### 0.0.1 Covering Spaces

**Definition 0.0.1:** Given $p : Y \to X$ continuous and onto, $p$ **is a covering map** (and $Y$ **is a covering space of** $X$) if $\forall\, x_0 \in X$, $\exists U_{x_0} \in \tau_X$ with $x_0 \in U_{x_0}$ such that

$$p^{-1}(U_{x_0}) = \dot{\bigcup_{y \in p^{-1}(x_0)}} V_y$$

such that $V_y \in \tau_Y$, and $V_y \cap V_{y'} = \emptyset$ for $y \neq y'$, and moreover, we have that $p\big|_{V_y} : V_y \to U_{x_0}$ is a homeomorphism.

Terminology: the $V_y$ sets are called "**sheets**" and $X$ the "**base space**".

1. Let $Y = X \times \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. Then $p : Y \to X$ defined by $(x, y) \mapsto x$ is a covering, in fact, an "$n$-sheeted" one.

2. $p : \mathbb{R} \to S^1$ by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is a covering. For example, pick $x_0 = (1, 0)$. Then $p^{-1}(x_0) = \mathbb{Z}$. We can pick $U_1 = U_{x_0} = S^1 \cap \{x > 0\} \in \tau_{S^1}$ around $x_0 = (1, 0)$, then $p^{-1}(U_{x_0}) = \bigcup_{n \in \mathbb{Z}} (n - 1/4, n + 1/4)$. This is a disjoint union of open sets. One can check that $p$ restricted to each of the open intervals $(n - 1/4, n + 1/4)$ is a homeomorphism onto $U_{x_0}$.

   The same set $U_{x_0}$ works for all points of $S^1$ with a positive $x$ coordinate.

   Using $U_2 = S^1 \cap \{x < 0\}$, $U_3 = S^1 \cap \{y > 0\}$, $U_4 = S^1 \cap \{y < 0\}$ and similar calculations as for $U_1 = U_{x_0}$ one can verify that indeed for every point $(x, y) \in S^1$ one can find an open set in $S^1$ (namely, $U_1, U_2, U_3$ or $U_4$) such that the pre-image of the open set as is required in the definition of a covering.

3. Let $Y = \mathbb{R}^2$ and $X = \mathbb{R}^2/\sim$, where $(x, y) \sim (x', y')$ if $x - x', y - y' \in \mathbb{Z}$.

   If $X$ has the quotient topology, then the quotient map $q : Y \to X$ is continuous and onto. We claim that $q$ is a covering map.

   For example, let $x_0 = q((0, 0)) \in X$. Then $q^{-1}(x_0) = \{(n, m)\} = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. Consider $U_{x_0} = q(B_{(0,0)}(1/4))$ in X. It contains $x_0$ and is an open set in the quotient, since

   $$q^{-1}(U_{x_0}) = \bigcup_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} B_{(n,m)}(1/4)$$

   which is the disjoint union of open sets, so open in $\mathbb{R}^2$. Then by the definition of quotient topology $U_{x_0}$ is open in $X$.

   The restriction $q|_{B_{(n,m)}(1/4)}$ is 1-1 and onto $U_{x_0}$, by construction. It is also continuous, since it is the restriction of a continuous map.

   Now, restrict further to a smaller <u>closed</u> ball $\overline{B}_{(0,0)}(1/5)$, that is: consider $W_{x_0} = q(\overline{B}_{(0,0)}(1/5))$. Then over this smaller set we have

   $$q^{-1}(W_{x_0}) = \bigcup_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} \overline{B}_{(n,m)}(1/5)$$

and the restrictions

$$q|_{\overline{B}_{(n,m)}(1/5)} : \overline{B}_{(n,m)}(1/5) \to W_{x_0}$$

are still 1-1, onto and continuous, but now $\overline{B}_{(n,m)}(1/5)$ is also compact. $W_{x_0}$ is Hausdorff, since it is a subspace of a Hausdorff space (since $X$ is Hausdorff (why?)). That means $q|_{\overline{B}_{(n,m)}(1/5)}$ is a homeomorphism onto $W_{x_0}$, for each $(n,m) \in \mathbb{Z} \times \mathbb{Z}$.

Finally, taking $U'_{x_0} = q(B_{(0,0)}(1/6))$ we get an open set around $x_0 = q((0,0)) \in X$ such that

$$q^{-1}(U'_{x_0}) = \bigcup_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} B_{(n,m)}(1/6)$$

is a disjoint union of open sets and the restrictions $q|_{B_{(n,m)}(1/6)}$ are homeomorphisms onto $U'_{x_0}$. So $B_{(n,m)}(1/6)$ are sheets above $U'_{x_0}$.

Similar calculations at other points show that $q$ is a covering map.

Note that since $X = \mathbb{R}^2/\sim$ is homeomorphic to the torus, we have that **the plane covers the torus**. The covering map $p = q$ can be described as $p((x,y)) = [(\{x\},\{y\})]$ where $\{x\}$ and $\{y\}$ are the fractional parts of $x$ and $y$, and $[(\{x\},\{y\})]$ denotes the equivalence class of $(\{x\},\{y\})$.

4. Let $Y = S^2$ the unit sphere in $\mathbb{R}^3$ and set $(x,y,z) \sim (-x,-y,-z)$. We claim that the quotient map $q : S^2 \to S^2/\sim$ is a covering map: pick $x_0 \in S^2/\sim$. That is: $x_0 = [(x,y,z)] = \{(x,y,z),(-x,-y,-z)$, it is the equivalence class of some point $(x,y,z) \in S^2$.

