Knot invariants: low dimensional topology and combinatorics

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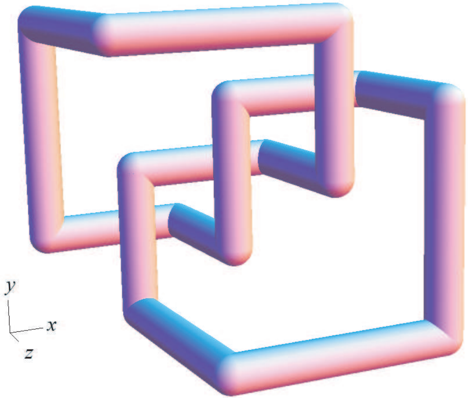
Definitions

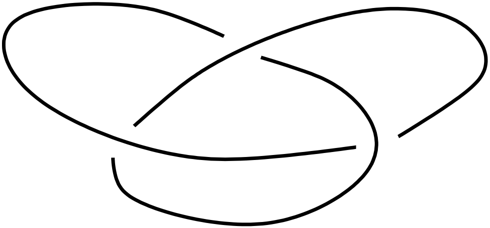
Differentiable embeddings of S^1 into $\mathbf{R}^3$ are called *knots*. The two knots K_1 and K_2 are regarded to be the same, if one can be moved into the other, i.e., there is a (differentiable) family of knots parametrized by $[1,2]$ which has K_1 for $t=1$ while K_2 for $t=2$. (In this case K_1 and K_2 are called *isotopic*.)

Az example: the trefoil knot



Projections

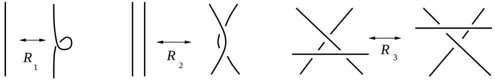
We use *planar projections* of knots to study them. The strand passing under at a crossing is interrupted, and with this convention the projection determines the knot up to isotopy. (It is not hard to see that any knot admits a projection with finitely many double points.)



Reidemeister's theorem

Theorem (Reidemeister's theorem)

Two projections correspond to isotopic knots if and only if the projections can be connected by a finite sequence of modifications R_1, R_2 *and* R_3*.*



An example: 3–colorability

Knot invariants can be defined from projections by showing that the invariant does not change under the Reidemeister moves R_i. An example: 3–colorability.

Definition

A projection is called 3–colorable if we can color the arcs in the projection with three colors R, B and G in such a way that every color appears, and at a crossing either all three or exactly one color is present.

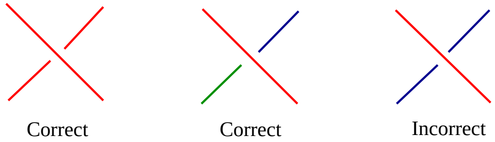


Figure: 3–colorability

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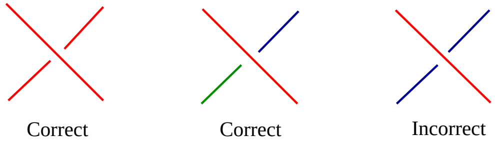
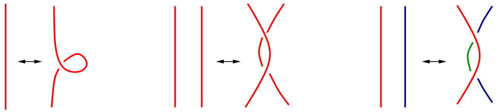


Figure: 3–colorability

The 3–colorable property

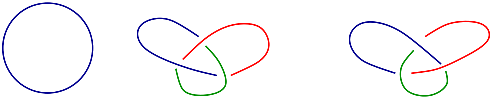
Theorem

3–colorability is the property of the knot and is independent of the chosen projection.



Corollary

The trefoil knot is different from the unknot. (But this property does not distinuish, for example, the trefoil from its mirror image.)



The genus of a knot

Fact (Seifert's Theorem)

Every knot in $\mathbf{R}^3$ *is the boundary of an (orientable) surface.*

A surface, in turn, can be characterized by the number of 'holes' it has. This number is called the *genus* of the surface.

