Homework 1.

- **Due Feb 16th, Friday (any time), by email at agnes.szilard@budapestsemesters.com, in pdf format. (No jpg etc is accepted.)**
- **Please, leave some blank space at the end of each problem or a wide margin so that there is space for comments.**
- **If you scan handwritten solutions, make sure to use a dark pen so that what you write is clearly visible.**

1. Let $X = \mathbb{R}$. Verify (ie prove) that
$$\tau = \{(a, \infty) \,|\, a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$
is a topology on $\mathbb{R}$.

2. We started discussing that a major source of topologies are "generated by metrics", that is: if you have a metric d on some set X, then we can define open balls, and then can define open sets (with respect to that metric), and the open sets form a topology on X. Prove this, that is: show that the sets that are open with respect to some metric d form a topology.

3. Consider the pair $(\mathbb{R}^2, d)$, where $\mathbb{R}^2$ denotes the Euclidean plane and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $d(x, y) = max\{|x_1 - x_2|, |y_1 - y_2|\}$ for all $x = (x_1, y_1)$, $y = (x_2, y_2) \in \mathbb{R}^2$.

   a.) Prove that d is a metric.

   b.) What is the circle of radius 1 around the origin with respect to this metric (that is: the set of points that are distance one from the origin)? Graph.

   What is $B_{(0,0)}(1)$ ie. the open ball of radius one around the origin?

   Characterize all open balls.

   c.) Give an argument that shows: the topology generated by this metric is the same as the standard (usual) topology.

4. However, not all topologies come from a metric.

   Give an argument that shows: if $X = \{a, b, c\}$ and τ is the anti-discrete topology, there does not exist a metric (distance function) that would generate this topology.

5. Consider the following

   Definition: For any topological space (X, τ_X), we say that a set $V \subset X$ is *closed* (with respect to that topology) if its complement $X \setminus V$ is open, i.e. $X \setminus V \in \tau_X$.

   Show that the following are true in *any* topological space:

   i.) the finite union of closed sets is closed;

   ii.) the arbitrary intersection of closed sets is closed.

   Hint: Use the de-Morgan Laws $X \setminus \cup U_\alpha = \cap(X \setminus U_\alpha)$ *and* $X \setminus \cap U_\alpha = \cup(X \setminus U_\alpha)$

   c.) In a space (X, τ_X) of your choice, give examples to show that the infinite union of closed sets may be closed, open, neither or both. Thus, in particular, the set of closed sets may not be closed with respect to arbitrary union.

EXTRA CREDIT Let $X = \mathbb{R}$ and d the standard metric on it. That is, if $x, y \in \mathbb{R}$, then $d(x, y) = |x - y|$.

Show that

$$d'(x,y) = \begin{cases} min\{|x-y|, 1\} & \text{if } x, y \in \mathbb{Q} \text{ or } x, y \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

is also a metric.

Does it generate the same topology on $\mathbb{R}$ as d?

If not, how are the two topologies related? One contains the other? Neither?