0.1 Classification of Surfaces

Our "longterm" aim for this semester is to discuss the classification of compact, connected surfaces. Recall that S is a surface if $S \subset \mathbb{R}^M$ for some $M > 0$ integer such that, for all $p \in S$, there exists $U \in \tau_S$, $p \in U$ which is homeomorphic to $\mathbb{R}^2$ or, equivalently, to the open disk D^2.

We are already familiar with some compact, connected surfaces: the sphere, the torus, Klein bottle, real projective plane.

Here is a method to make new surfaces.

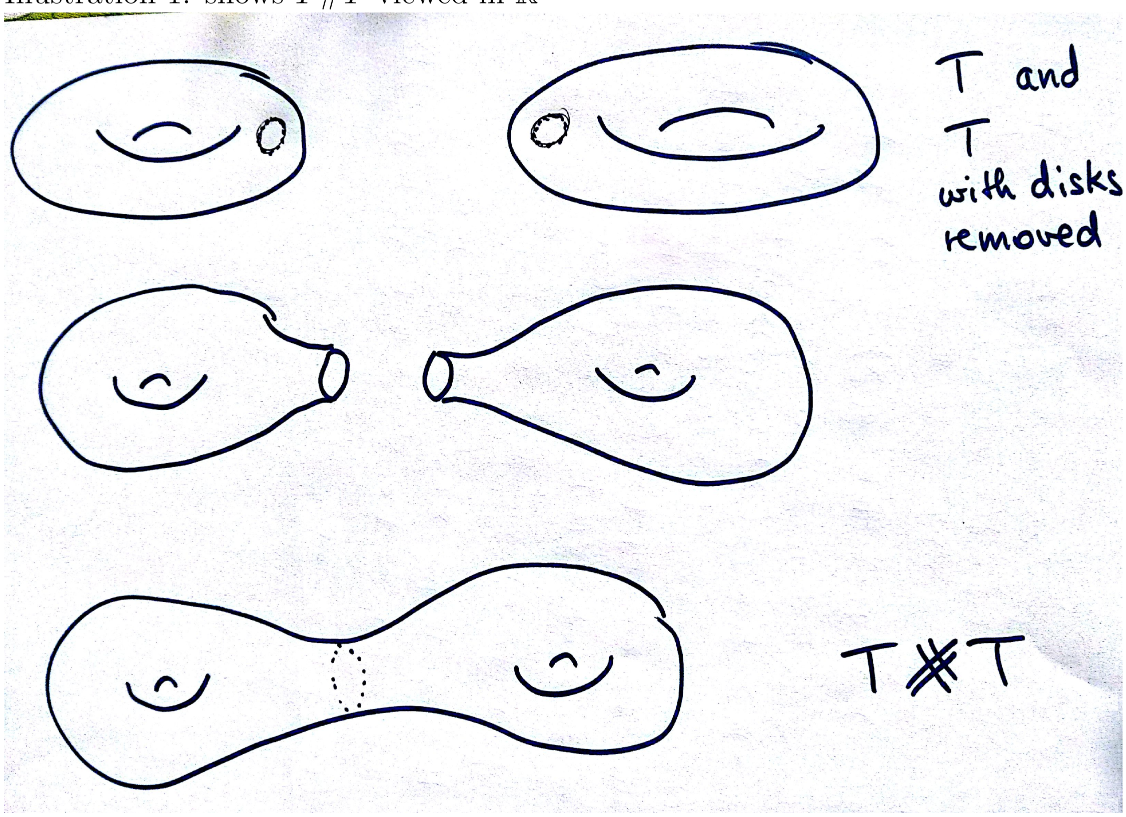
Definition 0.1.1: Given surfaces M and N, their **connected sum** $M\#N$ is the surface which is obtained by the following procedure:

Step 1: Cup out open disks from M and N. Each of the resulting spaces have boundaries homeomorphic to S^1.

Step 2: Glue $M \setminus D^2$ and $N \setminus D^2$ along their S^1 boundaries.

