Homework 7.

1. a.) Consider $f : X \to \mathbf{R}$ continuous and assume X is connected. Let $a, b \in f(X)$ and assume $a < b$. Show that for all $c \in (a, b)$ $\exists z \in X$ such that $f(z) = c$. *(This is the so called Generalized Intermediate Value Theorem.)*

   b.) Show that if $f : S^1 \to \mathbf{R}$ is continuous, then there is an $\underline{x} = (x_1, x_2) \in S^1$ for which $f(\underline{x}) = f(-\underline{x})$.

2. a.) Find all compact connected subsets of $\mathbf{R}$. Justify your answer.

   b.) Find all subsets of $\mathbf{R}$ that S^1 is homeomorphic to. Justify your answer.

3. Identify which surfaces in the classification theorem the following ones are homeomorphic to

   a.) The connected sum $K\#T$, where K is the Klein bottle and T is the torus.

   b.) A disk sewn onto the edge of a Mobius strip.
   (We did this problem already, using a cut-and-paste argument. But now use diagrams, as well as the Euler characteristic and orientability.)

   c.) The surface determined by the labelling scheme of a polygon

   as $abcdec^{-1}d^{-1}a^{-1}b^{-1}e^{-1}$

   d.) The surface which is called the "Purse of Fortunatus" in Lewis Caroll's novel *Sylvie and Bruno Concluded*, where Mein Herr teaches Lady Muriel how to construct such a purse by sewing together three handkerchiefs along their edges according to $cb^{-1}e^{-1}a^{-1}$, $bda^{-1}f^{-1}$ and $d^{-1}c^{-1}f^{-1}e^{-1}$.

   He then justifies the name of the purse as follows: *"Whatever is inside that purse is outside it; and whatever is outside it, is inside it. So you have all the wealth of the world in that little purse!"*

4. This exercise is related to platonic solids. A platonic solid is a convex polyhedron all of whose faces are regular n-gons, i.e. all the faces have the same number of sides, which in platonic solids are also of the same length. Note that $n \geq 3$. The word "convex" means that the inner angles between adjacent faces are less then 180 degrees.

   It is well-known that there are five platonic solids: the cube, the tetrahedron, the octahedron, the icosahedron and the dodecahedron.

   In this problem, we will prove, using the Euler characteristic, that these 5 are the only ones possible.

   Consider a platonic solid M. If it is placed inside a sphere and then expanded from within (ie it is mapped onto the sphere along rays from the center of the sphere), then it determines a triangulation of the sphere, where

   V =number of vertices of M, as well as of the triangulation;

   E =number of edges of M, as well as of the triangulation;

F =number of faces of M, as well as of the triangulation.

Let p denote the number of edges meeting at a vertex. Note that since M is a platonic solid p is the same for every vertex and $p \geq 3$.

a.) We have $pV = 2E$ and that $nF = 2E$ - why?

b.) Conclude that

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{p} = \frac{1}{E}$$

and because $E > 0$ this implies

$$\frac{1}{n} + \frac{1}{p} > \frac{1}{2}$$

c.) Use the last inequality to find upper bounds on n and p and thus, since these numbers are natural numbers, there is a finite list of possible values for them. What is this list? Use it to find all possible platonic solids. Identify them with the five known ones.

Extra Credit Wolfram Mathematica says that the doughnut surface (a subset of $\mathbf{R}^3$) can be parametrized by

x = (a+b cos(v)) $\cos u$
$y = (a + b\cos(v))\sin u$
$z = b\sin v$

for $(u, v) \in [0, 2\pi)$, which is a continuous bijection, if $a > b$. Assume this.

Use the above info to prove that the torus, as we defined it, is homeomorphic to the doughnut surface.

(Hint: use an appropriate commutative triangle.)