   Consider $W = B_(x,y,z)(\epsilon) \cap S^2$, which is open in $S^2$ and let $U_0$ be its image under $q$. Then

   $$q^{-1}(U_0) = W \cup -W$$

   where $-W = \{(-x,-y,-z) \mid (x,y,z) \in W$. Since both $W$ and $-W$ are open in $S^2$, $U_0$ is open in the quotient. Also, if $\epsilon$ is sufficiently small, then $W$ and $-W$ are disjoint.

   By a similar argument as in the previous example, we can find subsets $V_0 \subset W$ and $V_1 = -V_0 \subset -W$ open in $S^2$ such that the restrictions of $p = q$ to $V_0$ and $V_1$ are homeomorphisms.

   Note that since $S^2/\sim = \mathbb{R}P^2$, we have that **the sphere covers to real projective plane**. It is, in fact, **a two-sheeted covering**.

5. Consider the figure eight $S^1 \vee S^1 =: X$. See the handout of some coverings of this space.

**Definition 0.0.2:** Given a covering $p : Y \to X$ fix a basepoint $x_0 \in X$. Suppose $\alpha : [0,1] \to X$ is a path such that $\alpha(0) = x_0$. Then a path $\widetilde{\alpha} : [0,1] \to Y$ is a **lift of $\alpha$ starting at** $y_0$ if $p \circ \widetilde{\alpha} = \alpha$. Clearly, $y_0 \in p^{-1}(x_0)$.

**Lemma 0.0.3** (Path-lifting lemma)**:** Given a path $\alpha : [0,1] \to X$ with $\alpha(0) = x_0$, fix $y_0 \in p^{-1}(x_0)$. Then $\alpha$ has a *unique* lift starting at $y_0$.

We also have the following

**Lemma 0.0.4** (Homotopy lifting lemma)**:** Given a homotopy $H : [0,1] \times [0,1] \to X$ with $H(0,0) = x_0$, fix $y_0 \in p^{-1}(x_0)$. Then there exists a unique homotopy $\widetilde{H} : [0,1] \times [0,1] \to Y$ such that $H = p \circ \widetilde{H}$.

The $\widetilde{H}$ is called "the lift of $H$ starting at $y_0$".

**Corollary 0.0.5:** If $\alpha, \beta : [0,1] \to X$ are paths such that $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1)$ and $\alpha \cong \beta$, then there exist unique lifts $\widetilde{\alpha}, \widetilde{\beta} : [0,1] \to Y$ starting at a fixed $y_0 \in p^{-1}(x_0)$ by the path lifting lemma, moreover $\widetilde{\alpha} \cong \widetilde{\beta}$ by the homotopy lifting lemma. In particular, we must have $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$.

**Corollary 0.0.6** (Very Important)**:** Given paths $\alpha, \beta : [0,1] \to X$ such that $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1)$. If there exists a covering such that $\widetilde{\alpha}(1) \neq \widetilde{\beta}(1)$ for some lifts starting at the same $y_0 \in p^{-1}(x_0)$, then $\alpha$ and $\beta$ are NOT homotopic.

**Example 0.0.7:** The fundamental group of the figure eight is not abelian.

The "Very Important Corollary" can be used to define a **set map** a sort of "counting function" which helps determine several fundamental groups.

**Definition 0.0.8:** Given a covering $p : (Y, y_0) \to (X, x_0)$ (i.e. we assume p(y$_0$) $= x_0$), let $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ be the set map defined by $\langle\alpha\rangle \mapsto \widetilde{\alpha}(1)$, where $\widetilde{\alpha}(0) = y_0$.

Properties of $\phi$:

(i.) First of all, check that it is well-defined.

(ii.) If $Y$ is path-connected, then $\phi$ is surjective. (Why?)

(iii.) If $Y$ has a trivial fundamental group, then $\phi$ is injective.

*Proof.* Assume $\alpha, \beta : [0,1] \to X$ loops are such that $\phi(\langle\alpha\rangle) = \phi(\langle\beta\rangle)$. Then $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Then $\widetilde{\alpha} * \widetilde{\beta}^{-1} \in \pi_1(Y, y_0)$. So $\widetilde{\alpha} * \widetilde{\beta}^{-1} \simeq y_0$ and, therefore, $\widetilde{\alpha} \simeq \widetilde{\beta}$. That is to say, there exists a function $H : [0,1] \times [0,1] \to Y$ deforming $\widetilde{\alpha}$ to $\widetilde{\beta}$, then $p \circ H$ is a homotopy, showing that $\alpha \simeq \beta$.

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Therefore, if $Y$ is path-connected and $\pi_1(Y, y_0) = 0$, then $\phi$ is a bijection.

**Definition 0.0.9:** If $(W, \tau)$ is such that $W$ is path-connected and $\pi_1(W) = 0$, then we say that $W$ is **simply-connected**. If a covering $p : Y \to X$ is such that $Y$ is simply-connected, then $Y$ is called a **universal covering**.

There are very many reasons why a simply connected covering space is called a *universal* covering. For example, a universal covering contains "all the information" about the fundamental group of the base space via $\phi$.

In particular, using universal covers and $\phi$ one can show that

(1.) $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$.

(2.) $\pi_1(S^1) = \mathbb{Z}$.

(3.) For the torus $T$, $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$

(4.) The fundamental group of the figure eight is the free group on two generators.

However, not all topological spaces have universal covers.