The genus

Definition

The genus of K is the minimum of the genera of surfaces having K as boundary,

$$g(K) = \min\{g(F) \mid F \subset \mathbf{R}^3, \quad \partial F = K\}.$$

The genus $g(K)$ can be viewed as a measure of complexity of K, e.g., the trivial knot has genus 0, while the trefoil is of genus 1.
There are knots with arbitrarily large genus.

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Fibered knots

Definition

A knot K is called fibered if its complement can be presented as a family of surfaces, more precisely there is a map

$$\phi\colon S^3 - K \to S^1$$

such that for each $t \in S^1$ the fiber $\phi^{-1}(t)$ is homeomorphic to a fixed (non–compact) 2–dimensional surface. (Informally, these knots are built from 'lower dimensional pieces'.)

The Alexander polynomial

Suppose that V is a given projection of the knot K. Let $Cr(V)$ denote the set of crossings in the projection and $D_0(V)$ the set of domains (i.e. the connected components of the complement of the projection). Let us choose a distinguished arc (marked with an X) and let $D(V)$ denote those domains which are disjoint from the marked arc.

Definition

A bijection

$$\sigma : Cr(V) \to D(V)$$

is called a Kauffman state if for all $c_i \in Cr(V)$ we have $c_i \in \sigma(c_i)$.

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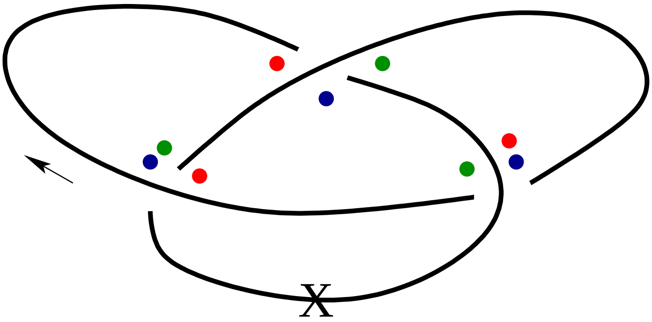
An example

The usual projection of the trefoil knot admits three Kauffman states, indicated by the red, green and blue dots on the diagram.



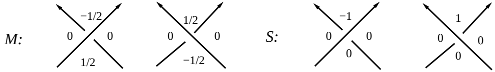
Let us fix an oriented, marked projection of a knot K and consider a Kauffman state σ. Let

$$M(\sigma) = \sum_{c_i \in Cr(V)} M(\sigma(c_i))$$

and

$$S(\sigma) = \sum_{c_i \in Cr(V)} S(\sigma(c_i)),$$

where



Define

$$\Delta_K(t) = \sum_{\sigma \in Kauff(V)} (-1)^{S(\sigma)} t^{M(\sigma)} \in \mathbf{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}].$$

Theorem

The Alexander polynomial $\Delta_K(t)$ is an invariant of the knot.

(Idea of the proof: show that the quantity does not change under the Reidemeister moves R_1, R_2, R_3. To achieve this, find convenient correspondences between Kauffman states before and after the moves.)

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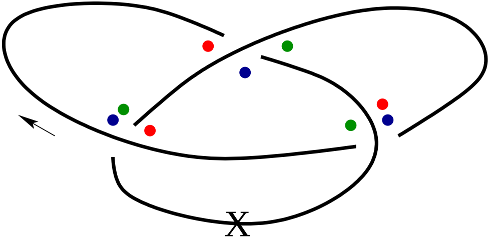
Example: the trefoil knot

$S(R) = 0, M(R) = -1; S(G) = 2, M(G) = 1; S(B) = 1, M(B) = 0$

The Alexander polynomial is therefore equal to $t - 1 + t^{-1}$.

For the mirror knot we have $(S, M) = (0, 1), (-2, -1), (-1, 0)$, hence the Alexander polynomial will be the same. (This is true for any knot and its mirror.)



