8 The Euler characteristic of a surface

Although we have shown that any compact surface is homeomorphic to a sphere, a sum of tori, or a sum of projective planes, we do not know that all these are topologically different. It is conceivable that there exist positive integers m and n, $m \neq n$, such that the sum of m tori is homeomorphic to the sum of n tori. To show that this cannot happen, we introduce a numerical invariant called the *Euler characteristic.*

First, we define the Euler characteristic of a triangulated surface. Let M be a compact surface with triangulation $\{T_1, \ldots, T_n\}$. Let

$$v = \text{total number of vertices of } M,$$

$$e = \text{total number of edges of } M,$$

$$t = \text{total number of triangles (in this case, } t = n).$$

Then,

$$\chi(M) = v - e + t$$

is called the *Euler characteristic* of M.

Examples

8.1 Figure 1.25 suggests uniform methods of triangulating the sphere, torus, and projective plane so that we may make the number of triangles as large as we please. Using such triangulations, the reader should verify that the Euler characteristics of the sphere, torus, and projective plane are 2, 0, and 1, respectively. He should also verify that the Euler characteristics are independent of the number of vertical and horizontal dividing lines in the diagrams for the sphere and torus, and of the number of radial lines or concentric circles in the case of the diagram for the projective plane.

Consideration of these and other examples suggests that $\chi(M)$ depends only on M, not on the triangulation chosen. We wish to suggest a method of proving this. To do this, we shall allow subdivisions of M into arbitrary polygons, not just triangles. These polygons may have any number

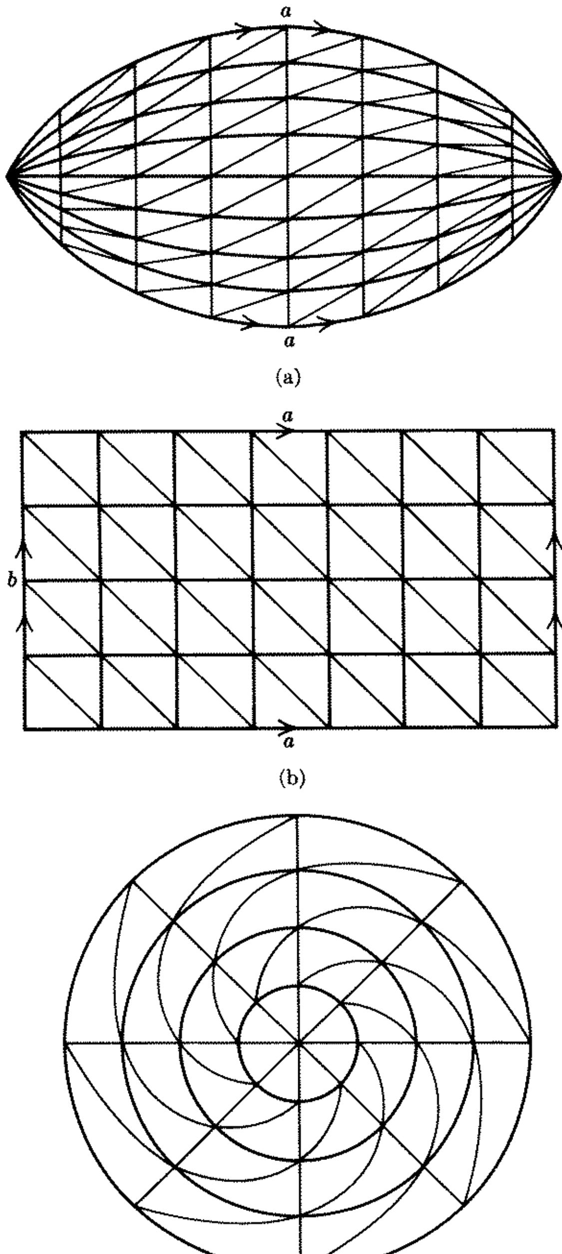

FIGURE 1.25 Computing the Euler characteristic from a triangulation. (a) Sphere. (b) Torus. (c) Projective plane.

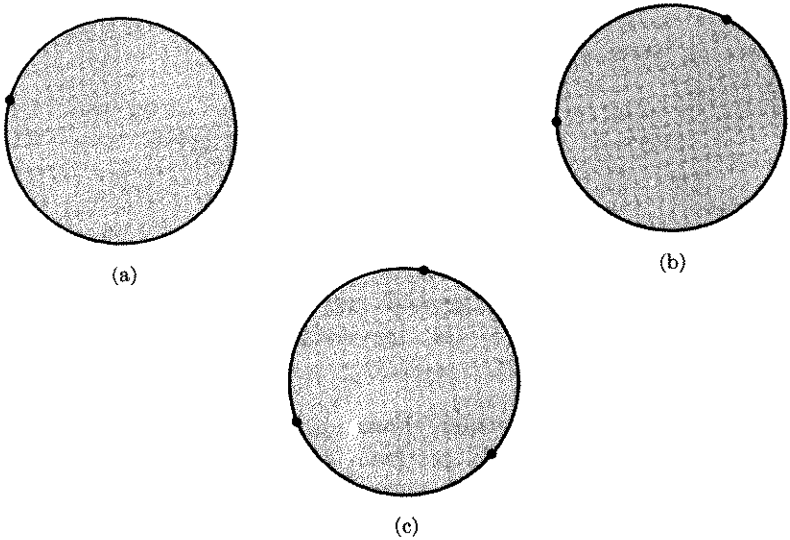

FIGURE 1.26 (a) A 1-gon. (b) A 2-gon. (c) A 3-gon.

