

The proof of the classification theorem of compact, connected surfaces relies on Rado's Theorem from 1925 that ensures:

Step 0: *Any compact, connected surface can be triangulated by finite many triangles.*

Using the triangulation one can show the following fact:

Step 1: *Any compact, connected surface can be represented as the quotient space of a polygon (some $2k$ -gon) whose edges are identified in pairs.*

From this, one then uses a series of clever cut-and-paste "tricks" to get such a polygon to normal form, i.e. a polygon whose edge labels either correspond to the word aa^{-1} (sphere case) or $a_1b_1a_1^{-1}b_1^{-1} \dots a_nb_na_n^{-1}b_n^{-1}$ (connected sum of n tori case) or $a_1a_1 \dots a_ma_m$ (connected sum of n projective planes case):

Second step. Elimination of adjacent edges of the first kind. We have now obtained a polygon D whose edges have to be identified in pairs to obtain the given surface S . These identifications may be indicated by the appropriate symbol; e.g., in Figure 1.16, the identifications are described by

$$aa^{-1}fb^{-1}f^{-1}e^{-1}gcc^{-1}g^{-1}dd^{-1}e.$$

If the letter designating a certain pair of edges occurs with *both* exponents, $+1$ and -1 , in the symbol, then we will call that pair of edges a pair of the *first kind*; otherwise, the pair is of the *second kind*. For example, in Figure 1.16, all seven pairs are of the first kind.

We wish to show that an adjacent pair of edges of the first kind can be eliminated, provided there are at least four edges in all. This is easily seen from the sequence of diagrams in Figure 1.17. We can continue this process until all such pairs are eliminated, or until we obtain a polygon with only two sides. In the latter case, this polygon, whose symbol will be aa or aa^{-1} , must be a projective plane or a sphere, and we have completed the proof. Otherwise, we proceed as follows.

Third step. Transformation to a polygon such that all vertices must be identified to a single vertex. Although the edges of our polygon must be

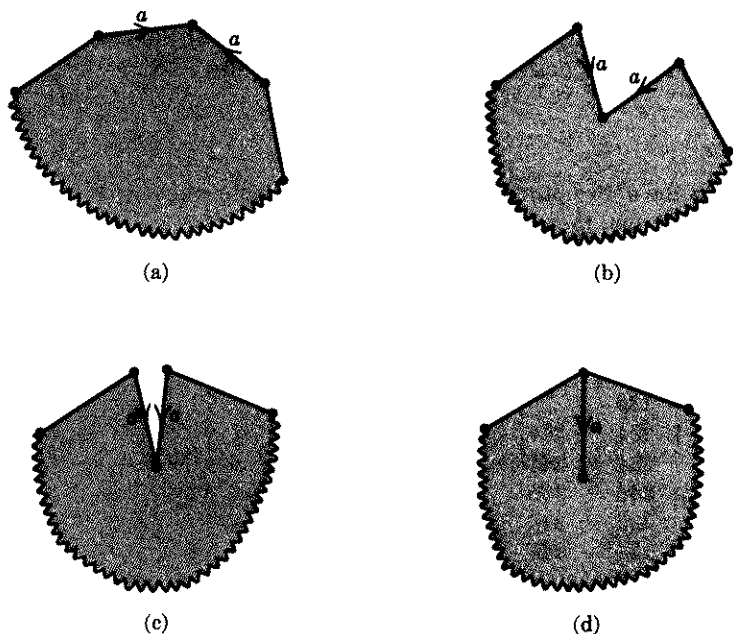


FIGURE 1.17 Elimination of an adjacent pair of edges of the first kind.