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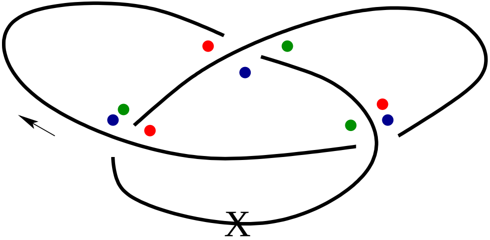

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Theorem

The Alexander polynomial $\Delta_K(t)$ of a knot K satisfies

- $\Delta_K(t) = \Delta_K(t^{-1})$, *and so* $\Delta_K(t) = a_0 + \sum_{i=1}^{n} a_i(t^i + t^{-i})$ $(a_n \neq 0)$
- $n \leq g(K)$
- *for a fibered knot K we have* $n = g(K)$ *and* $a_n = \pm 1$.

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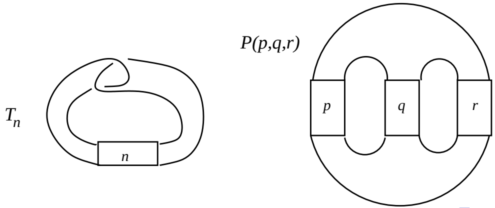
An example for a non–fibered knot

For n odd $\Delta_{T_n} = \frac{n+1}{2}t - n + \frac{n+1}{2}t^{-1}$, hence T_n is not fibered once $n > 1$. (Here n indicates n full left twists; for $n = 1$ the knot T_1 is the trefoil knot, which is fibered.)

$\Delta_{P(p,q,r)} = Dt - 2D + 1 + Dt^{-1}$, where
$D = ab + ac + bc + a + b + c + 1$ and
$p = 2a + 1, q = 2b + 1, r = 2c + 1$.
E.g. for the pretzel knot $K = P(-3, 5, 7)$ we have $\Delta_K(t) = 1$, although the knot is not trivial.



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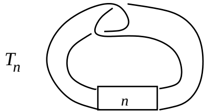

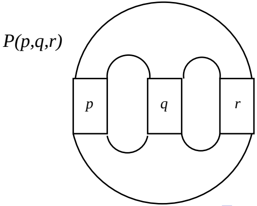

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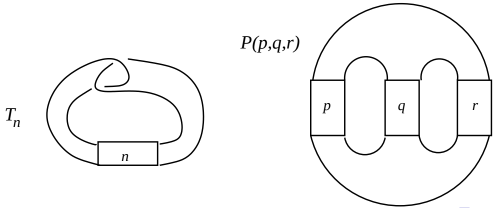

Chain complexes

Suppose that C_i $(i = 0, \dots, n+1)$ are given vector spaces and $\partial_i \colon C_i \to C_{i-1}$ linear maps. (Suppose that $\dim C_0 = \dim C_{n+1} = 0$.)

Definition

The pair (C_i, ∂_i) *is a chain complex if* $\partial_i \circ \partial_{i+1} = 0$ *(that is,* $\mathrm{im}\ \partial_{i+1} \leq \ker \partial_i$*) for all* $i = 1, \dots, n$. *The factor* $H_i = \ker \partial_i / \mathrm{im} \partial_{i+1}$ *is called the* i^{th} *homology group of the chain complex.*

A simple identity

Fact

$$\sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i$$

(For the chain complex $0 \to C_1 \xrightarrow{f} C_2 \to 0$ this statement reduces to $\dim H_2 - \dim H_1 = \dim \operatorname{coker} f - \dim \ker f = \dim C_2 - \dim \operatorname{im} f - \dim \ker f = \dim C_2 - \dim C_1$, called the homomorphism theorem in basic linear algebra.)

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The generators

Let $C_d(K,s)$ denote the $\mathbf{Z}_2$ vector space generated by the Kauffman states σ with $M(\sigma)=d$ and $S(\sigma)=s$.
Next we would like to define a chain complex $(C_d(K,s),\partial_d(K,s))_d$ for every fixed s.

