Wrap-up and practice problems for the final

The final is comprehensive with an emphasis on the 2nd part of the course. These sample problems below are primarily related to that. Please review problems we did before the midterm as well.

1. For each of the pairs of topological spaces X and Y, decide if X is homeomorphic to Y. Give reasons to your answers.

   i.) the set $X = \{(x, x^2, x^3) \mid 1 \le x \le 2\} \subset \mathbb{R}^3$ and $Y = \{(-\infty, \infty)\} \subset \mathbb{R}$

   ii.) the line $\mathbb{R}$ and the plane $\mathbb{R}^2$

   iii.) the plane $\mathbb{R}^2$ and the space $\mathbb{R}^3$

   iv.) the compact, connected surface whose polygonal diagram representation corresponds to the word $aba^{-1}cb^{-1}c$ versus $adbc^{-1}da^{-1}cb^{-1}$

   v.) $\mathbb{Z}$ with the cofinite topology versus $\mathbb{Z}$ with the co-countable topology

2. True or false?

   (a) A map $f : X \to Y$ where X is finite and Hausdorff, must be continuous.
   (b) If a set is compact, it must be closed.
   (c) The Equator is a retract of Earth.
   (d) If A and B are path-connected then $A \cup B$ and $A \cap B$ are connected.
   (e) The quotient $[0,1]/_{0\sim 1}$ is a subset of $\mathbb{R}$ that is homeomorphic to S^1
   (f) If a topological space X has a trivial fundamental group ie $\pi_1(X, x_0) = 0$, then any two paths $\alpha, \beta : [0,1] \to X$ starting at x_0 and having the same endpoint, must be homotopic.

3. Find a retract which is not a deformation retract.

4. a.) Show that if X is compact and F_i $(i = 1, 2, ...)$ are non-empty, closed subsets of X with $F_1 \supseteq F_2 \supseteq ... \supseteq F_n \supseteq ...$ then $\cap_{n=1}^{\infty} F_n \neq \emptyset$.

   This is the general version of Cantor's Nested Interval Theorem on $\mathbb{R}$.

   b.) Provide an example that shows: if X is not compact, then such an intersection may be empty.

4.* Suppose $p : Y \to X$ is a 7-sheeted covering, where Y is path-connected with a trivial fundamental group and X is path-connected. Show $\pi_1(X)$ must be Abelian.

5.* Show that $[0,1]$ is not homeomorphic to $[0,1] \times [0,1]$, but that there is an equivalence relation on $[0,1]$ such that $[0,1]/\sim$ is homeomorphic to $[0,1] \times [0,1]$.