In Section 3.2 we developed an extensive list of examples of compact surfaces: the 2-sphere, the torus, the projective plane, and various connected sums of these basic surfaces (including the Klein bottle and surfaces of higher genus). From Section 3.2 we have invariants to distinguish among various topological types of surfaces: the Euler characteristic and orientability. The following theorem provides the final ingredient needed for a classification of closed surfaces.

Classification Theorem for Closed Surfaces 3.33 *A closed, path-connected surface is homeomorphic to a 2-sphere, a connected sum of tori, or a connected sum of projective planes. The Euler characteristic and orientability of the surface distinguish among these possibilities.*

Proof. By Example 3.27 and Exercises 4 and 6 of Section 3.3, the connected sum of g tori is orientable and has Euler characteristic $2 - 2g$. Also, the connected sum of p projective planes is nonorientable and has Euler characteristic $2 - p$. Thus, the Euler characteristic and orientability of a closed, path-connected surface suffice to distinguish among these various possibilities.

By Lemma 3.32 we can represent the surface as a polygonal disk with edges identified in pairs. The standard proof of the classification theorem involves a series of steps to rearrange the edges until it is possible to read off the identifications as producing connected sums of tori and projective planes. A proof by Jerome Dancis simplifies this by removing connected sum components of tori and projective planes as soon as they are identified. This proof is an adaptation of that approach.

We proceed by induction on the number of pairs of edges to be identified in the boundary of the polygonal disk. If there is only one pair of edges, the edges in the boundary are identified according to the word aa^{-1} or the word aa. In the first case, the surface is a sphere; in the second case, it is a projective plane. If there are more than one pair of edges, we will consider five possibilities in the following paragraphs.

The simplest case is a pair of adjacent edges to be identified with opposing orientations. As illustrated in Figure 3.34, this pair of edges, corresponding to the sequence aa^{-1}, can be folded in and glued together to yield a polyhedral disk for the same surface, but with one less pair of edges.

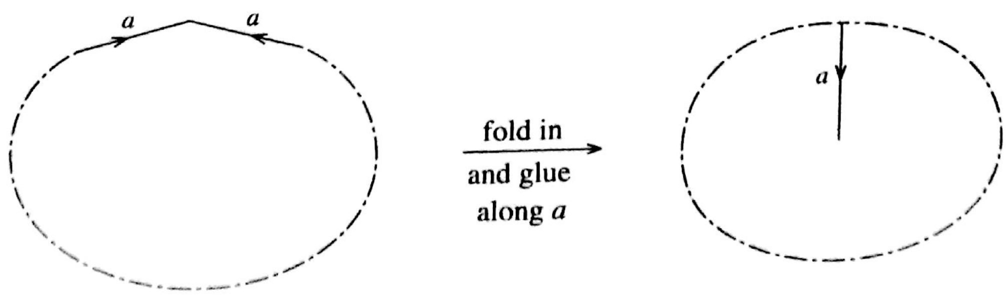

FIGURE 3.34
Identifying adjacent edges with opposing orientations.

An adjacent pair of edges to be identified with the same orientation (such as aa in Figure 3.35) would require some twisting of the disk to perform this glue job. Instead, the following argument allows us to recognize the surface as the connected sum of a projective plane and surface formed from the disk with this pair of edges removed. Notice that since the vertex at the head of a is the same as the vertex at the tail of a, the curve b in Figure 3.35 is actually a circle in the surface. Cut along this curve and glue in disks along the copies of b in each of the two resulting components. One of the pieces will be a disk with one less pair of edges to be identified. The other piece will be a disk with an aa pair to be identified; that is, it will be a projective plane. Notice that the operations we have performed are the reverse of the steps for forming the connected sum of the surface and a projective plane. Thus, we can reconstruct the original surface by forming this connected sum.