Fact

Since $\sum(-1)^d \dim C_d(K,s)=a_s$ (the s^{th} coefficient of the Alexander polynomial), therefore for any boundary map we will have

$$\sum_d (-1)^d \dim H_d(K,s)=a_s.$$

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To this end, we will need an alternative description of the Kauffman states.

Suppose that Σ is the boundary of the tubular neighbourhood of the projection in the 3–space. (For the trefoil knot, for example, this is a surface of genus 4.) Let the conturs of the 'inner' domains be called the α–curves. If the projection has k double points, then there are $k+1$ such curves. At a crossing the diagram shows our choice of a β–curve. In addition, at the distinguished edge we add one more β–curve, again as instructed by the diagram.

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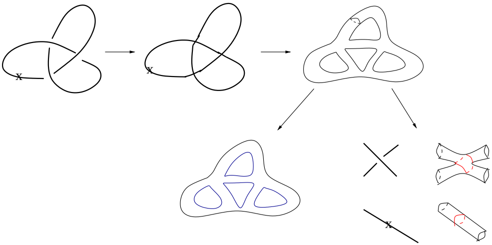
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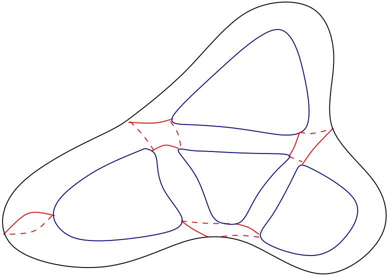


Obviously $\alpha_1 \times \ldots \times \alpha_n \subset \Sigma \times \ldots \times \Sigma = \Sigma^n$, and similarly for the β–curves. The order of the curves is based on a choice — choose therefore the tori in the 'symmetric power' $Sym^n(\Sigma)$, where we consider unordered n–tuples of points of Σ.
Result: $\mathbf{T}_\alpha, \mathbf{T}_\beta \subset Sym^n(\Sigma)$ n–dimensional tori (n^{th} powers of the circle) in the $2n$–dimensional space $Sym^n(\Sigma)$.

Fact

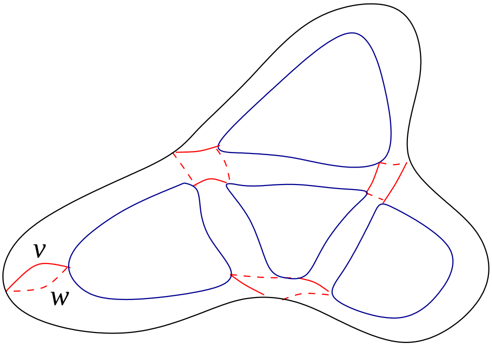
The Kauffman states of the projection are in 1-1 correspondence with the intersection points $\mathbf{T}_\alpha \cap \mathbf{T}_\beta$*.*

Idea: an element of $\mathbf{T}_\alpha \cap \mathbf{T}_\beta$ is an n–tuple $\{x_1, \ldots, x_n\}$ for which $x_i \in \alpha_i \cap \beta_{\pi(i)}$ for some permutation π. We do not have a choice on the β–circle near the distinguished edge, and on all other β–circles the choice of the α–intersection specifies a quadrant, and therefore a Kauffman state.

Let us fix two points w and v on the two sides of the β–curve near the distiguished edge.

