Interior, closure, boundary

Definition 0.0.1: Given (X, τ_X) and $A \subset X$,
the interior of A is

$$int(A) = \bigcup \{U \,|\, U \in \tau_X,\ U \subset A\}$$

the closure of A is

$$cl(A) = \bigcap \{V \,|\, X \setminus V \in \tau_X,\ A \subset V\}$$

and the boundary of A is

$$\partial A = cl(A) \setminus int(A)$$

From the definitions it is immediate that

a.) $int(A)$ is an open set

b.) $int(A) \subset A$

c.) A is open if and only if $int(A) = A$

d.) $int(A)$ is the "largest open subset of A" in the sense that if $M \in \tau_X$ and $int(A) \subset M \subset A$, then $int(A) = M$

e.) For $x \in X$ we have $x \in int(A)$ if and only if $\exists u \in \tau_X$ such that $x \in U \subset A$.

Similarly,

a.) $cl(A)$ is a closed set

b.) $cl(A) \supset A$

c.) A is closed if and only if $A = cl(A)$

d.) cl(A) is the "smallest closed set containing A" meaning that if Z is a closed set in X s.t. $A \subset Z \subset cl(A)$ then $Z = cl(A)$

In addition,

e.) $x \in cl(A)$ if and only if $\forall U \in \tau_X$, $x \in U$, we have $U \cap A \neq \emptyset$.

This is not immediate. We prove e.), by proving its contrapositive, that is, $x \notin$ cl(A) if and only if $\exists U \in \tau_X$ such that $U \cap A = \emptyset$ and $x \in U$.
By the definition of closure, if $x \notin cl(A)$, then there exists some closed V such that $A \subset V$ and $x \notin V$. Let $U = X \setminus V \in \tau_X$. Then $x \in U$ but $A \cap U = \emptyset$.
On the other hand, if there is some open set $U \in \tau_X$ such that $U \cap A = \emptyset$ and $x \in U$, then take the closed set $V = X \setminus U$. Then $A \subset V$, but $x \notin V$, so $x \notin cl(A)$.

Finally,

a.) since $\partial A = cl(A) \setminus int(A)$, we have $\partial A = cl(A) \cap (X \setminus int(A))$, so ∂A is a closed set.

b.) Also, for $x \in X$ we have $x \in \partial A$ if and only if $\forall U \in \tau_X$, $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$.

Exercise 0.0.2: What are $int(A)$, $cl(A)$, ∂A

a. if $X = \mathbb{R}$ with the usual topology and $A = [0,1)$, $A = \mathbb{Q}$, $A = [0,1) \cap \mathbb{Q}$?

b. if $X = \mathbb{R}^2$ with the usual topology and $A = [0,1) \times \{\frac{1}{2}\}$?

Exercise 0.0.3: Let $X = \mathbb{R}$ and $A = [0,1)$. What are $int(A)$, $cl(A)$, ∂A if τ_X is the discrete, anti-discrete, co-finite topology?

Exercise 0.0.4: Are the following statements true or false?

1. $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$ and
   $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

2. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ and
   $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$

3. B is closed $\iff$ $\partial B \subset B$

4. $cl(A) = X \setminus \text{int}(X \setminus A)$

5. $\partial(\partial(A) = \partial(A)$

6. $\text{int}B = B \setminus \partial B$

7. $\partial(\text{cl}(A)) = \partial A$

8. If A is open then $\partial A \subset (X \setminus A)$

9. $\partial(intA)) = \partial A$

Exercise 0.0.5: What is wrong with the following argument?
We show that

$$cl(\cup A_\alpha) \subset \cup cl(A_\alpha):$$

(i.e. find the mistake in the following argument)
If $\{A_\alpha\}$ is a collection of sets in X and if $x \in cl(\cup A_\alpha)$, then every open set U that contains x intersects $\cup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in cl(A_\alpha)$.

Is the statement nevertheless true?

Exercise 0.0.6: What is wrong with the following argument?
In a metric space (X,d) a unit ball S^1 around a point p (i.e. $S^1 = \{q \in X \,|\, d(p,q) = 1\}$) is closed, since it is the boundary of the open unit ball, and boundaries are always closed sets, as for any set A, one can show that $\partial A = cl(A) \cap cl(X \setminus A)$ (i.e. any boundary is the intersection of two closed sets).

The Hausdorff property

Definition 0.0.7: A topological space (X, τ_X) has the **Hausdorff property** if $\forall x \neq y$ in X, there exist $U_x, U_y \in \tau_X$ containing x and y, respectively, such that $U_x \cap U_y = \emptyset$.

