Consider $f: X \to \mathbb{R}$ continuous,
let $a, b \in f(X)$ s.t. $a < b$ and assume
that $\exists c \in (a, b)$ s.t. $\forall z \in X$, $f(z) \neq c$.
Then let $U = (-\infty, c) \cap f(X)$
$V = (c, +\infty) \cap f(X)$
We have that $a \in U$, $b \in V$ so $U, V \neq \phi$.
Clearly, $U \cap V = \phi$, since $(-\infty, c) \cap (c, +\infty) = \phi$,
Also $f(X) = U \cup V$, since $c \notin f(X)$
by assumption.
Note that $U, V \in \tau_{f(X)}$, by definition of subspace topologies, since $(-\infty, c), (c, +\infty)$ are open in $\mathbb{R}$.
Thus $f(X)$ is disconnected.
However, X is connected and continuous maps (f in this case) have to take connected sets to connected sets. $\lightning$

Let $f: S^1 \to \mathbb{R}$ be continuous and consider $g: S^1 \to \mathbb{R}$ where

$$g(\underline{x}) = f(\underline{x}) - f(-\underline{x}), \quad \forall\, \underline{x} = (x_1, x_2) \in S^1.$$

Note that
$$\begin{aligned} g(-\underline{x}) &= f(-\underline{x}) - f(-(-\underline{x})) \\ &= f(-\underline{x}) - f(\underline{x}) \\ &= -g(\underline{x}) \end{aligned}$$

Thus if $g(\underline{x}) > 0$ then $g(-\underline{x}) < 0$

and if $g(\underline{x}) < 0$ then $g(-\underline{x}) > 0$.

By the Generalized Intermediate Value Theorem then $\exists\, \underline{x}' \in S^1$ s.t.

$g(\underline{x}') = 0$. That is $f(\underline{x}') - f(-\underline{x}') = 0$

$\Rightarrow f(\underline{x}') = f(-\underline{x}')$ for that $\underline{x}'$.

In case $g(\underline{x}) = 0$ (neither > 0 nor < 0) we have $f(\underline{x}) = f(-\underline{x})$ for that $\underline{x} \in S^1$.