Fact

If we pass from w to v in the complement of the α–curves (simply cross the β near the distiguished edge), while from v to w in the complement of the β's, we recover the knot K we started our procedure with.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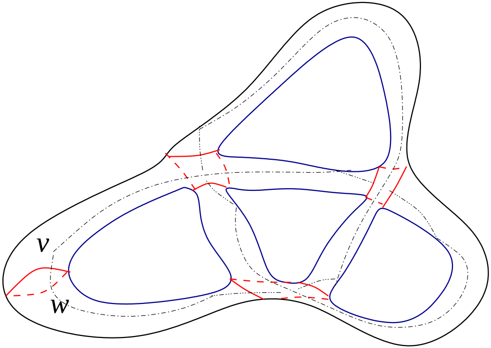

v
w

The differential

Recall that we are looking for a linear map

$$\partial_d = \partial_d(K,s)\colon C_d(K,s) \to C_{d-1}(K,s)$$

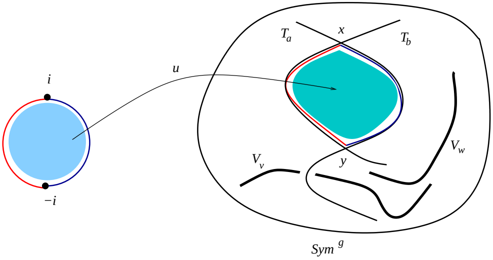
where $C_d(K,s)$ is generated by the corresponding Kauffman states. Fix generators $x \in C_d(K,s)$ and $y \in C_{d-1}(K,s)$; the matrix element

$$n_{xy} = \langle \partial_d x, y \rangle$$

will be defined as follows. Consider x, y as elements of the intersection $\mathbf{T}_\alpha \cap \mathbf{T}_\beta$. Let $S = \{z \in \mathbf{C} \mid z\overline{z} \leq 1\}$ be the unit circle.

Let n_{xy} denote the (mod 2) number of holomorphic (complex differentiable) maps $u\colon S \to Sym^n(\Sigma)$ for which

- $u(\{z \mid z\overline{z} = 1, \mathrm{Re}(z) < 0\}) \subset \mathbf{T}_\beta$
- $u(\{z \mid z\overline{z} = 1, \mathrm{Re}(z) > 0\}) \subset \mathbf{T}_\alpha$
- $u(i) = x$
- $u(-i) = y$
- If $V_v = \{v\} \times Sym^{n-1}(\Sigma)$, then $u(S) \cap V_v = \emptyset$
- If $V_w = \{w\} \times Sym^{n-1}(\Sigma)$, then $u(S) \cap V_w = \emptyset$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Some important theorems

The resulting homology theory: the knot Floer homology of K.

Theorem (Ozsváth–Szabó 2003-2005, Yi Ni, Juhász)

- *The homology $H_d(K,s)$ of the resulting chain complex is an invariant of the knot (so is independent of the chosen projection).*
- $\sum_d (-1)^d \dim H_d(K,s) = a_s$, *and $H_d(K,s)$ is isomorphic to $H_{d-2s}(K,-s)$.*
- $$g(K) = \max\{s \mid H_d(K,s) \neq 0 \quad \text{for} \quad \text{some} \quad d\}$$
- *The knot K is fibered if and only if $\sum_d H_d(K, g(K)) = \mathbf{Z}_2$, that is, for some d the group is isomorphic to $\mathbf{Z}_2$, and vanishes for all other d's (with $s = g(K)$).*

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- $\sum_d (-1)^d \dim H_d(K,s) = a_s$, *and $H_d(K,s)$ is isomorphic to $H_{d-2s}(K,-s)$.*
- $$g(K) = \max\{s \mid H_d(K,s) \neq 0 \quad \text{for} \quad \text{some} \quad d\}$$
- *The knot K is fibered if and only if $\sum_d H_d(K, g(K)) = \mathbf{Z}_2$, that is, for some d the group is isomorphic to $\mathbf{Z}_2$, and vanishes for all other d's (with $s = g(K)$).*

Some important theorems

The resulting homology theory: the knot Floer homology of K.

