① a) Using that the full ~~area~~ surface area of S^2 is 4π and the lune ~~of~~ α angle ~~to~~ has a proportional part we have

$$\frac{\alpha}{2\pi} = \frac{\text{area of lune}}{4\pi}$$

$\Rightarrow$ area of lune $= 2\alpha$

and so area of double lune $= 4\alpha$

b.) A geometric spherical triangle T of inner angles α, β, γ lies at the intersection of lunes of angles α, β and γ.

Now, consider double lunes since the double lunes of angle α, β and γ cover S^2:

double lune α $\cup$ double lune β $\cup$ double lune γ $= S^2$

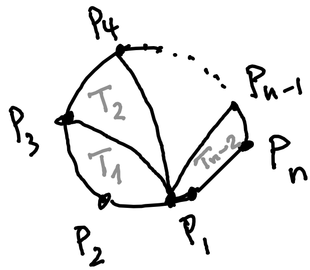
However, this union is not disjoint, the double lunes intersect in <u>two</u> antipodal geometric triangles of inner angles α, β, γ.

So for the areas we have

$$4\alpha + 4\beta + 4\gamma - 4\,(\text{area of } T) = 4\pi$$

$\Rightarrow A(T) = \alpha + \beta + \gamma - \pi$, where $A(T) =$ area of triangle T.

c.) Consider now a geometric n-gon K with vertices $P_1, \dots, P_n$.

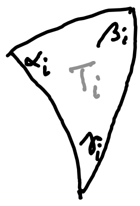


Fix P_1. For each P_i, $i \in \{3, \dots, n-1\}$ there is exactly one plane through P_1, P_i and O

(where O is the center of S^2).

~~the~~ Consider the intersection of this plane with $S^2 \Rightarrow$ get $n-2$ geometric triangles $\{T_1, \dots, T_{n-2}\}$.

Denote the inner angles of triangle T_i as $\alpha_i, \beta_i, \gamma_i$ for $i = 1, \dots, n-2$.



Also, denote the inner angles of the n-gon K as $\gamma_1, \dots, \gamma_n$.

Then we get for the areas

$$A(K) = \sum_{i=1}^{n-2} A(T_i) = \sum_{i=1}^{n-2} (\alpha_i + \beta_i + \gamma_i) - \pi$$

$$= \sum_{i=1}^{n} \gamma_i - (n-2)\pi$$

d.) If S^2 has a cell-decomposition into $P_1, \dots, P_k$ polygons s.t. P_i is an n_i-gon and its inner angles are $\gamma_1, \dots, \gamma_{n_i}$ we have

$$A(S^2) = \sum_{i=1}^{n} A(P_i) =$$

$$= \sum_{i=1}^{k} \left((\gamma_1 + \dots + \gamma_{n_i}) - (n_i - 2)\pi \right) =$$

$$= \underbrace{\sum_{i=1}^{k} (\gamma_1 + \dots + \gamma_{n_i})}_{1} - \underbrace{\sum_{i=1}^{k} n_i \pi}_{2} + \underbrace{\sum_{i=1}^{k} 2\pi}_{3} =$$

1 when you add up all the inner angles of all the polyons in the cell-decomposition you get 2π at each vertex, so considering the total sum you get $2\pi v$, where v = the number of vertices

2 $\sum_{i=1}^{k} n_i \pi = \pi \sum_{i=1}^{k} n_i$, where n_i = the # of vertices i.e. edges of P_i

So $\sum_{i=1}^{k} n_i = 2e$, where e = the number of edges in the whole cell-decomposition.

Since $\sum_{i=1}^{k} n_i$ adds the # of edges for each polygon, each edge is counted twice.

③ $\sum_{i=1}^{k} 2\pi = 2\pi \cdot k = 2\pi f$

since k = # of polygons = # of faces in the cell-decomposition

Thus we end up with

$$A(S^2) = 2\pi v - 2\pi e + 2\pi f = 2\pi \chi(S^2)$$

Since $A(S^2)$ is fixed, $\chi(S^2)$ has to be fixed, too.

Also, since $A(S^2) = 4\pi$, we get $\chi(S^2) = 2$.