Let

$A = \{(x,y) \mid x \in \mathbb{Q}, y \in \mathbb{Q}\}$ is not path-connected, because it is not connected.

Consider

$$U = \{(x,y) \mid x < \pi\} \cap A$$

$$V = \{(x,y) \mid x > \pi\} \cap A$$

both are open in A, since the sets we intersect A with are open in $\mathbb{R}^2$.

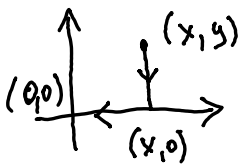
U, V are non-empty and

$A = U \cup V$. So A is indeed disconnected

Let $B = \{(x,y) \mid x \in \mathbb{Q}, \text{ or } y \in \mathbb{Q}\}$

This is path-connected.

WLOG assume $x \in \mathbb{Q}$.



Then the vertical line segment $s \mapsto (x, sy)$ is in B, and the horizontal line segment from $(x,0)$ to $(0,0)$ is in B.

Thus (x,y) can be connected to $(0,0)$ by a path.

(For $y \in \mathbb{Q}$ take a horizontal line segment first, then vertical.)

$\Rightarrow$ Every pt of B can be connected to $(0,0)$ by a path.

Then $\forall\ (x,y), (x',y') \in B$ can be connected by a path via $(0,0)$