denoting by r^*_{jk} the corresponding distances, we have

$$(9.1)\quad E(z_1^*, \dots, z_n^*) = \tfrac{1}{2} n^2 \binom{n}{2}^{-1} \left\{ n^{-2} \sum_{1\le j\ne k\le n} \log(1/r^*_{jk}) \right\},$$

which, applying the reasoning of §8, yields

$$(9.2)\quad E(z_1^*, \dots, z_n^*) \ge (n-1)^{-1} \sum_{h=2}^{n} (1/(h-1)h) \log(1/\delta_k^*).$$

As Fekete proved [9], the left side of (9.2) tends to $\log \Delta^{-1}$ as $n \to \infty$, Δ being the transfinite diameter of B. Owing to the known relation $\Delta = r(B)$, the proof of Theorem 4 is now completed.

§10

Theorem 3 is a remarkable special case of Theorem 4; so we turn now to prove the assertion preceding (4.2) concerning its best possibility. Let $0 < \epsilon < \frac{1}{100}$ be fixed and let

$$(10.1)\quad 2 \le n_1 < n_2 < \dots$$

a sequence of integers (to be determined later). Then every number in (0, 1) can be represented in the form

$$(10.2)\quad x = \sum_{v=1}^{\infty} c_v / n_1 n_2 \dots n_v, \quad 0 \le c_v \le n_v - 1.$$

Let our set B consist of x's having as c_v "digit" the values 0 or $n_v - 1$. We want to estimate δ_{2^k}, the 2^kth packing constant of this set. Having any 2^k points of our set, at least 2^{k-1} of these have common first digit, at least 2^{k-2} of these have identical first two digits, ..., at least 2 have first $k-1$ identical digits. Hence

$$(10.3)\quad \delta_{2^k} \le (n_k - 1)/n_1 n_2 \dots n_k \le 1/n_1 n_2 \dots n_{k-1}.$$

Owing to the monotonicity of the packing constants we have for $2^k \le v < 2^{k+1}$

$$\delta_v < 1/n_1 n_2 \dots n_{k-1}.$$

Hence

$$\begin{aligned} U_k &\stackrel{\text{def}}{=} \sum_{v=2^k}^{2^{k+1}-1} v^{-2+\epsilon} \log \delta_v^{-1} > \log(n_1 n_2 \dots n_{k-1}) \sum_{v=2^k}^{2^{k+1}-1} v^{-2+\epsilon} \\ &> 2^{-(k+1)(1-\epsilon)} 2^k \log(n_1 n_2 \dots n_{k-1}) \\ &> \tfrac{1}{4} \cdot 2^{-k(1-\epsilon)} \sum_{v=1}^{k-1} \log n_v. \end{aligned}$$

Choosing now

$$(10.4)\quad n_v = [2^{2^{v(1-\frac{1}{2}\epsilon)}}], \quad v = 1, 2, \dots,$$

we get

$$U_k > \tfrac{1}{20} \cdot 2^{-k(1-\epsilon)} \sum_{v=1}^{k-1} 2^{1-\frac{1}{2}\epsilon} > c \cdot 2^{\frac{1}{2}\epsilon k}$$

with a numerical constant c. Thus the series (4.2) indeed diverges.

If we can prove that the transfinite diameter of our set is positive then the proof of the assertion (4.2) is finished. This will be done by exhibiting for each integer l, 2^l elements x_v of our set so that

$$\text{(10.5)}\quad \left\{ \prod_{1\le j<k\le 2} |x_j - x_k| \right\}^{1/\binom{2^l}{2}} > c > 0$$

independently of l. For this purpose we choose for each $l \ge 3$ the points x_v as

$$\text{(10.6)}\quad \sum_{j=1}^{l} \epsilon_j / n_1 n_2 \dots n_j, \quad \epsilon_j = 0 \text{ or } n_j - 1.$$

We split the product in (10.5) into

$$\text{(10.7)}\quad \Pi_1\, \Pi_2 \dots \Pi_{l-1},$$

where Π_v is extended to all factors $|x_j - x_k|$ such that the first different digit in the expansion (10.6) of x_j and x_k occurs on the vth place. Each such factor is therefore (putting $n_0 = 1$)

$$\begin{aligned}\text{(10.8)}\quad &> (n_v - 1)/n_1 n_2 \dots n_v - \sum_{j=v+1}^{\infty} (n_v - 1)/n_1 n_2 \dots n_j \\ &= 1/n_1 n_2 \dots n_{v-1} - 2/n_1 n_2 \dots n_v \\ &= (1/n_1 n_2 \dots n_{v-1})(1 - 2/n_v).\end{aligned}$$