Notation: $M\#N = (M \setminus D^2) \cup_{S^1} (N \setminus D^2)$.

Illustration 1. shows $T\#T$ viewed in $\mathbb{R}^3$

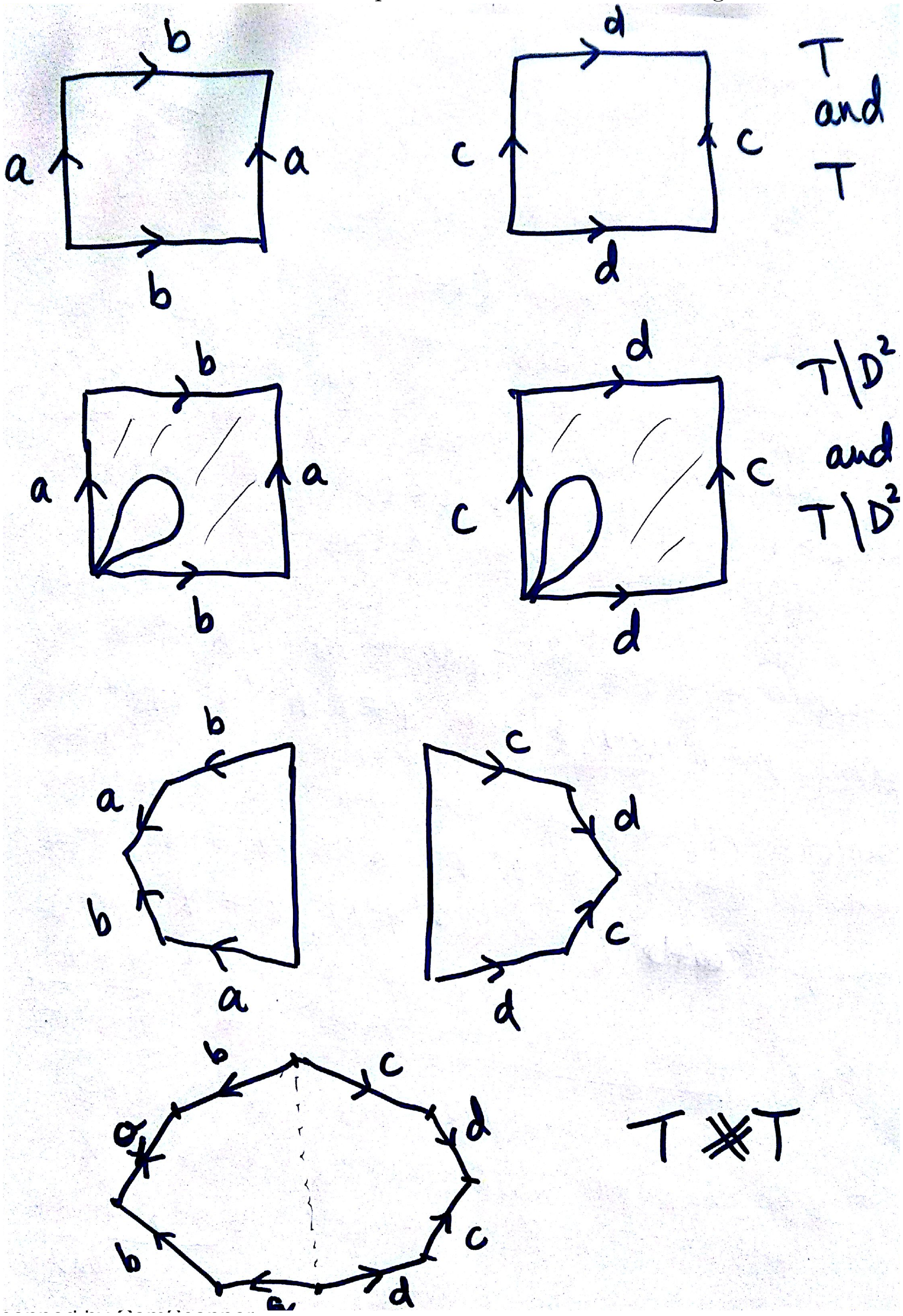


Note that this operation is independent of the actual size of disks cut out (up to homeomorphism).

