Intro to Topology – HW 9/10.

1. Using the examples of various fundamental groups we talked about (some intuitively) so far, find topological spaces X, Y and $f : X \to Y$ a continuous map to show that even if f is surjective, the induced homomorphism $f_* : \Pi(X, x_0) \to \Pi(Y, f(x_0))$ need not be surjective.

   Also, even if f is injective, f_* need not be injective.

2. I.a.) Is there a retraction $r : \mathbb{R} \to [0, 1]$?
   1.b.) Is there a retraction $r : \mathbb{R} \to (0, 1)$?
   1.c.) Is there a retraction $\mathbb{R}^2 \to \overline{D}^2$?
   1.d.) Is there a retraction $\mathbb{R}^2 \to S^1$?

   II. For the spaces below show explicitly (i.e. providing formulae for G) that there is a deformation retraction deforming X onto some $A \sim S^1$ of your choice
   a.) $X = S^1 \cup \{(x, 0) \mid x > 0\} \subset \mathbb{R}^2$
   b.) $X = \{(x, y, u, w) \mid x^2 + y^2 \leq 1, u^2 + w^2 = 1\}$ in $\mathbb{R}^4$. Also, what is X?

3. Consider $X = S^1$, the unit circle centered at the origin in $\mathbb{R}^2$ and fix $x_0 = (1, 0)$. Choose an integer $m \in \mathbb{Z}$. For that integer provide a formula for a loop (of your choice) based at x_0 which is homotopic to the "standard loop" $\beta_m(s) = (\cos 2\pi ms, \sin 2\pi ms)$, but is not equal to it.

   Prove that your loop is indeed homotopic to β_m.

4. We will define covering maps on Tuesday. Which of the following are covering maps? Explain briefly.
   a.) The quotient map $q : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ where $(x, y) \sim (x', y')$ if $x - x' \in \mathbb{Z}$ and $y = y'$.
   b.) The quotient map $q : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ where $(x, y) \sim (x', y')$ if $xx' > 0$ and $y = y'$.
   c.) The n-th power mapping of $\mathbb{C} \to \mathbb{C}$ given by $z \to z^n$. (Here z denotes a complex number.)
   d.) The n-th power mapping of $S^1 \to S^1$ given by $z \to z^n$. (Here $S^1 = \{z \in \mathbb{C} \mid ||z|| = 1\}$.)

   In part d.) and e.) I use complex numbers since then the maps have a simple description. But you can use if you wish $\mathbb{R}^2$ instead of $\mathbb{C}$, the unit circle in it and the maps $z = (x, y) = (r\cos\theta, r\sin\theta) \to (r^n \cos n\theta, r^n \sin n\theta) = z^n$.

5. Show that if $p : Y \to X$ is a covering map and X is connected, then the sets $p^{-1}(x)$ have the same cardinality for every $x \in X$.

(Hint: Fix $x_0 \in X$, let $|p^{-1}(x_0)| = \lambda$ and consider the set $A = \{x \in X \mid |p^{-1}(x)| = \lambda\}$. Show that A is both open and closed.)

6. The next formula is very useful to know:

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

a.) What is the isomorphism between the two groups? No proof is necessary, but write a formula for the candidate and its inverse.

With a good/proper notation, this is a straightforward problem. Make sure you state clearly what your functions/maps are.

b.) Show that the following two spaces, Z and W, are not homeomorphic by finding their fundamental groups.

Let Z be the quotient space of $[0,1] \times [0,1] \times [0,1]$, where we set $(x, 0, z) \sim (x, 1, z)$, $(0, y, z) \sim (1, y, z)$ and $(x, y, 0) \sim (x, y, 1)$.

On the other hand W is the quotient space of $[0,1] \times [0,1] \times [0,1]$ where we set $(x, 0, z) \sim (0, x, z)$, $(x, 1, z) \sim (1, x, z)$ and $(x, y, 0) \sim (x, y, 1)$.

(No precise proofs are necessary.)

Extra Credit When proving that the fundamental group $\pi_1(X, x_0)$ is a group indeed, one has to show (among other facts) that $< x_0 >$ is a left unit of the fundamental group $\Pi_1(X, x_0)$, that is $< x_o > * < \alpha > = < \alpha >$ for all α loops of a topological space X based at x_0. I posted a podcast proving this. Have a look at it, then prove that $< x_0 >$ is also a right unit of the fundamental group, that is
$< \alpha > * < x_0 > = < \alpha >$ for all α loops based at x_0.