In order to calculate the number of factors in Π_v, we observe that the first $v-1$ identical digits in x_j and x_k can be chosen on 2^{v-1} ways, the last $l-v$ digits of x_j resp. x_k can be chosen independently on 2^{l-v} ways. This gives rise to $(2^{l-v})^2 2^{v-1}$ factors in Π_v. Hence

$$\Pi_v > \{(1/n_1 n_2 \dots n_{v-1})(1 - 2/n_v)\}^{2^{v-1} 2^{2l-2v}},$$

and thus the product in (10.5) is at least

$$\begin{aligned}&\prod_{1\le v\le l-1} \{(1/n_1 n_2 \dots n_{v-1})(1 - 2/n_v)\}^{2^{2l-v}/2^l(2^l-1)} \\ &> \prod_{1\le v\le l-1} (1/n_1 n_2 \dots n_{v-1})(1 - 2/n_v)^{2^{-v+1}} \\ &> c_1 \exp\left\{ -2 \sum_{v=2}^{l-1} 2^{-v} (\log n_1 + \log n_2 + \dots + \log n_{v-1}) \right\}\end{aligned}$$

with a positive numerical c_1. Since, owing to the choice (10.4), indeed

$$\sum_{v=2}^{k-1} 2^{-v} (\log n_1 + \dots + \log n_{v-1}) < 2 \sum_{v=2}^{\infty} 2^{-v} \log n_v < c.$$

§11

Finally we turn to the proof of Theorem 4. We shall denote by $|s|$ the number of elements of a set s. We shall need the following

Lemma 1. *If $L_1, L_2, \dots, L_m$ are subsets of the finite set L and for $v = 1, 2, \dots, m$,*

$$\text{(11.1)}\quad |L_v| > ((m-1)/m)|L|,$$

then the intersection of all L_v's is not empty.

Proof. For the proof let

$$l_1^*, l_2^*, \ldots, l_n^*, \quad n = |L|,$$

be all elements of ω and let α_ν be the number of L_ν's which contain l_ν. Then we have, using (11.1),

$$n(m-1) < \sum_{j=1}^{m} |L_j| = \sum_{\nu=1}^{n} \alpha_\nu,$$

which implies that $\max_\nu \alpha_\nu \geq m$. This is equivalent to our assertion.

We shall need an easy corollary of the graph theorem (6.1), (6.2), which we shall formulate as

Lemma 2. *If in a graph G_N with N vertices every vertex has degree at least $((\gamma-2)/(\gamma-1))N$, then G_N contains a complete subgraph of order γ.*

Proof. The proof is easy. The degree condition implies that the number of edges in the graph is greater than the corresponding quantity in (6.2). This proves the lemma.

We shall also need

Lemma 3. *Let Γ_N be a graph with N vertices which does not contain any complete subgraph on K vertices, $3 \leq K \leq N$. Let*

$$\text{(11.2)} \quad 0 \leq \lambda \leq 1/(K-1).$$

Then the number of vertices of degree not exceeding $(1-\lambda)N$ in Γ_N is greater than

$$\text{(11.3)} \quad \{1 - \lambda(K-2)\}N.$$

Proof. For the proof we decompose the vertices of Γ_N into the disjoint classes Ω_1 and Ω_2, the first one containing all vertices of degree

$$\text{(11.4)} \quad > (1-\lambda)N$$

and Ω_2 the others. The essential observation is that Ω_1 cannot contain a complete subgraph of order $K-1$. Suppose namely that

$$\text{(11.5)} \quad Q_1, Q_2, \ldots, Q_{K-1}$$

were the vertices of such a subgraph. Denoting for $j = 1, 2, \ldots, K-1$ the set of vertices in Γ_N which are connected by an edge to Q_j by R_j, we have by (11.4),