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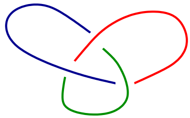
An example

For the two trefoils, with the standard three–crossing projections, simply from the gradings of the Kauffman states the differential vanishes (for any fixed s there is a single d with nontrivial $C_d(K,s)$). Therefore:

$$H_0(B,-1) = H_1(B,0) = H_2(B,1) = \mathbf{Z}_2$$

$$H_{-2}(J,-1) = H_{-1}(J,0) = H_0(J,1) = \mathbf{Z}_2$$



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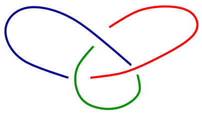

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Results

The proofs of the theorems above require deep complex (and almost–complex) geometry, resting on results of Gromov (1985) and Floer (1988) (who introduced *Floer homology*, aka Lagrangian intersection homology).

Theorem (Manolescu-Ozsváth-Sarkar, and independently Ozsváth-Stipsicz-Szabó and Ozsváth-Szabó)

The homology groups $H_d(K, s)$ can be computed by purely combinatorial means.

Ozsváth-Stipsicz-Szabó and Ozsváth-Szabó: extend the theory to *singular knots* (knots in $\mathbf{R}^3$ with transverse self–intersections).
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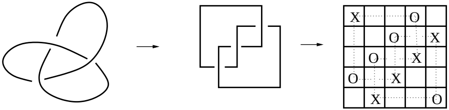
Grid diagrams

Observation: every knot projection can be slightly modified to have only horizontal and vertical segments and such that the vertical is always over the horizontal. Such a diagram, in turn can be depicted by an $n \times n$ grid.



The combinatorial complex

Identify bottom and top, and right and left edges, and recover the knot on the resulting torus.

The idea is then a simple adaptation of the previous definitions: regard the horizontal lines as α– while the vertical ones as β–curves. With this convention, elements of $\mathbf{T}_\alpha \cap \mathbf{T}_\beta$ correspond to permutations of $\{1, \ldots, n\}$ by assigning to i the index of the β–curve containing $x_i \in \alpha_i$.

