Intro to Topology – HW 3.

1. Give an argument that shows: for a top space (X, τ_X) and a subset $A \subset X$ with the subspace topology τ_A we have $V \subset A$ is closed wrto τ_A if and only if there is a $W \subset X$ that is closed wrto τ_X and $V = W \cap A$.

2. Read the three excerpts from "Messer and Straffin", "Munkres" as well as "Weeks"about quotient topologies and the torus/Mobius strip/projective plane case. These are posted separately.

3. Verify that the following limerick is true

   A matematician named Klein
   Thought the Möbius band was divine
   Said he, "If you glue
   the edges of two,
   you get a weird bottle like mine."

   (i.e. the Klein bottle can be obtained by gluing two Mobius strips along their bounding circles.)

4. Consider the unit square and all combinatorially possible *pairwise* identifications of its edges (up to rotation and mirror images).

   Identify what you got in each case, topologically.

   (No formal proofs are necessary, however, you may have to use a cut-and-paste argument on a diagram, to recognize what you got.)

5. a.) Consider the hexagon X whose edges are identified according to the word $abca^{-1}b^{-1}c^{-1}$.

   The identification provides a quotient space $(X/\sim, \tau_{x/\sim})$.

   i.) How are the vertices of the original hexagon identified in this space? Label the vertices and write it clearly which vertices of the original hexagon are in the same equivalence class.

   Note: if two edges are identified (glued), that means their beginning and endpoints -which are vertices - have to be the same. Thus identifying edges determines how the vertices are identified.

   ii.) The quotient is a surface, as every point in the quotient is contained in an open set that is homeomorphic to an open disc of $\mathbb{R}^2$.

   While if you consider images (under the quotient map) of points that are originally inside the hexagon or on an edge this is relatively straightforward to "see" intuitively, it is not so apparent for points "coming from" vertices.

   Label the corners of the hexagon and "show" that at each point in the quotient, that "comes from" a vertex of the hexagon, the corners glue in such a way that you get a disc around them.

(No formal proof is necessary just draw appropriate figures. However, make sure to label correctly, so that it is possible to follow what you are doing.)

iii.) Show that $abca^{-1}b^{-1}c^{-1}$ is actually a torus by a cut-and-paste argument.

I posted an illustration of how the doughnut surface (which is homeomorphic to the torus) can be glued from the hexagon.

Extra credit *We will discuss background needed for this problem on Tuesday.*

Start with the solid cube $X = \{(x, y, z) \mid x, z, y \in [-1, 1]\}$. Glue its faces as follows: $(1, y, z) \sim (-1, y, z)$, $(x, 1, z) \sim (-x, -1, z)$ and $(x, y, 1) \sim (x, y, -1)$. The resulting quotient space is also a product space. What product is it? No formal proof is necessary, however, explain your reasoning.

Find in this space a one sided torus.

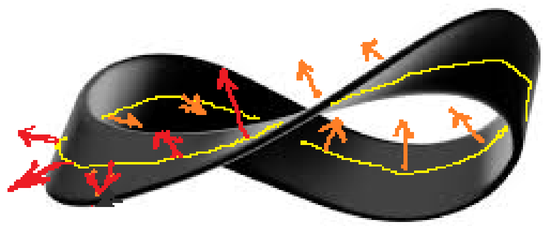
What does this mean? We have not proved it, but the quotient space above is a 3-manifold, ie locally it is like $\mathbb{R}^3$ (every point of it has an open set around it which is homeomorphic to $\mathbb{R}^3$). Inside this 3-manifold - the quotient space above - there lies a torus in a very special way: there is a loop (a closed path) in this torus so that if starting at a point P of that loop you move a normal vector along the loop all the way, it will arrive back pointing in the opposite direction.

When you consider a torus as a doughnut surface in $\mathbb{R}^3$ no matter what loop you take on this surface when you move a normal vector along it, the vector will point in the same direction as originally when you arrive back after traversing the loop. This is true for *all* loops on the doughnut surface lying in $\mathbb{R}^3$.

This phenomenon is referred to as the doughnut surface being two-sided in $\mathbb{R}^3$: each side corresponds to a choice of where the normal vector points originally.

On the other hand, consider the twisted strip that is colloquially called Mobius strip also, lying in $\mathbb{R}^3$. Some loops eg small circles are also such that if you move a normal vector along those loops, traversing them, the normal vector arrives back pointing in the same direction as when it started.

However, there is a special loop, as shown below with yellow, such that when you move a normal vector along that loop (shown red below), it arrives back pointing in the opposite direction. This phenomenon is referred to as "the Mobius strip being one-sided in $\mathbb{R}^3$".



So, to find a one sided torus in the above quotient space, you have to find a torus as well as a special loop in that torus, so that if you move a vector normal to the torus along that loop, traversing the entire loop, the normal vector will arrive back pointing in the opposite direction.

(You can show all this in the original cube, noting appropriate gluings.)