Example 0.0.8: a.) $\mathbb{R}^n$ with the usual topology is Hausdorff.
b.) $\mathbb{R}^n$ with the co-finite topology is not Hausdorff.
c.) Any set with the discrete topology is Hausdorff.
d.) Any set with the anti-discrete topology is not Hausdorff.

A non-Hausdorff space is "pathological" in many ways. For example, in a non-Hausdorff space, limits of sequences may not be unique. But what is convergence, in general?

Definition 0.0.9: Given any sequence of points $(x_n) \subset X$, we say that $lim_{n\to\infty} x_n = x \in X$ if, for all open sets U with $x \in U$, there exists some $N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n \in U$.

Example 0.0.10: Consider $X = \{a, b, c\}$ with $\tau_X = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. The constant sequence $\{a, a, a, a, a, a, a, ...\}$ converges to a. However, it also converges to b and c. Note that τ_X is not Hausdorff.

Exercise 0.0.11: Consider $X = \mathbb{R}$ and the sequence $x_n = \frac{1}{n}$. Where does x_n converge (if at all), if $\mathbb{R}$ has the discrete, anti-discrete, co-finite topology?

The Hausdorff property can be used to distinguish between topological spaces, as we have the following fact.

Proposition 0.0.12: The Hausdorff property is a topological invariant

Proof: Let $f : X \to Y$ be a homeomorphism. Suppose Y is Hausdorff. Then f is a continuous bijection. Then for $x \neq y$ in X, $f(x) \neq f(y)$ in Y. Since Y is Hausdorff, there exist $U \in \tau_Y$ and $V \in \tau_Y$, disjoint, such that $f(x) \in U$ and $f(y) \in V$. Hence, $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. These two sets are open in X, since f is continuous. Also, since f is a bijection, they do not intersect. So X is Hausdorff.

Example 0.0.13: $\mathbb{R}$ with the usual topology is not homeomorphic to $\mathbb{R}$ with the arrow topology, since the first one is Hausdorff, while the second is not. (In the second one, any two non-empty open sets intersect.)

Compactness

Definition 0.0.14: Given a topological space (X, τ), a subset $K \subset X$ is compact if *every* open cover of K has a finite subcover.

What do the terms "open cover" and "subcover" mean?

Definition 0.0.15: A **cover of** K is <u>a collection</u> $\{U_\lambda\}_\lambda \subset \mathcal{P}(X)$ such that $\bigcup_\lambda U_\lambda \supset K$. We have a **finite cover of** K if the number of sets in this collection is finite. An **open cover of** K is a cover such that every set within the cover is open in X. A collection of subsets of X, $\{W_\beta\}_\beta$, is **a subcover of a given cover** $\{U_\lambda\}_\lambda$, if it itself is a cover and such that $\forall \beta$, $\exists \lambda$ with $W_\beta = U_\lambda$.

Example 0.0.16: Consider $A = \mathbb{R}$ with the usual topology.

1. The collection $\{(-\infty, 1], (0, \infty)\}$ is a finite, but not open cover of A.
2. The collection $\{(-\infty, 1), (0, \infty)\}$ is a finite, open cover of A.
3. The collection $\{(-n, n)\}_{n=1}^{\infty}$ is an (infinite) open cover of A.

Exercise 0.0.17: The following sets are not compact. Show this by finding an open cover for each that has no finite subcover.

a.) $\mathbb{R}$, $(0, 1)$, $\{\frac{1}{n}\}, n = 1, 2...$ – in $\mathbb{R}$ with the usual topology;

b.) any infinite subset of X, itself an infinite set, with the discrete topology.

Example 0.0.18: The following sets are all compact:

a.) any subset of a set X with the anti-discrete topology

b.) any subset of X with a topology if X is finite

c.) any subset of a set X with the co-finite topology

We get a vast amount of examples of compact sets recalling the well-known Heine-Borel theorem from real analysis.

Theorem 0.0.19 (Heine-Borel)**:** In $\mathbb{R}^n$ a set K is compact if and only if it is closed and bounded.

(Recall that a set K is bounded if you can put it in a ball i.e. $\exists B_x(r)$ (open) ball around a point $x \in K$ such that $K \subset B_x(r)$. Equivalently, $\exists M > 0$ such that $\forall x, y \in K$ we have $d(x, y) < M$.)