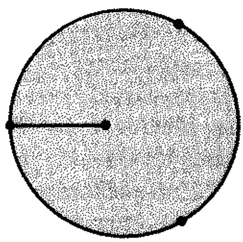

FIGURE 1.27 An allowable kind of edge.

n of sides and vertices, $n \geqq 1$ (see Figure 1.26). We shall also allow for the possibility of edges that do not subdivide a region, as in Figure 1.27. In any case, the interior of each polygonal region is required to be homeomorphic to an open disc, and each edge is required to be homeomorphic to an open interval of the real line, once the vertices are removed (the closure of each edge shall be homeomorphic to a closed interval or a circle). Finally, the number of vertices, edges, and polygonal regions will be finite. As before, we define the Euler characteristic of such a subdivision of a compact surface M to be

$$\chi(M) = (\text{no. of vertices}) - (\text{no. of edges}) + (\text{no. of regions}).$$

It is now easily shown that the Euler characteristic is invariant under the following processes:

(a) Subdividing an edge by adding a new vertex at an interior point (or, inversely, if only two edges meet at a given vertex, we can amalgamate the two edges into one and eliminate the vertex).
(b) Subdividing an n-gon, $n \geqq 1$, by connecting two of the vertices by a new edge (or, inversely, amalgamating two regions into one by removing an edge).
(c) Introducing a new edge and vertex running into a region, as shown in Figure 1.27 (or, inversely, eliminating such an edge and vertex).

The invariance of the Euler characteristic would now follow if it could be shown that we could get from any one triangulation (or subdivision) to any other by a finite sequence of "moves" of types (a), (b), and (c). Suppose we have two triangulations

$$\mathfrak{I} = \{T_1, T_2, \ldots, T_m\}$$

$$\mathfrak{I}' = \{T'_1, T'_2, \ldots, T'_n\}$$

of a given surface. If the intersection of any edge of the triangulation $\mathfrak{I}$ with any edge of the triangulation $\mathfrak{I}'$ consists of a finite number of points and a finite number of closed intervals, then it is easily seen that we can get from the triangulation $\mathfrak{I}$ to the triangulation $\mathfrak{I}'$ in a finite number of such moves; the details are left to the reader. However, it may happen that an edge of $\mathfrak{I}$ intersects an edge of $\mathfrak{I}'$ in an infinite number of points, like the following two curves in the xy plane:

$$\{(x, y) : y = 0 \quad \text{and} \quad -1 \leqq x \leqq +1\},$$

$$\{(x, y) : y = x \sin \frac{1}{x} \quad \text{and} \quad 0 < |x| \leqq 1\} \cup \{(0, 0)\}.$$

If this is the case, it is clearly impossible to get from the triangulation $\mathfrak{I}$ to the triangulation $\mathfrak{I}'$ by any finite number of moves. It appears plausible that we could always avoid such a situation by "moving" one of the edges slightly. This is true, and can be proved rigorously. However, we do not attempt such a proof here, for several reasons: (a) The details are tedious and involved. (b) The Euler characteristic can be defined for more general spaces than surfaces and its invariance can be proven by the use of homology theory. In these more general circumstances, the type of proof we have suggested is not possible. (c) We shall use the Euler characteristic to distinguish between compact surfaces. We shall achieve this purpose with complete rigor in a later chapter by the use of the fundamental group.

Proposition 8.1 *Let S_1 and S_2 be compact surfaces. The Euler characteristics of S_1 and S_2 and their connected sum, $S_1 \# S_2$, are related by the formula*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

PROOF: The proof is very simple; assume S_1 and S_2 are triangulated. Form their connected sum by removing from each the interior of a triangle, and then identifying edges and vertices of the boundaries of the removed triangles. The formula then follows by counting vertices, edges, and triangles before and after the formation of the connected sum.

Using this theorem, and an obvious induction, starting from the known results for the sphere, torus, and projective plane, we obtain the following values for the Euler characteristics of the various possible compact surfaces:

Surface	*Euler characteristic*
Sphere	2
Connected sum of n tori	$2 - 2n$
Connected sum of n projective planes	$2 - n$
Connected sum of projective plane and n tori	$1 - 2n$
Connected sum of Klein Bottle and n tori	$-2n$

Note that the Euler characteristic of an orientable surface is always even, whereas for a nonorientable surface it may be either odd or even.

Assuming the topological invariance of the Euler characteristic and Theorem 5.1, we have the following important result:

Theorem 8.2 *Let S_1 and S_2 be compact surfaces. Then, S_1 and S_2 are homeomorphic if and only if their Euler characteristics are equal and both are orientable or both are nonorientable.*

This is a topological theorem par excellence; it reduces the classification problem for compact surfaces to the determination of the orientability and Euler characteristic, both problems usually readily soluble. Moreover, Theorem 5.1 makes clear what are all possible compact surfaces.

Such a complete classification of any class of topological spaces is very rare. No corresponding theorem is known for compact 3-manifolds, and for 4-manifolds it has been proven (roughly speaking) that no such result is possible.

We close this section by giving some standard terminology. A surface that is the connected sum of n tori or n projective planes is said to be of *genus* n, whereas a sphere is of *genus* 0. The following relation holds between the genus g and the Euler characteristic χ of a compact surface:

$$g = \begin{cases} \frac{1}{2}(2 - \chi) & \text{in the orientable case,} \\ 2 - \chi & \text{in the nonorientable case.} \end{cases}$$

Theorem 3.24 *Suppose A and B are triangulated so that $A \cap B$ is also triangulated. Then $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.*

Proof. For vertices, edges, and faces, the number in $A \cup B$ is the number in A plus the number in B minus the number in $A \cap B$. Hence the alternating sum of these quantities yields the formula for the Euler characteristic. ❄