Intro to Topology – HW 4.

1. Show that if U is closed in X and V is closed in Y then $U \times V$ is closed in $X \times Y$.

   (It is assumed that X has topology τ_X, Y has topology τ_Y and $X \times Y$ has the product topology.)

2. Let (X, τ_X), (Y, τ_Y) be topological spaces and equip the product $X \times Y$ with the product topology. In class we discussed that the projections maps $proj_x : X \times Y \to X$, $proj_x((x, y)) = x$ and $proj_y : X \times Y \to Y$, $proj_x((x, y)) = y$ are continuous. Now prove that they are also open maps (that is: they take open sets to open sets).

3. Decide if the following statements are true or false (i.e. for all sets A, B and in any topology):

$$cl(A \cap B) = cl(A) \cap cl(B)$$

$$cl(A \cup B) = cl(A) \cup cl(B)$$

   If false, give a counterexample to the statement.

   If true, prove it **two ways**:

   first ("proof 1") by element chasing - that is, show that if an element x is in the left hand side (LHS) set, then it is in the right hand side set (RHS) and vice-versa, and also that $x \in cl(D)$ for some set D if and only if $\forall U \in \tau$ such that $x \in U$ we have $D \cap U \neq \emptyset$.

   Then ("proof 2") by using "properties of closure":

   $\forall A \subset X$ we have

   (1) $cl(A)$ is a closed set in X

   (2) $A \subset cl(A)$

   (3) A is closed if and only if $cl(A) = A$

   (4) if W is a closed set for which $A \subset W \subset cl(A)$ then $W = cl(A)$

   At each step of proof 2, state which of properties 1-4 you are using.

4. In $\mathbb{R}$ with the standard (or usual) topology consider

$$A = (0, 1) \cup (1, 2) \cup \{(2, 3) \cap \mathbb{Q}\} \cup \{4\}$$

   i.) What are $cl(A)$, $int(A)$, ∂A?

   ii.) Check that $cl(int(A)) = cl(int(cl(int(A)))$. i.e. work out both sides.

   iii.) Show that, in fact, $cl(int(A)) = cl(int(cl(int(A)))$ in general. (i.e. this is true in all topologies, for any A).

EXTRA CREDIT Pythagorean triples are natural numbers $a, b, c \in N$ satisfying the equation $a^2+b^2 = c^2$. The Pythagorean triple that is probably most commonly known is 3, 4, 5.

In this problem you can work out formulae to generate such triples, starting a rational number, using stereographic projection.

a.) Work out a formula for stereographic projection for $n = 1$, that is $\phi : S^1 \setminus \{(0, 1)\} \longrightarrow \mathbb{R} = \{(x, 0) \mid x \in \mathbb{R}\}$, where $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and its inverse, to show that there is a one-to-one correspondence between those points of the unit circle take away the North pole whose both coordinates are rational and rational points of the x-axis (ie points of the form $(s, 0)$ with $s \in \mathbb{Q}$).

b.) Consider now $a, b, c \in \mathbb{Z}$ (not $\mathbb{N}$), not all zero.

Show that Pythagorean triples determine points of the unit circle whose both coordinates are rational. How?

Find an inverse procedure, ie formulae which given a "rational point" of S^1, provide a Pythagorean tripple.

c.) If k is not zero and $m, k \in \mathbb{Z}$, then $\frac{m}{k} \in \mathbb{Q}$. Use formulae you worked out in part a) and b) to find formulae which for a pair of non-zero integers $m, k \in \mathbb{Z}$ provide a Pythagorean triple $a, b, c \in Z$. (These formulae were first already known to Euclid).

d.) Is it possible to get all Pythagorean triples this way?