Thus the following are also examples of compact sets:

Example 0.0.20: a.) in R with the usual topology: a point, finite many points, $[0, 1]$

b.) in R^2 with the usual topology: the closed unit disk $\overline{D^2}$, the square $[0, 1] \times [0, 1]$, the unit circle S^1

We get even more examples of compact sets using the following observation:

Proposition 0.0.21: If $f : X \to Y$ is continuous and onto with X compact, then Y is compact.

Proof. Indeed, let $\{U_\alpha\}_\alpha \subset \tau_Y$ be an open cover of Y. So $\bigcup_\alpha U_\alpha \supset Y$. Consider $\{f^{-1}(U_\alpha)\}_\alpha$. Since f is continuous, each set in this collection is in τ_X. Also, $\bigcup_\alpha f^{-1}(U_\alpha) = X$ so $\{f^{-1}(U_\alpha)\}_\alpha$ is an open cover of X. We know X is compact, so there is some finite subcover $\{f^{-1}(U_{\alpha_i})\}_{\alpha_i}$, $i = 1, ..., n$; that is, $\bigcup_{i=1}^n f^{-1}(U_{\alpha_i}) = X$.

Correspondingly we have for $\{U_{\alpha_i}\}_{\alpha_i}$ that $\bigcup_{i=1}^n U_{\alpha_i} = Y$, so $\{U_{\alpha_i}\}_{\alpha_i}$ is a finite subcover of the original open cover of Y, so Y is compact. ■

In other words, the image of a compact set with respect to a continuous function is also compact. Thus we have more examples of compact sets:

Example 0.0.22: Recall that the torus, Mobius strip, the Klein bottle were all defined as quotients $[0,1]\times[0,1]/\sim$ of the square $[0,1]\times[0,1]$, which is a compact set in $\mathbb{R}^2$ by the Heine-Borel theorem. Recall also, that the quotient topology is defined such that the quotient map

$$q:[0,1]\times[0,1]\to[0,1]\times[0,1]/\sim$$

$q(x)=[x]=$ the equivalence class of x, is automatically continuous and q is an onto map.

By the proposition above, the torus, Mobius strip and the Klein bottle are all compact.

Similarly, $\mathbb{R}P^2$ is compact as it is defined as a quotient of the closed unit disk which is compact.

In addition, as a direct consequence of the proposition we have

Proposition 0.0.23: Compactness is a topological invariant. That is: if $X\sim Y$ (X and Y are homeomorphic, then X is compact if and only if Y is compact.

We can now use compactness to distinguish between topological spaces.

Example 0.0.24: $[0,1]$ and [0,1) are not homeomorphic.

This is because $[0,1]$ is compact (by the Heine-Borel theorem, since it is closed and bounded), but $[0,1)$ is not compact (by the Heine-Borel theorem, since it is not closed).

Let us consider the Heine-Borel theorem again. Recall that it only holds in $\mathbb{R}^n$.
Already in metric spaces there are closed and bounded sets that are not compact. One example of such is an infinite set with the discrete topology (why?).

Here is another one:

Exercise 0.0.25: Let $K=[0,\pi]\cap\mathbb{Q}$ in $X=\mathbb{Q}$ with the subspace topology of $\mathbb{R}$. This is a closed set (why?) and bounded (why?), but it has an open cover that does not contain a finite subcover (provide one) and thus K is not compact.

What about the other direction? If K is compact in a general topological space, does it follow that K is closed? (Note that "boundedness" makes no sense in general as it assumes distance.)

The answer is "no".

Example 0.0.26: Let $X=\{a,b,c\}$ and $\tau_X=\{\emptyset,\{a\},\{a,b\},\{a,b,c\}\}$. In this topology every set is compact since we have only finite many open sets to use for covers to begin with.

However, e.g. $\{b\}$ is not a closed set, since its complement is not open.

Note that in this example the topology is not Hausdorff - and that is a crucial fact as we have the following lemma:

Lemma 0.0.27: If (X,τ_X) is Hausdorff and $K\subset X$ is compact then K is closed.

Proof. We want to show that $X\setminus K$ is open. This can be done by showing that for every $x\in X\setminus K$ there is an open set U around it which lies entirely in $X\setminus K$. Thus $X\setminus K$ is the union of such open sets and is therefore itself open.

So pick $x \in X \setminus K$. Since X is Hausdorff, to each $y \in K$ there are disjoint open sets U_y containing x and V_y containing y. The collection

$$\{V_y \; : \; y \in K, V_y \cap U_y = \emptyset\}$$

is clearly an open cover of K.

