Homework 8.

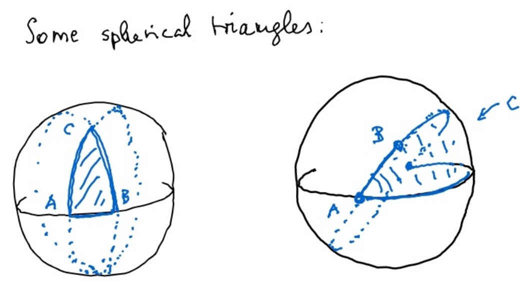
1. In this problem you will prove - for a very special case - that the Euler characteristic of a compact connected surface is independent of cell decomposition. What you get is also a "baby version" of an intriguing theorem, the Gauss-Bonnet theorem which shows (among other things) that the topology and geometry of compact connected surfaces are "intertwined", determine each other (see Part III, which is extra credit).

   So this problem not entirely topological, but uses geometry - measurements such as angle, area.

   Consider the sphere $S^2 = \{(x, y, z) \,|\, x^2 + y^2 + y^2 = 1\} \subset \mathbb{R}^3$ as before.

   We will also need the following definitions:

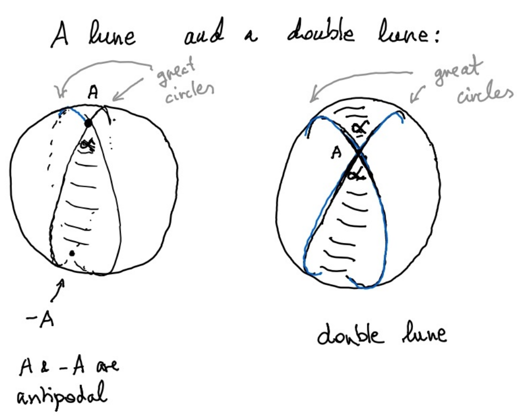
   - A great circle of S^2 is the intersection of a plane through the origin and the sphere. (So for example, the Equator is a great circle of the Earth, but the Arctic circle is not.)
   - A geometric spherical triangle of S^2 is a closed subset homeomorphic to a usual triangle of the plane and in addition each of its 3 edges are subsets of great circles.



   - A geometric spherical polygon P, an n-gon, of the sphere is a closed subset homeomorphic to a usual polygon, an n-gon of the plane whose edges are all subsets of great circles.

   PART I. Start by working out a formula for the area of a geometric spherical triangle.

   a.) A "double lune of angle α" of the sphere is shown in the figure below. It is part of the sphere that is enclosed by two great circles whose planes intersect at angle α.

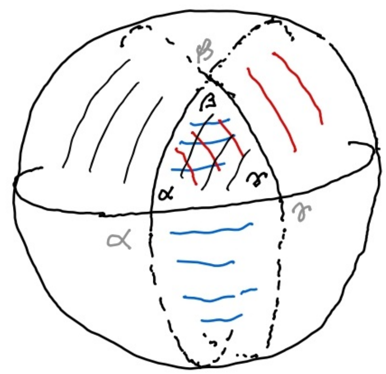


   Use proportions to find a formula for the area of a double lune in S^2, in terms of its angle α.

b.) A geometric spherical triangle is shown in the figure below - it lies at the intersection of three lunes. Use this, as well as the fact that the corresponding double lunes cover S^2 to prove that the area of a geometric spherical triangle with angles α, β, γ on the unit sphere is

$$A(\text{triangle}) = \alpha + \beta + \gamma - \pi$$

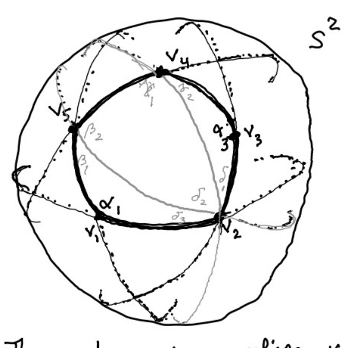
A geometric spherical triangle
is at the intersection of 3 lunes:



(Muse over the differences between the spherical and Euclidean "world", implied by your formula.)

c.) Find a formula for the area $A(P)$ a geometric spherical polygon P with n vertices (and sides) in S^2. *(Hint: use the figure below.)*

Geometric spherical polygon; a pentagon



The pentagon has vertices v_1, v_2, v_3, v_4, v_5
and angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$.

PART II. Now, work out a formula relating the surface area of S^2 and the Euler characteristic: Assume that the sphere S^2 has a cell-decomposition consisting of geometric spherical polygons $\{P_1, ...P_k\}$. Then the surface area of S^2 is

$$A(S^2) = \sum_{i=1}^{k} A(P_i)$$

Using that each polygon P_i has n_i vertices and e_i edges, so in fact $n_i = e_i$, $\forall i = 1, ..., k$, find a formula for the surface area $A(S^2)$ of S^2 in terms of its Euler characteristic $\chi(S^2)$.

