HW 10.

1. Consider the mapping $p : Y = \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0,0)\} = X$ where $p(r,\theta) = (r\cos(\theta), r\sin(\theta))$.
   Think through for yourself that this is a covering.
   Now choose a basepoint $y_0 \in Y$ for $x_0 = (2,0) \in X$ i.e. $y_0 \in p^{-1}(x_0)$ and consider the paths in X

   $$\alpha(t) = (2-t, 0) \quad t \in [0,1]$$

   $$\beta(t) = ((1+t)\cos(2\pi t), (1+t)\sin(2\pi t))) \quad t \in [0,1]$$

   $$\gamma(t) = \alpha * \beta$$

   Find the liftings of α, β^{-1}, γ starting at a point of your choice y_0.

   Note that $<\gamma> \in \Pi_1(X, x_0)$. Use the covering to argue that it is not the trivial class (=identity element) of $\Pi_1(X, x_0)$.

   And a little calculus: sketch γ.

2. *This problem illustrates why it is crucial in the path-lifting lemma and the homotopy lifting lemma to choose where your lifts start. Since, as you can show, as soon as two lifts of a path or of a homotopy agree at a point, they are the same:*

   Suppose $f : Z \to X$ is a continuous map into the base-space of a covering $p : Y \to X$ and Z is connected. Let $\tilde{f}, \tilde{\tilde{f}} : Z \to Y$ be two continuous lifts of f s.t. $\tilde{f}(z) = \tilde{\tilde{f}}(z)$ for some $z \in Z$. Then $\tilde{f}(z) = \tilde{\tilde{f}}(z)$ for all $z \in Z$.

3. Let S and S' be compact connected surfaces and assume that $p : S \to S'$ is an n-sheeted covering where $n > 0$ is an integer.

   a.) Show that a triangulation $T_{S'}$ of the base space S' can be lifted to a triangulation T_S of the covering space S.

   b.) Use (a) to find a relationship between the Euler characteristic of S and S'.

   c.) Are the following statements true or false?

   (a) There is a 3-sheeted cover of the torus by the torus.
   (b) The two-torus $T\#T$ may double-cover the torus.
   (c) The only oriented surface that can cover itself with 3 sheets is the torus.
   (d) The torus can only double-cover the torus among the orientable surfaces.
   (e) The torus double-covers the Klein bottle.

4. Actually, in the previous problem, given S and S' such that $p : S \to S'$ is an n-sheeted covering, we only need to assume that S' is compact since the compactness of S follows. Prove this.

5. (a) Find an example to show that if $f : Y \to X$ is some continuous map, the induced homomorphism $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ may not be 1-1.

   But if $f = p : (Y, y_0) \longrightarrow (X, x_0)$ is a covering map, then the induced homomorphism of fundamental groups is one-to-one.
   (b) [Extra Credit] Suppose $p : (Y, y_0) \longrightarrow (X, x_0)$ is an n-sheeted covering, Y is path-connected. Show that $p_*(\pi_1(Y))$ is a subgroup of index n in $\pi(X)$.