K is compact, so there exist $y_1, ..., y_n \in K$ such that $\bigcup_{i=1}^n V_{y_i} \supset K$. Let $U = \cap_{i=1}^n U_{y_i}$. U is open since it is a finite intersection of open sets. Clearly, $x \in U$.

Finally, $[\bigcup_{i=1}^n V_{y_i}] \cap [\cap_{i=1}^n U_{y_i}] = \emptyset$. Since if z is in this intersection then, in particular, it is in the first set and so $z \in V_{y_i}$ for some i. On the other hand, $z \in U_{y_j}$ for all $j = 1, .., n$ so, in particular, $z \in U_{y_i}$ for the same i as well. But then $V_{y_i} \cap U_{y_i} = \emptyset$.

■

Note that in fact, we proved

Corollary 0.0.28: If $K \subset X$, X Hausdorff, K compact, and $x \notin K$, then $\exists U, V$ open sets such that $K \subset U$, $x \in V$ and $U \cap V = \emptyset$.

We have the following very important theorem which is an important tool in showing that two spaces are homeomorphic. We will use it to show that "if you glue the endpoints of an interval, you get a circle".

Theorem 0.0.29: Suppose $f : X \to Y$ is continuous and bijective. Also assume X compact and Y is Hausdorff. Then f is a homeomorphism.

Proof. We must show that $f^{-1} : Y \to X$ is continuous. We will use an equivalent statement that $g : A \to B$ is continuous if and only if $\forall\, V$ closed in B, $g^{-1}(V)$ is closed in A. That is, we must check that, $\forall\, V$ closed in X, $f(V) = (f^{-1})^{-1}(V)$ is closed in Y.

Let $V \subset X$ be closed. We claim that V is compact since X is compact. We see this since, taking $\{U_\alpha\}$ as an arbitrary open cover of V, we have that $\{U_\alpha\} \cup \{X \setminus V\}$ is an open cover of X. Hence there exists a finite subcover of X, denoted by $\{U_{\alpha_i}\}_{i=1}^n \cup \{X \setminus V\}$. Then $\{U_{\alpha_i}\}_{i=1}^n$ is a finite open subcover of V, so V is compact.

Thus, since V is compact and f is continuous, $f(V)$ is compact. But Y is Hausdorff and we just proved that a compact set in a Hausdorff space is closed. Hence $f(V)$ is closed.

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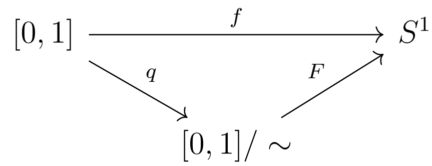
Corollary 0.0.30: Let $X = [0, 1]$ with the subspace topology of the usual topology of $\mathbb{R}$. Then $X/\sim = [0, 1]/0 \sim 1$ is homeomorphic to S^1

Proof. Consider $f : [0, 1] \longrightarrow S^1$, $f(t) = (\cos(2\pi t), \sin(2\pi t))$. The map f is continuous (by calculus), onto, but not 1-1 as $f(0) = f(1)$.

Let $q : [0, 1] \longrightarrow [0, 1]/_{0\sim 1}$ be the quotient map i.e. $q(t) = [t] =$ the equivalence class of t. Recall that q is continuous.

Let $F : [0, 1]/_{0\sim 1} \longrightarrow S^1$ be defined by $F([t]) = f(t) = (\cos(2\pi t), \sin(2\pi t))$ so that $F \circ q = f$ i.e. the following diagram commutes.



Then we need to verify that

- F is well-defined: one can check directly that $F([0]) = F([1])$ i.e. F does not depend on the representative of an equivalence class.
- F is bijective: F is onto, since f is onto; F is 1-1 by construction: f is not 1-1 at $t = 0$ and 1, however, those are in one equivalence class in $[0, 1]/\sim$
- F is continuous: we showed before that $F : X/\sim \longrightarrow Y$ is continuous if and only if $F \circ q : X \to Y$ is continuous. But $F \circ q = f$ which is continuous "by calculus".
- $[0, 1]/\sim$ is compact: it is the image of a compact set, namely $[0, 1]$ under a continuous map, namely q. (Note that $[0, 1]$ is compact, since it is closed and bounded in R.)
- S^1 is Hausdorff: it is a subspace of $\mathbb{R}^2$ which itself is Hausdorff. (Every subspace of a Hausdorff space is Hausdorff.)

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