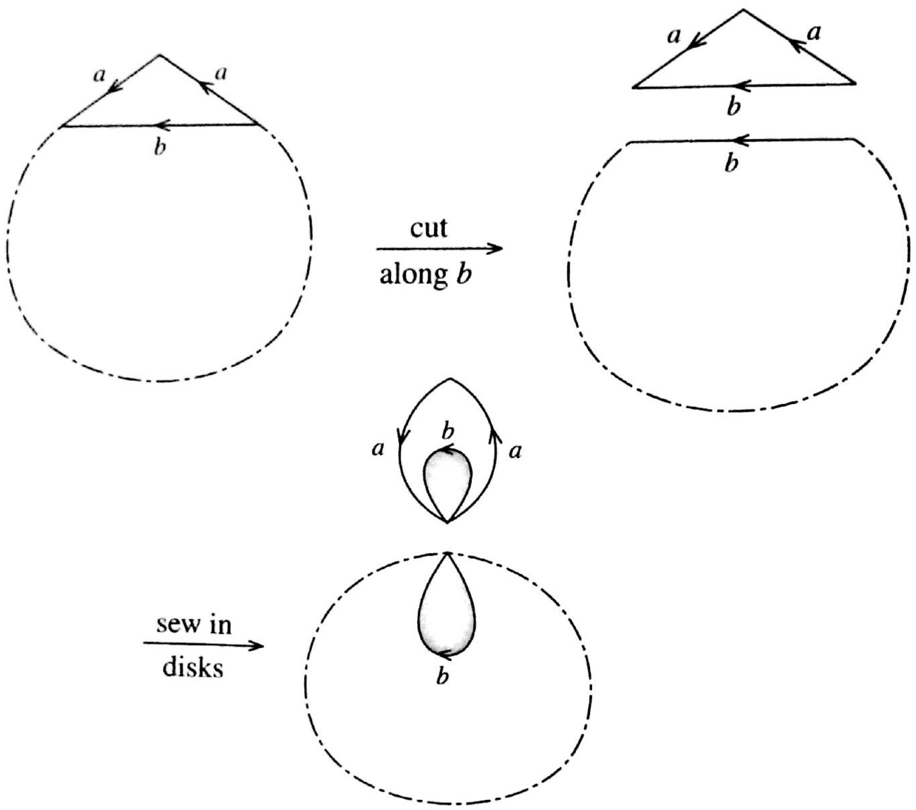

FIGURE 3.35
An adjacent pair of edges oriented in the same direction gives a projective plane as a connected sum component.

Next consider a nonadjacent pair of edges with the same orientation. Figure 3.36 illustrates this case with edges labeled a. Cut the disk along an arc b joining the heads of the two edges. Glue the two pieces together along a. This disk has the same number of pairs of edges, but it has an adjacent pair of edges bb with the same orientation. Thus, we can proceed as in the previous case to cut off a projective plane as a connected sum component leaving a disk with one less pair of edges.

The final two cases arise from nonadjacent pairs oriented in opposing directions. The two edges (labeled a in Figures 3.37 and 3.38) separate the boundary of the disk into two arcs, one joining the heads of the two a edges and one joining their tails.

For the first of these two cases, suppose that none of the edges in either of these arcs is identified with any edge in the other arc. Let c_1 and c_2 denote these two arcs. Glue the disk

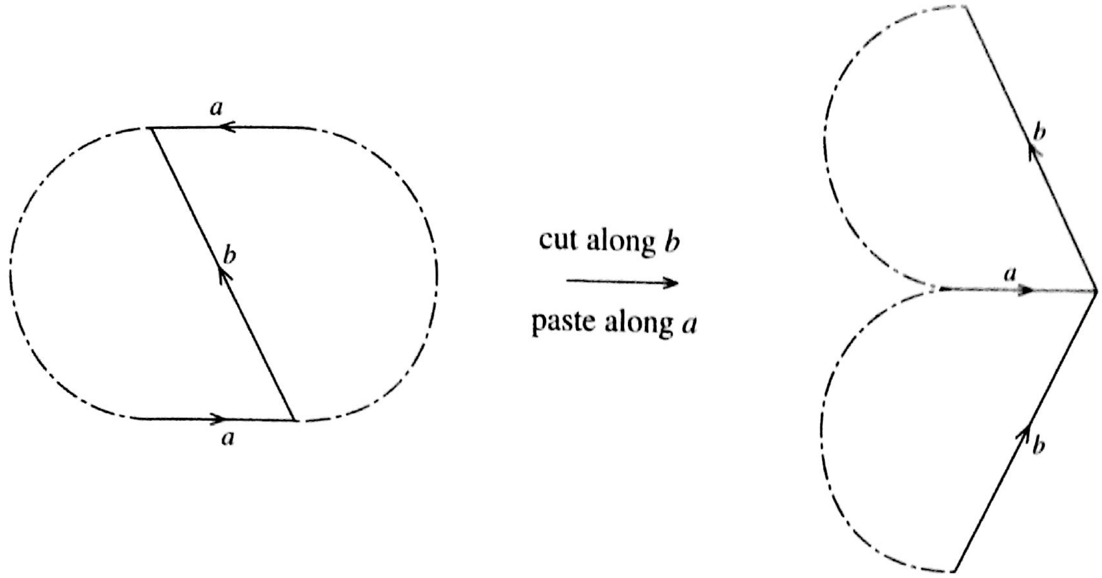

FIGURE 3.36
A nonadjacent pair of edges oriented in the same direction converted into an adjacent pair of edges.