Convince yourself that *all* points of the connected sum of $M\#N$ – so those too that "come from" identifying pairs of points on the bounding circles of $M \setminus D^2$ and $N \setminus D^2$ –

have open sets around them that are homeomorphic to D^2, the open unit disk centered at the origin, in $\mathbb{R}^2$.

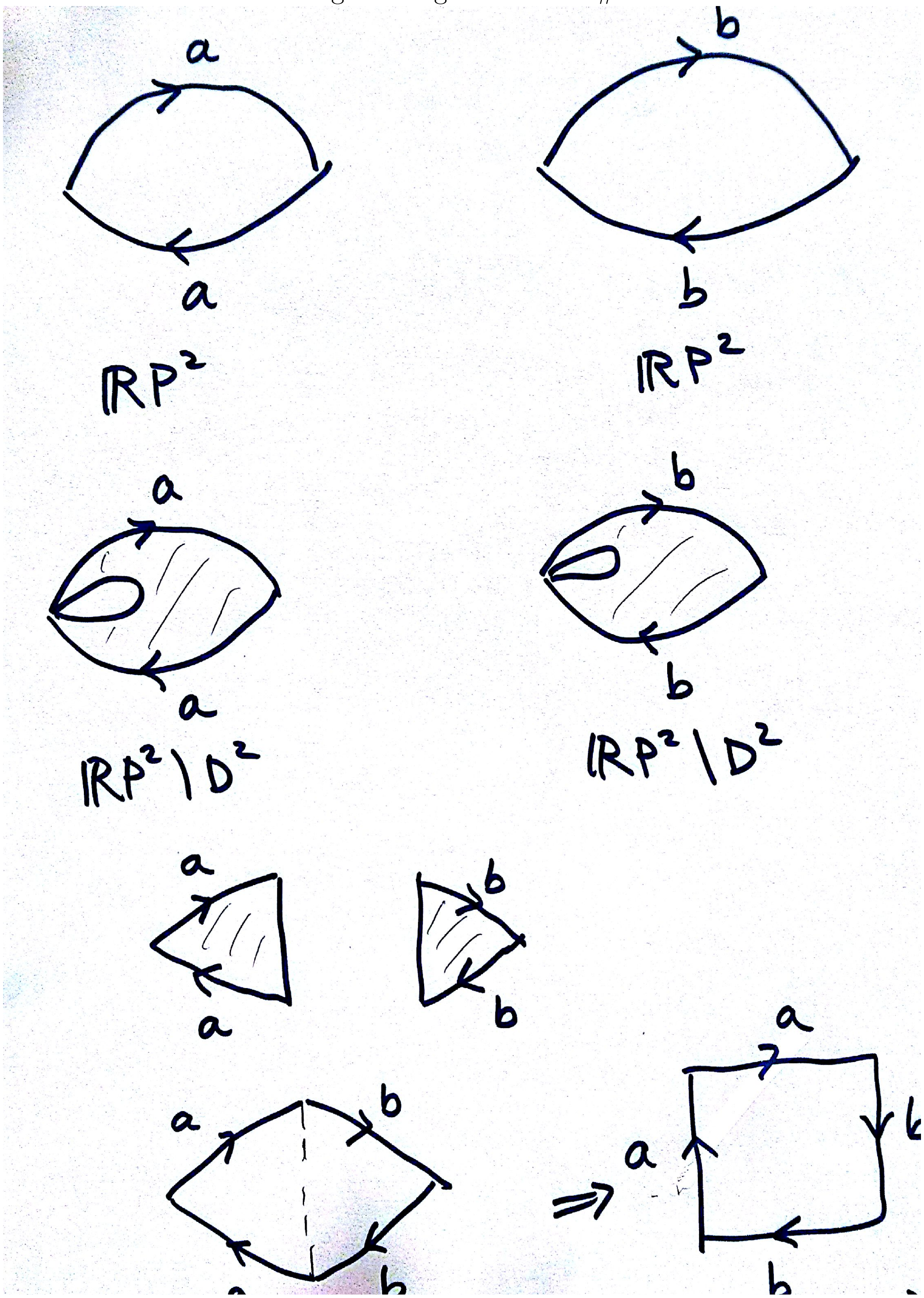
Illustration 2. shows the same operation on identification diagrams.