Observe that, since the surface area of the unit sphere is constant, the Euler characteristic must be constant as well, independent of what (in this case, somewhat restricted) cell decomposition you start with!

In particular, since we know from calculus that S^2 has surface area 4π, the usual value for $\chi(S^2)$ follows - this is a good way to check if your formula is correct!

PART III. [Extra credit.]

How should the last formula be modified if instead of the unit sphere, you start with a sphere of radius $r > 0$ ie

$$S^2_r = \{(x, y, z) \,|\, x^2 + y^2 + y^2 = r^2\}.$$

Just write the formulae for i.) the area of a geometric triangle of S^2_r

ii.) the area of a geometric polygon of S^2_r in terms of the inner angles of the polygon and r

iii.) a formula for $A(S^2_r)$ in terms of $\chi(S^2_r)$ and r

Observe that since $A(S^2_r) = 4\pi r^2$, $\chi(S^2_r)$ is still constant, independent of r, as expected.

The Gauss-Bonnet theorem provides a relationship between *the curvature* κ of a compact connected surface M and its Euler characteristic $\chi(M)$ and states that

$$\int_M \kappa \, dA = 2\pi\chi(M).$$

The curvature is an important characteristic of a surface, which in general, changes pointwise. Think of a "blob", for example, that is homeomorphic to a sphere. We have, from multivariate calculus, that a sphere of radius r, has constant curvature $\frac{1}{r}$, on the other hand, the curvature of a random blob changes pointwise, so we have a curvature function $\kappa : M \to \mathbb{R}$.

Note that the Gauss-Bonnet theorem says the "geometric" left side of the formula is determined by the "topological" right side. Eg. you cannot create a blob with just any curvature, the "overall curvature" has to be constant.

Now, check that for spheres, the Gauss-Bonnet formula gives your formula iii. above.

Also, provide *heuristics* to show the Gauss-Bonnet theorem is true, by working out a "partial sum"

$$\int_M \kappa \, dA \approx \sum \kappa_i A(P_i).$$

where you can use

- a finite cell-decomposition of M into polygons P_i, for $i = 1, ..., n$, each having area $A(P_i)$ and (approximately) constant curvature κ_i
- assuming your formula from ii.) above holds after rewriting it in terms of κ_i instead of r_i.

2. Let $S^2 = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\}$ as before and fix the basepoint $x_0 = (1, 0, 0)$. Show explicitly that the loop $\alpha : [0, 1] \to S^2$ taking $s \in [0, 1]$ to $(\cos 2\pi s, \sin 2\pi s, 0) \in S^2$ is homotopic to the constant loop $x_0 : [0, 1] \to S^2$ taking $s \in [0, 1]$ to $(1, 0, 0) \in S^2$.

   (ie find such a homotopy)

3. Consider the paths $\alpha, \alpha' : [0,1] \to X$ from point a to b and paths $\beta, \beta' : [0,1] \to X$ from b to c. In order to show that $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ imply $\alpha * \beta \simeq \alpha' * \beta'$, we used the homotopy

$$H(s,t) = \begin{cases} F(2s,t) & \text{if } s \in [0, \frac{1}{2}] \\ G(2s-1,t) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$

where F is the homotopy for $\alpha \simeq \alpha'$ and G is the homotopy for $\beta \simeq \beta'$.

a.) So what is F? Recite the definition. Similarly, write down everything you know about G.

b.) What are $H(0,t)$, $H(1,t)$, $H(s,0)$, $H(s,1)$, $H(1/2,t)$, $H(s,1/2)$?

c.) H is continuous because of the Pasting Lemma, which we will prove in class. It says

> *Suppose $X = A \cup B$ where A and B are both closed subsets of X. Suppose $f : A \to C$ and $g : B \to C$ are continuous mappings such that $f(x) = g(x)$ for all $x \in A \cap B$. Then the "piecewise defined function" $h : X \to C$ given by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is well-defined and continuous.*

How does the lemma apply in case of the homotopy H above?

(ie. $h = H$ and what are X, A, B, f, g? Is $f = g$ on $A \cap B$?)

EXTRA CREDIT Consider notes for the proof of the classification of compact, connected surfaces as was discussed in class, posted separately, as well as our discussion itself. Make a flowchart of how the diagram of a 2m-gon whose edges are identified in pairs can be changed, by "tricky" cut-and-paste steps, to one of the "standard diagrams" (or normal forms) of the surfaces given in the classification theorem of compact connected surfaces.

(An example of a very simple flowchart is posted separately.)