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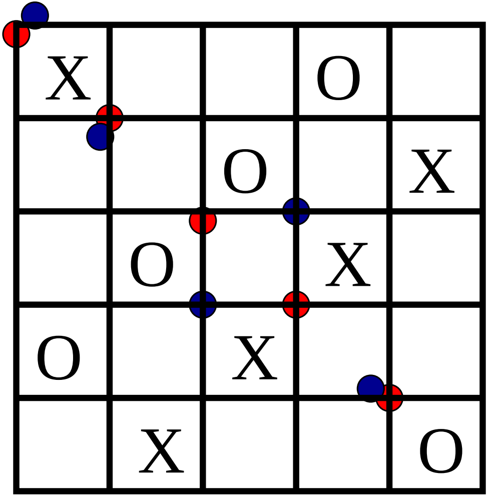
Define $n_{xy} = 0$ if the two permutations *do not* differ by a transposition.
If the difference of x and y is a transposition, then they determine four rectangles in the grid. Two of them are from x to y and the other two are from y to x (depending on the induced orientation of the α–curves). Now define n_{xy} as the mod 2 number of *empty* rectangles from x to y. (The degrees d and s admit a similar, slightly more complicated definition.)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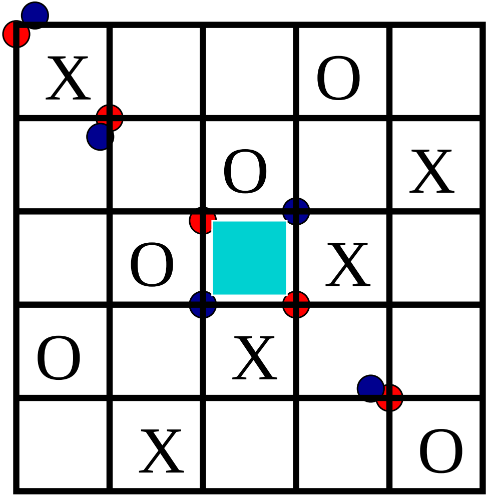
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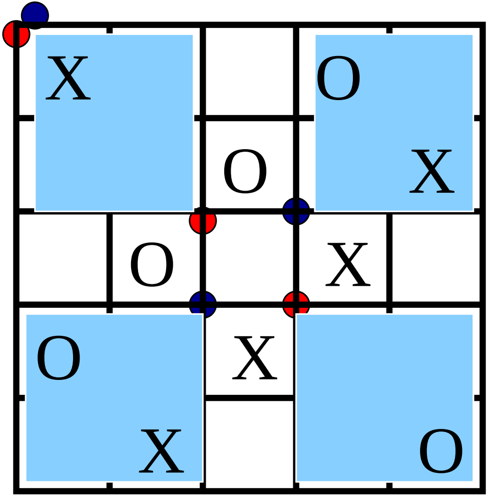
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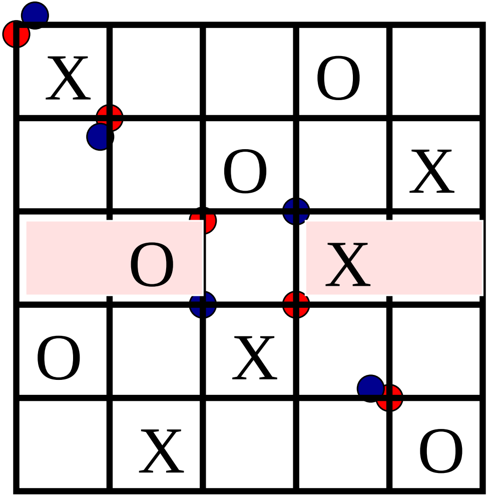
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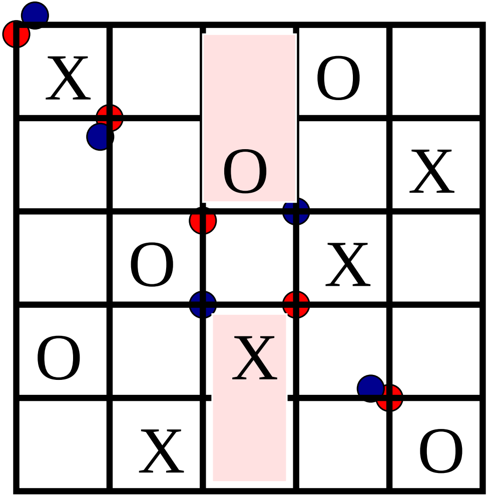
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Theorem (Manolescu-Ozsváth-Sarkar)

The homology of the resulting chain complex is isomorphic to the vector space $H_d(K, s) \otimes (\mathbf{Z}_2 \oplus \mathbf{Z}_2)^{\otimes(n-1)}$, consequently determines the knot Floer homology of the knot K.

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The invariance of this homology theory (i.e. the independece of the chosen grid presentation) can be shown by purely combinatorial means.

The proof rests on a result of Cromwell, determining moves on grid diagrams which connect any two grid presentations of the same knot (adapting the Reidemeister moves to the current situation).

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Further results

Further possible generalizations/modifications/variatons:

- Instead of $\mathbf{Z}_2$–coefficients we use $\mathbf{Z}$–coefficients. (This requires orientation conventions, which has been worked out by Manolescu-Ozsváth-Szabó-Thurston.)
- Instead of considering knots in $\mathbf{R}^3$, take embedded images of S^1 into arbitrary 3–manifolds. The holomorphic theory has been worked out by Ozsváth–Szabó and Rasmussen (independently). When considering the *trivial knot* (the bounding circle of an embedded disk) we get invariants of 3–manifolds, the *Ozsváth–Szabó homology groups*.
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An additional result

Theorem (Ozsváth–Stipsicz–Szabó, 2008)

The theories over the truncated polynomial rings $\mathbf{Z}_2[U]/(U^2=0)$ *and* $\mathbf{Z}_2[U]/(U^3=0)$ *can be computed combinatorially.*