Thus $T\#T$ can be given as the quotient space of an octagon whose edges are identified according to the word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.
Similarly, by induction, we can build an infinite family of surfaces gluing n tori together to obtain $\#nT = T\#T\#...\#T$. These surfaces can be obtained as quotients of a $4n$-gon with edges identified according to $a_1b_1a_1^{-1}b_1^{-1}...a_nb_na_n^{-1}b_n^{-1}$.

Another family is obtained by gluing real projective planes together.

Illustration 3 shows how to get a diagram for $\mathbb{R}P^2 \# \mathbb{R}P^2$



By induction, $\#m\mathbb{R}P^2$ can be given as the identification space of a $2m$-gon with edges identified according to $a_1a_1...a_ma_m$.
Since the surfaces $\#nT$ and $\#m\mathbb{R}P^2$ are each obtained as quotients of compact, connected polygons, they themselves are compact, connected.

The classification of compact, connected surfaces is:

Theorem 0.1.2: *Any* compact, connected surface is homeomorphic to exactly one of

(a) S^2

(b) $\#nT$ $\forall n \in \mathbb{N}$

(c) $\#m\mathbb{R}\mathrm{P}^2$ $\forall m \in \mathbb{N}$.

Recall that, by definition, a surface is non-orientable if it contains a Mobius strip. Thus the surfaces in (c) are non-orientable and the surfaces in (b) and S^2 are orientable.
We have:

Exercise 0.1.3: Orientability is a topological invariant of surfaces. That is: if X and Y are homeomorphic surfaces then X is non-orientable if and only if Y is non-orientable.

Thus no surface in the (c) family is homeomorphic to a surface in (b) or S^2.
In order to distinguish the surfaces given in family (b) or (c) we will use a combinatorial invariant, called *the Euler characteristic* or *Euler number*.
In order to define the Euler characteristic we need the following:

Definition 0.1.4: Given a surface M, a **finite triangulation** of M is the set $\{T_1, T_2, ..., T_n\}$ such that

- $T_i \subset M$ is closed $\forall i$
- $\forall i$ T_i is homeomorphic to a regular triangle in $\mathbb{R}^2$
- if $i \neq j$ then $T_i \cap T_j$ is either empty or a single vertex or an entire edge
- $M = \cup_{i=1}^{n} T$

Definition 0.1.5: Given a finite triangulation of M, the **Euler characteristic** of M is defined as $\chi(M) = V - E + F$, where V is the number of vertices (up to identification) in the triangulation, E is the number of edges (up to identification), and F is the number of faces (i.e. the number of triangles).

We have the following facts:

1. The Euler characteristic of a surface M is independent of the triangulation of M.
2. The Euler characteristic is a topological invariant.

Furthermore, χ together with orientation distinguish the surfaces in the classification theorem: as a consequence of a crucial theorem of T. Radó from 1925, every compact connected surface is finitely triangulable and thus they each have an Euler characteristic.

Lemma 0.1.6: Let S_1 and S_2 be two different compact connected surfaces in the list given in the classification theorem. Then they are non-homeomorphic because either their Euler characteristic or their orientability is different.

Proof: We already discussed that orientability distinguishes the surfaces that are connected sums of projective planes from surfaces that are connected sums of tori or S^2.

Now, suppose S_1 and S_2 are finitely triangulated. In step 1 of forming the connected sum of S_1 and S_2, remove open faces of a triangle in each, instead of open disks. Then glue the edges of the triangles pairwise, instead of the circle boundaries of disks. Up to homeomorphism, you still get $S_1 \# S_2$. and the following very useful formula which relates the Euler characteristic of the connected sum of surfaces:

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

As a consequence, using induction, the following is a table of the Euler characteristics of compact surfaces.

