Intro to Topology – HW 5.

1. Show that in a Hausdorff space (X, τ_X) limits are unique. That is, if for a sequence of points $x_n \in X$, $n = 1, 2...$ we have $x_n \to a$ and $x_n \to b$, then $a = b$.

2. a.) In real analysis one often finds points of a closure using convergence, since in $\mathbb{R}^n$ we have $x \in cl(A)$ if and only if $\exists$ a sequence $x_n \to x$ with $x_n \in A$. Verify that the same is true in any metric space.

   (Hint: Think through the $\mathbb{R}^n$ case and generalize for any metric. Recall that $x \in cl(A)$ if and only if $\forall U$ open sets s.t. $x \in U$ we have $U \cap A \neq \emptyset$.)

   b.) However, in general this is not true! That is, $cl(A)$ and the set of limit points of A, for $A \subset X$ in a topological space (X, τ_X), may be different.

   Explain why by working out the following example:

   let $X = \mathbb{R}$, and let $\tau_X =$ *the co-countable topology* on X, where the open sets are the $\emptyset$, X and those sets whose complement is countable (possibly countably infinite). Take $A = \mathbb{R} \setminus \{p\}$ for some $p \in \mathbb{R}$.

   What is $cl(A)$ and why?

   What is the set of limit points of A and why? (A point $x \in X$ is a limit point if $\exists (x_n)$ sequence in X such that $x_n \to x$.)

   Conclude that the set of limit points of A is not the same as $cl(A)$.

3. Let (X, d) be a metric space and τ_d =the topology generated by d.

   Prove that if $K \subset X$ is compact then K is closed.

   Here is how you can start: if a set in a metric space (X, d) is not closed, then it cannot be compact, because it has an open cover with no finite subcover.

   Specifically, if a set K is not closed, then K is a proper subset of its closure, so $\exists x \in cl(K)$ such that $x \notin K$. Then consider

   $$A_n = \{y \in X \mid d(x, y) > \frac{1}{n}\} \quad \forall n \in \mathbb{N}$$

   Prove that the collection $\{A_n\}$ is a cover of K that has no finite subcover.

   Also, A_n must be open $\forall n \in \mathbb{N}$ - why?

   Thus K is not compact.

EXTRA CREDIT Let X have the τ_X=co-finite topology.

Prove that a sequence (x_n) $n \in \mathbb{N}$ in this space is divergent if and only if it has at least two eventually constant subsequences.

Prove that