#4. Claim: If X is compact,

F_i $(i=1,2\ldots)$ are non-empty, closed subsets of X with

$F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq$ then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$

Pf: By contradiction, assume

$\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Consider $X \setminus F_i \in \tau_X \ \forall i \in \mathbb{N}$.

(We have $X \setminus F_i \in \tau_X$, since F_i is closed $\forall i$.)

By deMorgan Laws

$\bigcup_{i=1}^{\infty}(X \setminus F_i) = X \setminus \bigcap_{i=1}^{\infty} F_i = X$, since we assumed $\bigcap_{i=1}^{\infty} F_i = \emptyset$

That means $\{X \setminus F_i\}_{i=1}^{\infty}$ is an open cover of X.

Since X is compact, this cover has a

finite subcover:

$\{X\setminus F_{i_1}, \dots, X\setminus F_{i_k}\}$ for some $k \in \mathbb{N}$.

where we can assume $i_1 < \dots < i_k$.

That is $\displaystyle\bigcup_{j=1}^{k} X\setminus F_{i_j} \supseteq X$

and since X is the entire space,

$$\bigcup_{j=1}^{k} X\setminus F_{i_j} = X$$

But the left-hand-side is

$$\bigcup_{j=1}^{k} X\setminus F_{i_j} = X\setminus F_{i_k}$$

Since $F_1 \supseteq F_2 \supseteq \dots$ and $i_1 < \dots < i_k$

imply $F_{i_1} \supseteq F_{i_2} \supseteq \dots \supseteq F_{i_k}$

so

$$X\setminus F_{i_1} \subseteq X\setminus F_{i_2} \subseteq \dots \subseteq X\setminus F_{i_k}.$$

Thus $X\setminus F_{i_k} = X \implies F_{i_k} = \phi$. But we assumed $F_n \neq \phi \ \forall n$.

b.) Let $X = \mathbb{R}$ and

$F_n = [n, +\infty)$

Then clearly $F_n \neq \phi$, F_n are closed

$F_1 \supseteq F_2 \supseteq \cdots$,

but $\bigcap_{n=1}^{\infty} F_n = \phi$.