$$\text{(11.6)} \quad |R_j| > (1-\lambda)N, \quad j = 1, 2, \ldots, K-1.$$

This implies, owing to (11.2), that

$$|R_j| > \frac{K-2}{K-1} n.$$

Hence Lemma 1 is applicable to the R_j's with $m = K-1$. Consequently, Γ_N would contain a vertex Q^* which is contained in all R_j's, i.e. is connected to all Q_j's. But then $(Q_1, Q_2, \ldots, Q_{K-1}, Q^*)$ would be a *complete* subgraph of order K in Γ_N, in contradiction to our assumptions. Thus Ω_1, contains no complete subgraphs of order $K-1$. But then the application of Lemma 2 to $G_N = \Omega_1$ with $\gamma = K-1$ implies that the degree of at least one vertex in Ω_1 with respect to Ω_1 is

$$< \frac{K-3}{K-2} |\Omega_1|$$

and hence its degree with respect to Γ_N is less than

$$N - |\Omega_1| + \frac{K-3}{K-2}|\Omega_1| = N - \frac{1}{K-2}|\Omega_1|.$$

This yields in connection with (11.4) that

$$(1-\lambda)N < N - \frac{1}{K-2}|\Omega_1|,$$

i.e.

$$|\Omega_1| < \lambda(K-2)N.$$

This implies indeed that more than $(1-\lambda(K-2))N$ vertices in Γ_N have degree $\leq (1-\lambda)N$.

We shall use Lemma 3 in the following form:

If Γ_N contains no complete subgraphs of order K and λ satisfies (11.2), then the complementary graph $\overline{\Gamma_N}$ contains more than

$$(11.7) \quad (1-\lambda(K-2))N$$

vertices of degree (with respect to $\overline{\Gamma_N}$)

$$(11.8) \quad > \lambda N.$$

§12

Now we can turn to the proof of Theorem 7. Let $P_1, P_2, \ldots, P_n$ be in an f from the family F with packing constants δ_v and critical indices $i_1, i_2, \ldots$. For a fixed $v \geq 2$, corresponding to $P_1, \ldots, P_n$, we define a graph G with vertices P'_j, $j = 1, 2, \ldots, n$, as follows:

The edge $P'_j P'_k$ occurs in G if and only if

$$(12.1) \quad \overline{P_j P_k} > \delta_{i_{v+1}} (= \delta_{i_v+1}).$$

We easily see, as before, that G does not contain a complete subgraph of order $i_{v+1} = i_v + 1$. Thus, applying Lemma 3 in the form (11.7), (11.8) with $\Gamma_N = G$, $N = n$, $K = (i_v + 1)$, we get that for

$$(12.2) \quad 0 \leq \lambda \leq 1/i_v$$

more than

$$(12.3) \quad (1-\lambda(i_v - 1))n$$

points P_j have the property that the inequality

$$\overline{P_j P_k} \leq \delta_{i_v+1}$$

holds for more than λn points P_k, $k \neq j$. Thus the positivity and monotonicity of $g(x)$ implies that

$$(12.4) \quad n^{-1} \sum_{\substack{k=1 \\ k \neq j}}^{n} g(\overline{P_k P_j}) \geq \lambda g(\delta_{i_v+1}).$$

By usual passage to limit we obtain that the inequality

$$(12.5) \quad G(P) \geq \lambda |D| g(\delta_{i_{v+1}})$$

holds in the set *f* of *F* with the possible exception of a set of measure

$$(12.6) \quad \leq \lambda(i_v - 1)|D|.$$

Replacing λ by $(1/i_v)\gamma$, $0 \leq \gamma \leq 1$, this yields that the inequality

$$(12.7) \quad G(P) \geq (\gamma|D|/i_v)g(\delta_{i_{v+1}}) \geq \gamma|D|(g(\delta_{i_{v+1}})/(i_{v+1}-1)i_{v+1})i_{v+1}$$

holds in each set *f* of the family *F* with exception of a set of measure

$$\leq \gamma(1 - 1/i_v)|D| = \gamma(1 - 1/(i_{v+1} - 1))|D|.$$

Defining r by (5.7), the inequality (12.7) implies on choosing v so that $i_{v+1} = r$ that

$$G(P) \geq \gamma|D| \sum_{v=2}^{r} g(\delta_v)/(v-1)v$$

at most holds in D with the exception of a set of measure

$$\gamma|D|(1 - 1/(r-1)).$$

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