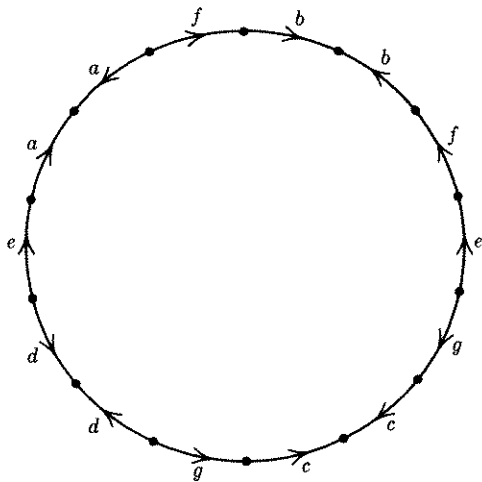


FIGURE 1.16 Simplified version of polygon shown in Figure 1.15.

identified in pairs, the vertices may be identified in sets of one, two, three, four, Let us call two vertices of the polygon *equivalent* if and only if they are to be identified. For example, the reader can easily verify that in Figure 1.16 there are eight different equivalence classes of vertices. Some equivalence classes contain only one vertex, whereas other classes contain two or three vertices.

Assume we have carried out step two as far as possible. We wish to prove we can transform our polygon into another polygon with all its vertices belonging to one equivalence class.

Suppose there are at least two different equivalence classes of vertices. Then, the polygon must have an adjacent pair of vertices which are nonequivalent. Label these vertices P and Q . Figure 1.18 shows how to proceed. As P and Q are nonequivalent, and we have carried out step two, it follows that sides a and b are *not* to be identified. Make a cut along the line labeled c , from the vertex labeled Q to the other vertex of the edge a (i.e., to the vertex of edge a , which is distinct from P). Then, glue the two edges labeled a together. A new polygon with one less vertex in the equivalence class of P and one more vertex in the equivalence class of Q results. If possible, perform step two again. Then carry out step three to reduce the number of vertices in the equivalence class of P still further, then do step two again. Continue alternately doing step three and step two until the equivalence class of P is eliminated entirely. If more than one equivalence class of vertices remains, we can repeat this procedure to reduce the number by one. If we continue in this manner, we ultimately obtain a polygon such that all the vertices are to be identified to a single vertex.

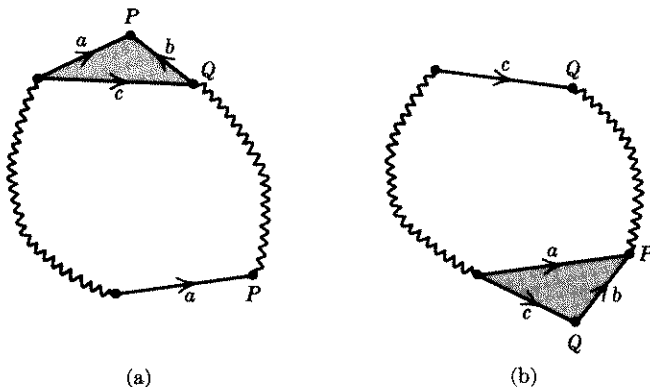


FIGURE 1.18 Third step in the proof of Theorem 5.1.

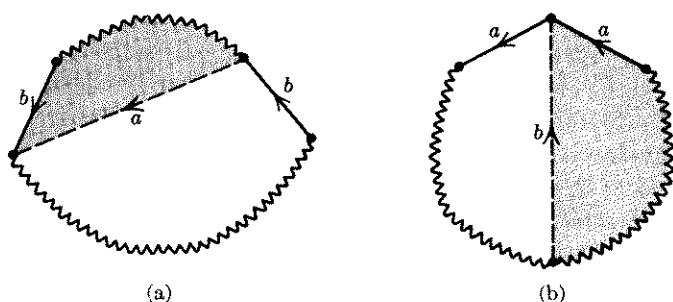


FIGURE 1.19 Fourth step in the proof of Theorem 5.1.

Fourth step. How to make any pair of edges of the second kind adjacent. We wish to show that our surface can be transformed so that any pair of edges of the second kind are adjacent to each other. Suppose we have a pair of edges of the second kind which are nonadjacent, as in Figure 1.19(a). Cut along the dotted line labeled a and paste together along b . As shown in Figure 1.19(b), the two edges are now adjacent.