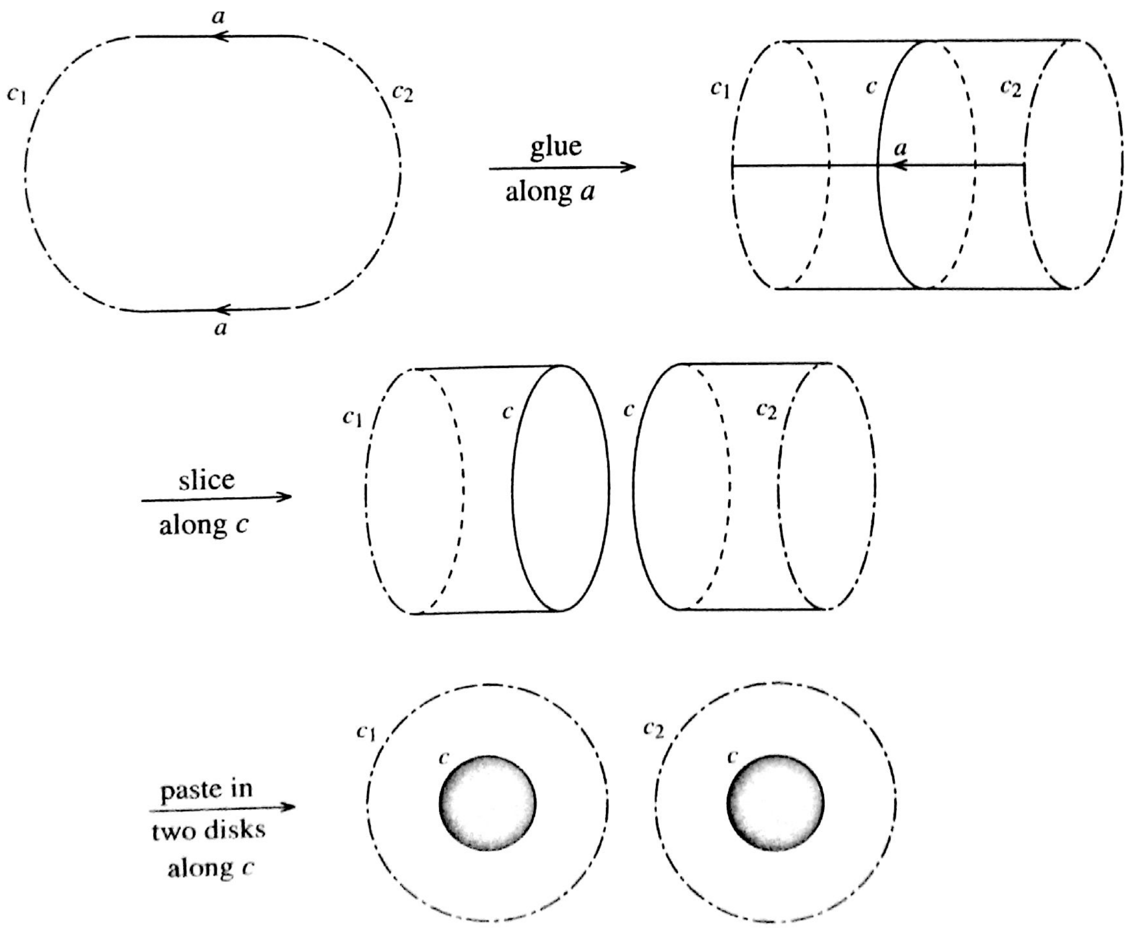

FIGURE 3.37
A nonadjacent pair of edges oriented in opposing directions.