Surface	*Euler Characteristic*
Sphere	2
Torus	0
$\mathbb{R}\mathrm{P}^2$	1
$\#n$-tori	$2 - 2n \ \forall n \in \mathbb{N}$
$\#m$-$\mathbb{R}\mathrm{P}^2$	$2 - m \ \forall m \in \mathbb{N}$

Definition 0.1.7: When referring to $\#n\mathbb{T}$ or $\#m\mathbb{R}\mathrm{P}^2$, we call n or m the **genus** of the surface.

0.1.1 Working with cell-decompositions

Note that instead of using finite triangulations of compact, connected surfaces, one can calculate the Euler number using "cell-decompositions" of surfaces, which simplify calculations in many examples. A cell-decomposition may be defined recursively:

Definition 0.1.8: Given a compact, connected surface M, its finite cell-decomposition consists of

- 0-cells $\mathsf{C}^0 = \{C_1^0, ..., C_k^0\}$ which are merely points $C_i^0 \in M$, $i = 1, .., k$;
- 1-cells $\mathsf{C}^1 = \{C_1^1, ..., C_m^1\}$ where each $C_i^1 \subset M$, $i = 1, .., m$ is closed, and $C_i^1 \setminus [\{C_j^0\}_{j=1}^k \cup \{C_j^1\}_{j \neq i}]$ is homeomorphic to $(0, 1)$;
- 2-cells $\mathsf{C}^2 = \{C_1^2, ..., C_l^2\}$ where each $C_i^2 \subset M$, $i = 1, .., l$ is closed, and $C_i^2 \setminus [\{C_j^0\}_{j=1}^k \cup \{C_j^1\}_{j=1}^m \cup \{C_j^2\}_{j \neq i}]$ is homeomorphic to D^2 (the open unit disk in $\mathbb{R}^2$);
- and $M = \cup \mathsf{C}^0 \cup \cup \mathsf{C}^1 \cup \cup \mathsf{C}^2$

Note that the above conditions imply that two different cells of the same dimension should be disjoint or intersect in lower dimensional cells only.

Clearly, every compact, connected surface has a finite cell-decomposition, since any of their finite triangulations provide a cell-decomposition.

Lemma 0.1.9: For every compact connected surface M, we have $\chi(M) = V - E + F$ where V =number of 0-cells; E =number of 1-cells; F =number of 2-cells.

Proof (outline): Show that each of the following moves (and their inverses) on a given cell decomposition leaves the alternating sum $V - E + F$ invariant:

1. Subdividing an edge by adding a vertex.
2. Subdividing a face by connecting two vertices with a new edge.
3. Introducing a new vertex in the interior of a face and a new edge connecting that vertex to an existing vertex adjacent to that face.

Since any cell decomposition can be turned into a triangulation by moves 1-3 and/or their inverses, we must have $\chi(M) = V - E + F$.

Remark 0.1.10: Moves 1-3 can also be used to show that χ is independent of triangulation.

Example 0.1.11: Let $M = S^2$ the unit sphere in $\mathbb{R}^3$.

Consider the cell decomposition of S^2 which consists of one 0-cell $C^0 = p = (1, 0, 0)$, one 1-cell that is the great circle $S^2 \cap xy-$plane and two 2-cells that are the upper and lower hemispheres of S^2. Then $\chi(M) = 1 - 1 + 2 = 2$.

Consider the cell decomposition of S^2 which consists of one 0-cell $C^0 = p = (1, 0, 0)$, and one 2-cell that is S^2. Then $\chi(M) = 1 - 0 + 1 = 2$.

Example 0.1.12: Consider the torus. Note that its diagram $aba^{-1}b^{-1}$ determines a cell-decomposition that consists of one 0-cell which is the equivalence class of a vertex of the square; two 1-cells which are the edges a and b and finally, one 2-cell, which is the torus itself. Then $\chi(T) = 1 - 2 + 1 = 0$.