Intro to Topology – Hw2.

Always give reasons.

1. Consider a metric space (X, d) (that is a set X with a metric d on it) and fix $p \in X$ as well as $r > 0$. Define a closed ball centered at p, radius r as

   $$\overline{B_p(r)} = \{q \in X \mid d(p, q) \leq r\}$$

   Show that a closed ball is a closed set.

2. a.) For an infinite set X consider the collection of subsets

   $$\tau = \{\emptyset, X\} \cup \{U \subset X \mid X \setminus U \text{ is finite }\}$$

   .

   Show that τ is a topology on X. (It is called the *co-finite* topology on X.)

   b.) Let $X = \mathbb{N}$ and consider $g : \mathbb{N} \to \mathbb{N}$ given by

   $$g(x) = \begin{cases} 2 & \text{if } x \text{ is even} \\ 4 & \text{if } x \text{ is odd} \end{cases}$$

   If $\mathbb{N}$ is equipped with the co-finite topology both as a source (domain) and target (range), is g continuous?

3. Let $f : X \to Y$ be a mapping between topological spaces (X, τ_X) and (Y, τ_Y). Prove that f is continuous if and only if $\forall V \subset Y$ such that V is closed, we have $f^{-1}(V)$ is closed in X.

4. Let $f : X \to Y$ be a mapping between topological spaces (X, τ_X) and (Y, τ_Y). For each of the types of functions I.) and II.) below decide if the given properties are equivalent to f being continuous. If yes, prove it.

   For the one(s) not equivalent, decide if continuity implies the given property or vice-versa. If one implies the other, prove it. If not, provide an example showing it does not.

   Property I.) f takes open sets to open sets. (Such a map is called an *open map.*)

   Property II.) f is such that $U \subset X$ is open whenever $f(U)$ is open.

EXTRA CREDIT Given a metric space (X, d), let τ_d be the topology generated by that metric. Consider also a subset $A \subset X$ and $d' = d_A$ the metric d restricted to this subset. Show that the following two topologies are the same:

$$\tau_{d,A}, \text{ the subspace topology of } \tau_d \text{ on } A$$

and

$$\tau_{d'}, \text{ the metric topology on } A \text{ generated by } d' = d_A$$