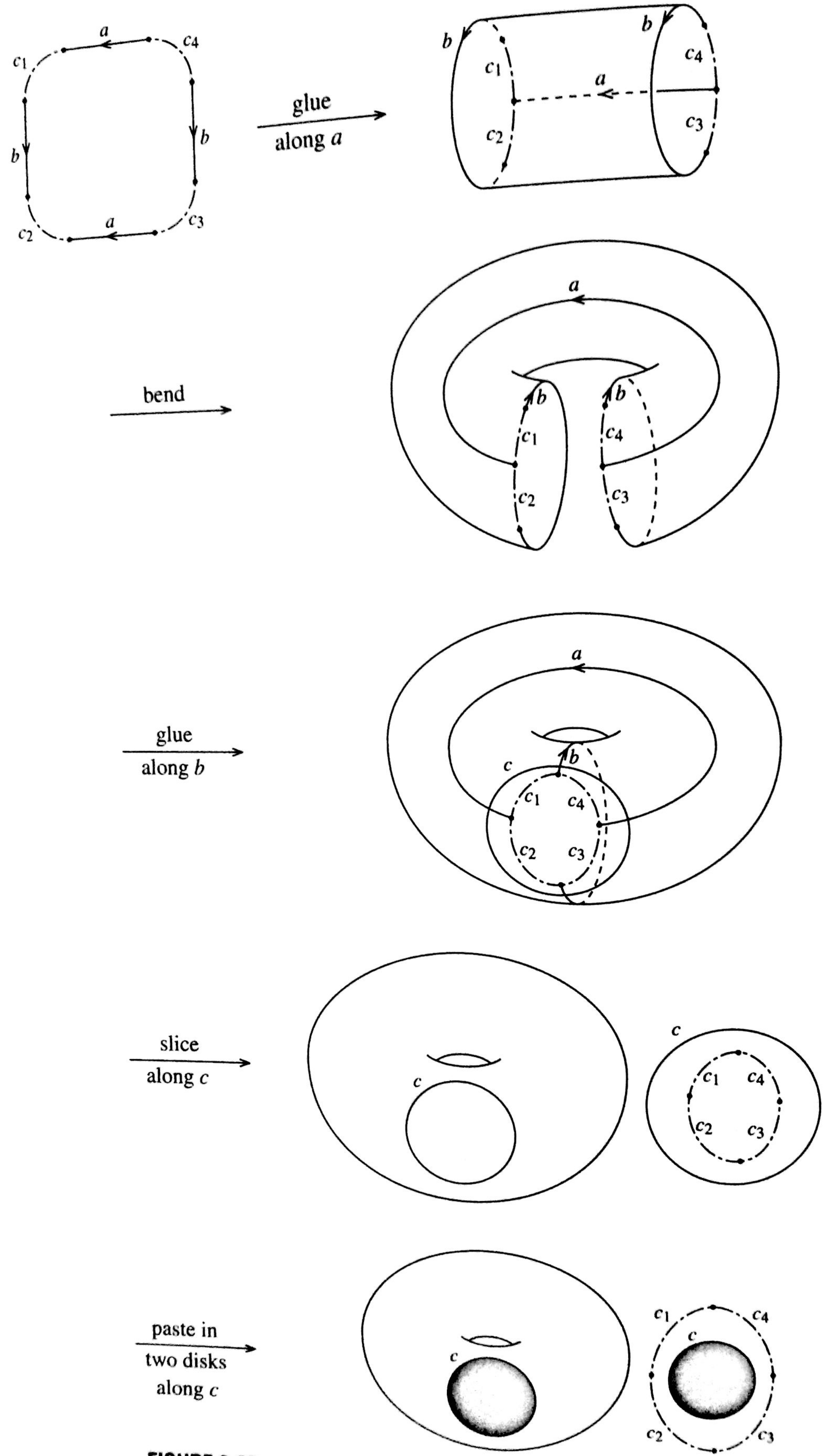

FIGURE 3.38

Two pairs of edges that alternate along the boundary of the disk.

together along the edges labeled a to form a cylinder. Notice how the two ends of c_1 and the two ends of c_2 have been glued together so that they form the boundary circles of the cylinder. Slice the cylinder along a curve c that separates c_1 and c_2. Finally, paste in two disks along the copies of c in the two pieces of the cylinder. The result is two disks whose boundaries c_1 and c_2 consist of edges to be identified in pairs to form two surfaces. The operation of slicing and pasting in the two disks is the inverse of the operation of forming a connected sum. Thus, the two disks represent surfaces whose connected sum is the original surface. Notice that each of the two disks has fewer pairs of edges than the original disk.

Last, but not least, is the case of nonadjacent pairs of edges oriented in opposing directions that are interlaced with another pair of edges as we go around the boundary of the disk. For the sake of simplicity, we can assume that all pairs with the same orientation have been removed by the procedures described in the second and third cases. Thus, the two pairs of edges are as indicated by a and b in Figure 3.38 with four arcs c_1, c_2, c_3, and c_4 connecting them to form the boundary of the disk. Glue the disk together along the edges labeled a and b to form a torus with a hole. Notice how the four arcs c_1, c_2, c_3, and c_4 join together to form the boundary curve of the hole. Slice along a curve c that separates this boundary curve from the rest of the torus. Finally, paste in two disks along the copies of c in the two pieces. The result is a torus and a disk whose boundary consist of edges to be identified in pairs to form a surface. The operation of slicing and pasting in the two disks is the inverse of the operation of forming a connected sum. Thus, the torus and disk represent surfaces whose connected sum is the original surface. Notice that the disk has fewer pairs of edges than the original disk.

Iteratively applying processes described in the previous five cases will decompose the original surface as a connected sum of tori and projective planes. If any projective planes appear, we can use Exercises 7 and 8 of Section 8 to replace $T^2 \# P^2$ by $P^2 \# P^2 \# P^2$. Repeat this as often as necessary to write the surface as a connected sum of projective planes. ❄