Intro to Topology – Hw6

1. In class we showed claim 1: if K is compact in a Hausdorff space X, then for any $x \in X \setminus K$ there exist open sets U and V with $U \cap V = \emptyset$ such that $K \subset U$ and $x \in V$.

   That is: in a Hausdorff space, a compact set can be separated from a point.

   As a "next level", now show claim 2: if X is Hausdorff, then for any two compact, disjoint sets K and M in X there exist disjoint open sets U and V such that $K \subset U$ and $M \subset V$.

   That is: in a Hausdorff space, two disjoint compact sets can be separated by disjoint open sets.

   No need to start from scratch, but you can use and modify the proof of claim 1. See your class notes and the posted notes for a proof. Note that in class the first proof had to be corrected.

2. We also showed that the quotient $X = [0,1]/0 \sim 1$ is homeomorphic to S^1.

   Generalize that proof to show that

   $$S^1 \times S^1 = \{(u,v,x,y)\,|u^2+v^2=1,\,x^2+y^2=1\} \;\subset\; \mathbb{R}^4$$

   is homeomorphic to the torus i.e. the identification space $\mathbb{T} = I^2/\sim$, where $I^2 = [0,1] \times [0,1]$ and the equivalence relation $\sim$ is $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$.

3. Show that if (X,d) is a metric space and X has countable many points (but more than 2 and possibly countably infinite) and τ_d is the topology generated by d, then (X,τ_d) must be disconnected.

4. Are the following subsets of $\mathbb{R}^2$ path-connected? If not path-connected, are they connected?

   $$A = \{(x,y)\,|\,x,y \in \mathbb{Q}\}$$

   $$B = \{(x,y)\,|\,x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$$

Extra Credit Is the following set path-connected, connected or neither?

$$C = \{(x,y)\,|\,x,y \in \mathbb{Q} \text{ or } x,y \in \mathbb{R} \setminus \mathbb{Q}\}$$

You may find this difficult, I really suggest that you work on it in groups. However, please write up your solutions on your own.