Continue this process until all pairs of edges of the second kind are adjacent. If there are no pairs of the first kind, we are finished, because the symbol of the polygon must then be of the form $a_1a_1a_2a_2 \dots a_na_n$, and hence S is the connected sum of n projective planes.

Assume to the contrary that at this stage there is at least one pair of edges of the first kind, each of which is labeled with the letter c . Then we assert that there is at least one other pair of edges of the first kind such that these two pairs separate one another; i.e., edges from the two pairs occur alternately as we proceed around the boundary of the polygon (hence, the symbol must be of the form $c \dots d \dots c^{-1} \dots d^{-1} \dots$, where the dots denote the possible occurrence of other letters).

To prove this assertion, assume that the edges labeled c are not separated by any other pair of the first kind. Then our polygon has the appearance indicated in Figure 1.20. Here A and B each designate a whole sequence of edges. The important point is that any edge in A must be identified with another edge in A , and similarly for B . No edge in A is to be identified with an edge in B . But this contradicts the fact that the initial and final vertices of either edge labeled “ c ” are to be identified, in view of step number three.

Fifth step. Pairs of the first kind. Suppose, then, that we have two pairs of the first kind which separate each other as described (see Figure 1.21). We shall show that we can transform the polygon so that the four sides in question are consecutive around the perimeter of the polygon.

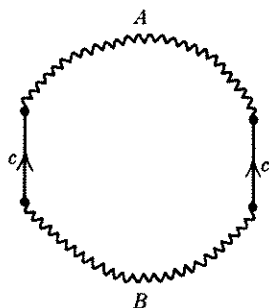


FIGURE 1.20 A pair of edges of the first kind.

First, cut along c and paste together along b to obtain Figure 1.21(b). Then, cut along d and paste together along a to obtain (c), as desired.

Continue this process until all pairs of the first kind are in adjacent groups of four, as $cdc^{-1}d^{-1}$ in Figure 1.21(c). If there are no pairs of the

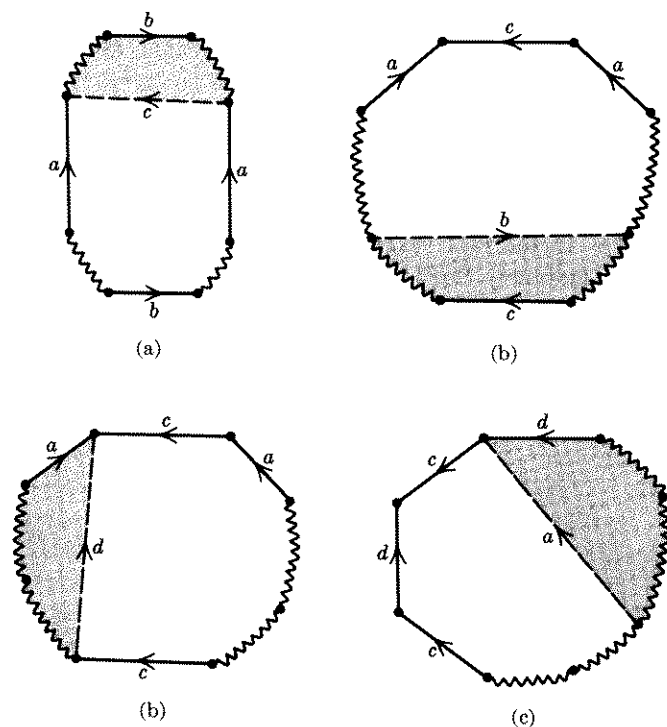


FIGURE 1.21 Fifth step in the proof of Theorem 5.1.

second kind, this leads to the desired result because, in that case, the symbol must be of the form

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_n b_n a_n^{-1} b_n^{-1}$$

and the surface is the connected sum of n tori.

It remains to treat the case in which there are pairs of both the first and second kind at this stage. The key to the situation is the following rather surprising lemma:

Lemma 7.1 *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*